EXISTENCE AND UNIQUENESS THEOREMS
FOR EXTENSIONS OF
ZERO-DIMENSIONAL COMPACT METRIC SPACES

BY

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1. **Introduction.** One of the central problems in the theory of Boolean algebras is the classification of all countable Boolean algebras. Using the Stone representation theorem, this objective can be translated into topological form: classify the homeomorphism types of all zero-dimensional, compact metric spaces. The purpose of this paper is to prove an existence theorem and a uniqueness theorem which shed some light on this classification problem.

For any metric space X, and for any ordinal number ξ, let $X^{(\xi)}$ denote the ξth topological derivative of X. The definition of this concept will be recalled in detail below. The space X is called *perfect* if $X^{(1)} = X$. The opposite extreme occurs when $X^{(\xi)} = \varnothing$ for some ξ. In this case, X is said to be *scattered*. Any nonempty, perfect, zero-dimensional compact metric space is homeomorphic to the Cantor discontinuum $\mathscr{D}$. A scattered, zero-dimensional compact metric space is homeomorphic to the ordered space $\omega^\mu \cdot n + 1$ for some uniquely determined, countable ordinal number μ, and some unique natural number n. Thus, the classification problem for zero-dimensional, compact metric spaces has a satisfactory solution for either perfect or scattered spaces. The spaces between these extremes have a unique decomposition as a disjoint union of a nonempty, closed, perfect subspace, and a nonempty, open, scattered subspace. The classification of these subspaces can be handled by the results for the perfect and scattered cases, so that the problem is one of determining how the Cantor discontinuum can be extended by adjoining a particular scattered space. It is to this question that our main theorems are addressed.

THEOREM 1.1. *Let X and Y be zero-dimensional, compact metric spaces. Let $Z \subseteq X$ and $W \subseteq Y$ be closed subsets such that $X - Z$ and $Y - W$ are scattered. Suppose that $\varphi: Z \to W$ is a homeomorphism. Then φ can be extended to a homeomorphism of X onto Y if and only if*
  (a) $X - Z \simeq Y - W$, *and*
  (b) $\varphi(Z \cap (X^{(\xi)} - Z)^-) = W \cap (Y^{(\xi)} - W)^-$ *for all ordinal numbers ξ.*

The conditions (a) and (b) of Theorem 1.1 are plainly necessary; the essential content of the theorem is the fact that they are also sufficient.

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THEOREM 1.2. *Let Z be a zero-dimensional, compact metric space, and suppose that $Z \supseteq Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_\xi \supseteq \cdots$, $\xi < \mu$, where each Z_ξ is closed and nonempty. Then there is a zero-dimensional compact metric space $X \supseteq Z$ such that*

(a) *$X - Z \simeq \omega^\mu$, and*

(b) *$Z \cap (X^{(\xi)} - Z)^- = Z_\xi$ for all $\xi < \mu$.*

This result clearly shows the complexity of the class of zero-dimensional, compact metric spaces.

2. Terminology and background.   We will use the term *Boolean space* to designate a zero-dimensional, compact Hausdorff space. The name *Stone space* will mean a locally compact, zero-dimensional, Hausdorff space. The class of all metrizable Boolean spaces will be denoted by $\mathfrak{S}$. By the Urysohn metrization theorem, a Boolean space X belongs to $\mathfrak{S}$ if and only if X satisfies the second axiom of countability.

It is well known that every $X \in \mathfrak{S}$ is homeomorphic to a closed subset of the Cantor discontinuum $\mathscr{D}$. This fact has the useful consequence that if $X \in \mathfrak{S}$, then there is a metric d for X that satisfies: $d(x, y) = d(x, z)$ implies $y = z$, for all x, y, and z in X. Indeed, $\mathscr{D}$ has such a metric, and therefore so does every closed subset of $\mathscr{D}$. (See [1] for the proof of this fact.) A metric with this property will be called a *U.D. metric* (U.D. standing for unique distance). Any U.D. metric on X determines retractions of X onto each of its nonempty, closed subspaces Y. Specifically, let π be the mapping that sends each $x \in X$ to the unique point of Y which lies closest to x. We will refer to π as the "nearest point retraction of X onto Y." Note that by the definition of π, $d(x, \pi(x)) = d(x, Y)$, for all $x \in X$.

The transfinite sequence of topological derivatives are defined recursively for an arbitrary space X as follows. For any subset Y of X, let Y' denote the set of all accumulation points of Y, that is, Y' is the complement in Y of the set of points which are isolated in the relative topology of Y. Then we define

$X^{(0)} = X$,

$X^{(\xi+1)} = (X^{(\xi)})'$,

$X^{(\eta)} = \bigcap_{\xi < \eta} X^{(\xi)}$, when η is a limit ordinal.

Several elementary observations follow from this definition:

$X^{(\xi)}$ is closed in X;

$\xi \leq \eta$ implies $X^{(\xi)} \supseteq X^{(\eta)}$;

$X^{(\xi)} - X^{(\xi+1)}$ is the set of isolated points of $X^{(\xi)}$ (with its relative topology);

$(X^{(\xi)})^{(\eta)} = X^{(\xi+\eta)}$;

$Y \subseteq Z$ implies $Y^{(\xi)} \subseteq Z^{(\xi)}$.

These facts will be used without citation. Slightly deeper properties are listed in the following proposition.

LEMMA 2.1. *Assume that X is a Hausdorff space.*

(a) *There is a smallest ordinal number λ such that $X^{(\lambda)} = X^{(\lambda+1)}$; if $\xi \geq \lambda$, then $X^{(\xi)} = X^{(\lambda)}$.*

(b) *If X is compact, and $X^{(\xi)} = \varnothing$ for some ξ, then there is a nonlimit ordinal λ such that $X^{(\lambda)} = \varnothing$ and $X^{(\lambda-1)}$ is finite and not empty.*

(c) *If Y is open in X, then $Y^{(\xi)} = Y \cap X^{(\xi)}$ for all ξ.*

(d) *If $X = \bigcup_{i \in I} Y_i$, where each Y_i is open in X, and $Y_i \cap Y_j = \varnothing$ for $i \neq j$, then $X^{(\xi)} = \bigcup_{i \in I} Y_i^{(\xi)}$.*

(e) *If $\varphi \colon X \to Y$ is a homeomorphism, then $\varphi(X^{(\xi)}) = Y^{(\xi)}$.*

The proofs of these facts can be found in [2] and [3].

As is customary, each ordinal number will be identified with the set of all smaller ordinals. Consequently, ordinal numbers are themselves well ordered sets. This convention can cause some confusion. For example in a discussion of an ordinal number ξ, it is sometimes necessary to decide from context whether ξ is to be considered as the label for an ordered set, or rather as an object of another set. Usually, this decision can be made without difficulty.

If V is a totally ordered set, and if $u < v$ are elements of V, then the various intervals determined by u and v are denoted in the usual way:

$$[u, v] = \{x \in V \mid u \leqq x \leqq v\}, \quad [u, v) = \{x \in V \mid u \leqq x < v\},$$

$$(u, v] = \{x \in V \mid u < x \leqq v\}, \quad \text{and} \quad (u, v) = \{x \in V \mid u < x < v\}.$$

Summary of notation.

(a) $\mathfrak{S} =$ class of all metrizeable Boolean spaces (i.e., zero-dimensional, compact metric spaces).

(b) $\simeq$ represents homeomorphism of topological spaces.

(c) $\sim$ represents order isomorphism of ordered sets.

(d) If Y is a subset of the topological space X, the closure of Y in X is denoted by Y^-.

(e) If Y is a subset of a space with a metric d, then the diameter of Y is denoted by diam (Y). That is, diam $(Y) = \sup \{d(x, y) \mid x, y \in Y\}$.

(f) Let $\varphi_i \colon X_i \to Y$, $i \in I$, be a set of mappings with disjoint domains ($X_i \cap X_j = \varnothing$ for $i \neq j$). Define $\bigcup_{i \in I} \varphi_i \colon \bigcup_{i \in I} X_i \to Y$ by $(\bigcup_{i \in I} \varphi_i)(x) = \varphi_j(x)$ for all $x \in X_j$. For the case of two mappings φ and ψ with disjoint domains, this combined mapping will be written $\varphi \cup \psi$.

3. **Scattered Stone spaces.** For the proof of Theorem 1.1 it is necessary to know something about the structure of scattered Stone spaces with a countable basis. The result that is needed will be presented in this section.

DEFINITION 3.1. Let X be a Stone space. Define

$$\lambda(X) = \min \{\xi \mid X^{(\xi+1)} = X^{(\xi)}\},$$

$$\mu(X) = \min \{\xi \mid X^{(\xi)} - X^{(\lambda(X))} \text{ is compact}\}.$$

Note that $\mu(X)$ is well defined and $\leqq \lambda(X)$. Moreover, the space X is scattered exactly in the case that $X^{(\lambda(X))} = \varnothing$, and indeed $\lambda(X)$ is the first vanishing derivative of X.

PROPOSITION 3.2. *Let X be a scattered Stone space with a countable basis of open sets. Then there is a countable ordinal number α and a natural number n such that X is homeomorphic to $(\omega^\alpha \cdot n + 1) - \{\omega^{\mu(X)}\}$ with the relative topology inherited from the order topology of $\omega^\alpha \cdot n + 1$. Moreover, either*

(a) $\lambda(X) = \mu(X) = \alpha$ *and* $n = 1$, *or*

(b) $\lambda(X) = \alpha + 1$, $\mu(X) = \alpha$, *and* $n = |X^{(\alpha)}| + 1$, *or*

(c) $\lambda(X) = \alpha + 1$, $\mu(X) < \alpha$, *and* $n = |X^{(\alpha)}|$.

Proof. Let Y be the one point compactification of X, say $Y = X \cup \{\infty\}$. Then $Y \in \mathfrak{S}$, and $Y^{(\lambda(X)+1)} = \varnothing$ by Lemma 2.1(c) and the fact that X is scattered and open in Y. Consequently, there is a countable ordinal α, a natural number n, and a homeomorphism $\theta : Y \to \omega^\alpha \cdot n + 1$ (see [3] or [4]). Let $\theta(\infty) = \xi$, where $\xi \leq \omega^\alpha \cdot n$. Without loss of generality, it can be assumed that $\xi = \omega^\beta$ for some ordinal number $\beta \leq \alpha$. In fact, any ξ can be written in the form $\eta + \omega^\beta$, and it is easy to see that there is a homeomorphism of $\omega^\alpha \cdot n + 1$ onto itself which carries $\eta + \omega^\beta$ to ω^β. By Lemma 2.1(c) and (e), $\theta(X^{(\xi)}) = (\omega^\alpha \cdot n + 1)^{(\xi)} - \{\omega^\beta\}$ for all ξ. The results in §4 of [3] shows that

(i) $\omega^\beta \in (\omega^\alpha \cdot n + 1)^{(\xi)}$ if and only if $\xi \leq \beta$, and

(ii) $(\omega^\alpha \cdot n + 1)^{(\alpha)} = \{\omega^\alpha \cdot 1, \ \omega^\alpha \cdot 2, \ \omega^\alpha \cdot 3, \ldots, \ \omega^\alpha \cdot n\}$.

From (i) it is clear that $X^{(\xi)} - X^{(\lambda(X))} = X^{(\xi)}$ is compact if and only if $\xi \geq \beta$. Hence, $\beta = \mu(X)$. Using this observation and (ii), the alternatives (a), (b), or (c) are easily obtained.

It is evident that the natural number $n = n(X)$ defined in Proposition 3.2 is an invariant of the space X, and that the triple $(\lambda(X), \mu(X), n(X))$ completely determines X. Thus, we obtain a complete system of invariants for scattered Stone spaces with countable bases of open sets.

Now consider the situation described in Theorem 1.1, that is, let $X \in \mathfrak{S}$, and suppose that Z is a closed subset of X such that $X - Z$ is scattered. Then $X - Z$ is a scattered Stone space with a countable basis. By Proposition 3.2, there is a homeomorphism $\theta : (X - Z) \to (\omega^\alpha \cdot n + 1) - \{\omega^\mu\}$, where $\mu = \mu(X - Z)$, and either

(a) $\mu = \alpha = \lambda(X - Z)$ and $n = 1$, or

(b) $\lambda(X - Z) = \alpha + 1$, $\mu = \alpha$, and $n = |(X - Z)^{(\alpha)}| + 1$, or

(c) $\lambda(X - Z) = \alpha + 1$, $\mu < \alpha$, and $n = |(X - Z)^{(\alpha)}|$.

Let $U = \theta^{-1}(\{\xi \mid \omega^\mu < \xi \leq \omega^\alpha \cdot n\})$. Then U is a compact open subset of X, and in the respective cases (a), (b) and (c) we have

$$U = \varnothing, \qquad U \simeq \omega^\alpha \cdot (n-1) + 1, \quad \text{and} \quad U \simeq \omega^\alpha \cdot n + 1.$$

It is clear that $Z \subseteq X - U$, and $Z \cap ((X - U)^{(\xi)} - Z)^- = Z \cap (X^{(\xi)} - Z)^-$ for all ξ. From this observation, it follows that for the proof of Theorem 1.1 we may assume without loss of generality that $X - Z$ and $Y - W$ are both homeomorphic to the ordered set consisting of all ordinal numbers less than ω^μ.

4. Proof of 1.2. Theorems 1.1 and 1.2 are both proved by transfinite induction on μ. We begin with 1.2, whose proof is somewhat easier than the proof of 1.1.

It is convenient to get a different approach to the problem. Our first lemma accomplishes this.

LEMMA 4.1. *Let $X \in \mathfrak{S}$, and suppose that $Z \supseteq Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_\xi \supseteq \cdots$, $\xi < \mu$, where each Z_ξ is closed and nonempty. Assume that there exists a continuous mapping $\varphi \colon \omega^\mu \to Z$ satisfying*

(a) *$\varphi((\omega^\mu)^{(\xi)}) \subseteq Z_\xi$ for all $\xi < \mu$, and*

(b) *for all $\xi < \mu$, if $y \in Z_\xi$, then there exists $\{\eta_i \mid i < \omega\} \subseteq (\omega^\mu)^{(\xi)}$ such that $\lim_{i \to \omega} \eta_i = \omega^\mu$ and $\lim_{i \to \omega} \varphi(\eta_i) = y$.*

Then there exists $X \in \mathfrak{S}$ satisfying $X \supseteq Z$, $X - Z \simeq \omega^\mu$, and $Z \cap (X^{(\xi)} - Z)^- = Z_\xi$ for all $\xi < \mu$.

Proof. Define X to be the following subspace of $(\omega^\mu + 1) \times Z$:

$$X = \{(\eta, \varphi(\eta)) \mid \eta < \omega^\mu\} \cup (\{\omega^\mu\} \times Z).$$

Using the continuity of φ, it is easy to show that X is closed in $(\omega^\mu + 1) \times Z$. Hence, $X \in \mathfrak{S}$. Plainly, we can identify each $y \in Z$ with $(\omega^\mu, y) \in X$, so that Z is embedded in X. The continuity of φ implies that the inverse of the continuous map $(\eta, \varphi(\eta)) \to \eta$ $(\eta < \omega^\mu)$ is continuous. Thus, the open set $U = \{(\eta, \varphi(\eta)) \mid \eta < \omega^\mu\}$ is homeomorphic to ω^μ, and by Lemma 2.1(c),

$$X^{(\xi)} - Z = U \cap X^{(\xi)} = U^{(\xi)} = \{(\eta, \varphi(\eta)) \mid \eta \in (\omega^\mu)^{(\xi)}\}.$$

Therefore, by (a), $(X^{(\xi)} - Z)^- \subseteq (\omega^\mu + 1) \times Z_\xi$ and $Z \cap (X^{(\xi)} - Z)^- \subseteq Z_\xi$.

On the other hand, if $y \in Z_\xi$, then by (b) there is a sequence $\{\eta_i \mid i < \omega\} \subseteq (\omega^\mu)^{(\xi)}$ such that $\lim_{i \to \omega} \eta_i = \omega^\mu$ and $\lim_{i \to \omega} \varphi(\eta_i) = y$. Therefore $\{(\eta_i, \varphi(\eta_i)) \mid i < \omega\}$ is a sequence in $X^{(\xi)} - Z$ which satisfies $\lim_{i \to \omega} (\eta_i, \varphi(\eta_i)) = (\omega^\mu, y)$. This shows that $Z \cap (X^{(\xi)} - Z)^- = Z_\xi$ for all $\xi < \mu$.

Our task is now reduced to the job of constructing a continuous mapping $\varphi \colon \omega^\mu \to Z$ which satisfies 4.1(a) and 4.1(b).

It is helpful to introduce some notation. Let V be an ordered set, and $\{v_i \mid i < \omega\} \subseteq V$. We will write $\lim_{i \to \omega} v_i = \infty$ as an abbreviation for the sentence "for all $v \in V$, there exists $n < \omega$ such that $i > n$ implies $v_i \geq v$."

Let V be an ordered set (topologized with the order topology), Z a topological space, and $\varphi \colon V \to Z$ a continuous mapping. For each ordinal number ξ, define

$$\Phi_\xi(\varphi) = \bigcap_{v \in V} (\varphi(V^{(\xi)} \cap \{w \in V \mid w \geq v\}))^-.$$

The case which interests us occurs when V is countable and Z is a Boolean space with a countable basis. For this situation, $\Phi_\xi(\varphi)$ can be characterized as the set of all $y \in Z$ such that there exists a sequence $\{v_i \mid i < \omega\} \subseteq V^{(\xi)}$ satisfying $\lim_{i \to \omega} v_i = \infty$, and $\lim_{i \to \omega} \varphi(v_i) = y$. In particular, the condition (b) of Lemma 4.1 can be reformulated as: $\Phi_\xi(\varphi) = Z_\xi$ for all $\xi < \mu$.

LEMMA 4.2. *Let V be an ordered set, $V = \bigcup_{i \in I} V_i$, where each V_i is open in V, $V_i \cap V_j = \varnothing$ for $i \neq j$, and each V_i is cofinal in V. Suppose that for each $i \in I$, φ_i is a*

continuous mapping of V_i to the space Z. Define $\varphi = \bigcup_{i \in I} \varphi_i$. Then φ is a continuous mapping of V to Z, and $\Phi_\xi(\varphi) \supseteq \bigcup_{i \in I} \Phi_\xi(\varphi_i)$ for all ξ. If I is finite, then $\Phi_\xi(\varphi) = \bigcup_{i \in I} \Phi_\xi(\varphi_i)$.

Proof. If N is open in Y, then $\varphi^{-1}(N) = \bigcup_{i \in I} \varphi_i^{-1}(N)$ is open in V (since each V_i is open, and each φ_i is continuous). Thus, φ is continuous. Using the fact that V_i is confinal in V, we obtain

$$\begin{aligned}
\Phi_\xi(\varphi_i) &= \bigcap_{v \in V_i} (\varphi_i(V_i^{(\xi)} \cap \{w \in V_i \mid w \geq v\}))^- \\
&= \bigcap_{v \in V} (\varphi_i(V_i^{(\xi)} \cap \{v \in V_i \mid w \geq v\}))^- \subseteq \bigcap_{v \in V} (\varphi(V^{(\xi)} \cap \{w \in V \mid w \geq v\}))^- \\
&= \Phi_\xi(\varphi).
\end{aligned}$$

Since $i \in I$ was arbitrary, we have $\Phi_\xi(\varphi) \supseteq \bigcup_{i \in I} \Phi_\xi(\varphi_i)$. For the proof of the last statement, it can be assumed that $I = \{0, 1\}$. In this case,

$$\begin{aligned}
\Phi_\xi(\varphi) &= \bigcap_{v \in V} (V^{(\xi)} \cap \{w \in V \mid w \geq v\})^- \\
&= \bigcap_{v \in V} ((\varphi_0(V_0^{(\xi)} \cap \{w \in V \mid w \geq v\}))^- \cup (\varphi_1(V_1^{(\xi)} \cap \{w \in V \mid w \geq v\}))^-),
\end{aligned}$$

since $V^{(\xi)} = V_0^{(\xi)} \cup V_1^{(\xi)}$ by 2.1. Using the fact that if $\{A_v \mid v \in V\}$ and $\{B_v \mid v \in V\}$ are nested families of sets (i.e. $v < v'$ implies $A_v \supseteq A_{v'}$ and $B_v \supseteq B_{v'}$), then $\bigcap_{v \in V} (A_v \cup B_v) = (\bigcap_{v \in V} A_v) \cup (\bigcap_{v \in V} B_v)$, we obtain the desired result:

$$\begin{aligned}
\Phi_\xi(\varphi) &= \left(\bigcap_{v \in V} (\varphi_0(V_0^{(\xi)} \cap \{w \in V \mid w \geq v\}))^-\right) \cup \left(\bigcap_{v \in V} ((V_1^{(\xi)} \cap \{w \in V \mid w \geq v\}))^-\right) \\
&= \Phi_\xi(\varphi_0) \cup \Phi_\xi(\varphi_1)
\end{aligned}$$

(because V_0 and V_1 are cofinal in V).

LEMMA 4.3. *Let μ be a countable ordinal number, and let $0 \leq \beta < \alpha < \mu$. Then there exist subsets V_0, V_1 of ω^μ such that*
  (a) *$V_0 \cap V_1 = \varnothing$, $V_0 \cup V_1 = \omega^\mu$,*
  (b) *V_0 and V_1 are open and cofinal in ω^μ,*
  (c) *$V_0 \sim \omega^\alpha$, $V_1 \sim \omega^\mu$, and*
  (d) *$V_1 \subseteq [\omega^\beta, \omega^\mu)$.*

Proof. Let $\{\gamma_i \mid i < \omega\}$ be a strictly increasing sequence of ordinal numbers satisfying $\gamma_0 \geq \omega^\beta$ and $\lim_{i \to \omega} \gamma_i = \omega^\alpha$. Let $\{\delta_i \mid 1 \leq i < \omega\}$ be a strictly increasing sequence of ordinal numbers such that $\delta_1 \geq \omega^\alpha$ and $\lim_{i \to \omega} \delta_i = \omega^\mu$. Define

$$V_0 = [0, \gamma_0] \cup \bigcup_{1 \leq i < \omega} (\delta_i, \delta_i + \gamma_i], \quad V_1 = (\gamma_0, \delta_1] \cup \bigcup_{1 \leq i < \omega} (\delta_i + \gamma_i, \delta_{i+1}].$$

Plainly, (a), (b), and (d) are satisfied, and (c) is easily obtained from the identities $\sum_{i < \omega} \gamma_i = \omega^\alpha$ and $\sum_{1 \leq i < \omega} \delta_i = \omega^\mu$.

LEMMA 4.4. *Let μ be a countable limit ordinal, and let $\{\alpha_i \mid i < \omega\}$ be a strictly increasing sequence of ordinal numbers such that $\alpha_0 > 0$ and $\lim_{i \to \omega} \alpha_i = \mu$. Then there*

is a family $\{V_i \mid i < \omega\}$ *of subsets of* ω^μ *such that*

(a) $V_i \cap V_j = \varnothing$ *if* $i \neq j$, *and* $\bigcup_{i < \omega} V_i = \omega^\mu$,

(b) *all* V_i *are open and cofinal in* ω^μ, *and*

(c) $V_i \sim \omega^{\alpha_i}$ *for all* i.

Proof. Applying 4.3 with $\beta = 0$, $\alpha = \alpha_0$ and μ, we can obtain sets V_0 and U_0 which are open and cofinal in ω^μ, and satisfy $V_0 \cap U_0 = \varnothing$, $V_0 \cup U_0 = \omega^\mu$, $V_0 \sim \omega^{\alpha_0}$, and $U_0 \sim \omega^\mu$. Reapply 4.3 to the set U_0 with $\beta = \alpha_0$, $\alpha = \alpha_1$, and μ. We obtain sets V_1 and U_1 which are open and cofinal in U_0 (consequently, also open and cofinal in ω^μ), and satisfy $V_1 \cap U_1 = \varnothing$, $V_1 \cup U_1 = U_0$, $V_1 \sim \omega^{\alpha_1}$, $U_1 \sim \omega^\mu$, and U_1 does not meet the initial segment W_0 of U_0, such that $W_0 \sim \omega^{\alpha_0}$. This last condition implies that $U_1 \subseteq [\omega^{\alpha_0}, \omega^\mu)$ (see, for example, [5, Lemma 3.4, p. 8, and Proposition 3.2, p. 7]). Continue this process. After $n+1$ steps we have sets $V_0, V_1, \ldots, V_n, U_n$ which are open and cofinal in ω^μ, and satisfy $V_i \cap V_j = \varnothing$ for $i \neq j$, $V_i \cap U_n = \varnothing$ for all i, $V_0 \cup V_1 \cup \cdots \cup V_n \cup U_n = \omega^\mu$, $V_i \sim \omega^{\alpha_i}$ for all i, $U_n \sim \omega^\mu$, and $U_n \subseteq \{\eta < \omega^\mu \mid \eta \geqq \omega^{\alpha_n - 1}\}$. Since $\lim_{i \to \omega} \omega^{\alpha_i} = \omega^\mu$, it is clear that if the construction is repeated indefinitely, the collection $\{V_i \mid i < \omega\}$ will satisfy the conditions (a), (b), and (c).

We now proceed with the proof of 1.2. Let Z and Z_ξ, $\xi < \mu$, be given as in the statement of the theorem. It will be shown by induction on μ that there exists a continuous mapping $\varphi : \omega^\mu \to Z$ which satisfies the conditions 4.1(a) and 4.1(b). If $\mu = 0$, there is nothing to prove.

Case: $\mu = 1$. Let $\{y_i \mid i < \omega\}$ be a dense subset of Z_0. Let $\rho : \omega \to \omega \times \omega$ be one-to-one and onto, and let $\pi : \omega \times \omega \to \omega$ be the projection $\pi(r, s) = r$. Define $\varphi(n) = y_{\pi\rho(n)}$. Note that for any $i < \omega$, $(\pi\rho)^{-1}(\{i\})$ is an infinite subset of ω. From this observation and the denseness of $\{y_i \mid i < \omega\}$, it follows that φ satisfies 4.1(b). Clearly, 4.1(a) is satisfied. Finally, φ is automatically continuous, since ω is discrete in its order topology.

Induction step: $\mu = \alpha + 1$. By 4.3 there exist subsets V_0, V_1 which are open and cofinal in ω^μ, and satisfy $V_0 \cap V_1 = \varnothing$, $V_0 \cup V_1 = \omega^\mu$, $V_0 \sim \omega^\alpha$, and $V_1 \sim \omega^\mu$. By the induction hypothesis, there exists a continuous mapping $\varphi_0 : V_0 \to Z$ such that $\varphi_0(V_0^{(\xi)}) \subseteq Z_\xi$ and $\Phi_\xi(\varphi_0) = Z_\xi$ for $\xi < \alpha$. Since $V_0 \sim \omega^\alpha$, it is clear that $V_0^{(\alpha)} = \varnothing$. Since $V_1 \sim \omega^\mu$, there is a one-to-one, order preserving mapping θ of V_1 onto $\omega^\mu = \omega^{\alpha+1}$. By the case $\mu = 1$, there is a continuous mapping $\psi : \omega \to Z$ such that $\psi(\omega) \subseteq Z_\alpha$ and $\Phi_0(\psi) = Z_\alpha$. Define $\chi : \omega^\mu \to Z$ by $\chi(\omega^\alpha \cdot n) = \psi(n)$ for $n < \omega$, and $\chi(\omega^\alpha \cdot n + \gamma) = \psi(n+1)$ for $n < \omega$ and $0 < \gamma < \omega^\alpha$. Since χ is constant on each of the closed intervals $(\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)]$, this mapping is continuous. Let $\varphi_1 = \chi\theta : V_1 \to Z$. Plainly, $\varphi_1((\omega^\mu)^{(\xi)}) \subseteq Z_\alpha$ for all $\xi < \mu$, and it is easy to see that $\Phi_\xi(\varphi_1) = Z_\alpha$ for all $\alpha < \mu$. Since θ and χ are continuous, so is φ_1. Finally, let $\varphi = \varphi_0 \cup \varphi_1$. Using Lemma 2.1, we have $\varphi((\omega^\mu)^{(\xi)}) = \varphi(V_0^{(\xi)} \cup V_1^{(\xi)}) = \varphi_0(V_0^{(\xi)}) \cup \varphi_1(V_1^{(\xi)}) \subseteq Z_\alpha \cup Z_\xi = Z_\xi$ for all $\xi \leqq \alpha$ (that is, all $\xi < \mu$). In particular, $\Phi_\xi(\varphi) \subseteq Z_\xi$ for all $\xi < \mu$, since each Z_ξ is closed. On the other hand, by 4.2, φ is continuous and satisfies $\Phi_\xi(\varphi) = \Phi_\xi(\varphi_0) \cup \Phi_\xi(\varphi_1)$

$=Z_\xi \cup Z_\alpha = Z_\xi$ for $\xi < \alpha$, and $\Phi_\alpha(\varphi) = \Phi_\alpha(\varphi_0) \cup \Phi_\alpha(\varphi_1) = \varnothing \cup Z_\alpha = Z_\alpha$. Thus, φ satisfies 4.1(b).

Induction step: $\mu = \textit{limit ordinal}$. Since μ is countable, there is a sequence $\{\alpha_i \mid i < \omega\}$ of ordinal numbers such that $\alpha_0 > 0$ and $\lim_{i \to \omega} \alpha_i = \mu$. Choose $\{V_i \mid i < \omega\}$ as in 4.4. By the induction hypothesis, there exist continuous mappings $\varphi_i \colon V_i \to Z$ such that $\varphi_i(V_i^{(\xi)}) \subseteq Z_\xi$ and $\Phi_\xi(\varphi_i) = Z_\xi$ for all $\xi < \alpha_i$. Since $V_i^{(\xi)} \sim (\omega^{\alpha_i})^{(\xi)} = \varnothing$ for $\xi \geqq \alpha_i$, it follows that $\varphi_i(V^{(\xi)}) = \varnothing$ and $\Phi_\xi(\varphi_i) = \varnothing$ for $\xi \geqq \alpha_i$. Let $\varphi = \bigcup_{i < \omega} \varphi_i$. Then φ is continuous, and by an argument like the one given in the nonlimit case, $\varphi((\omega^\mu)^{(\xi)}) \subseteq Z_\xi$, and $\Phi_\xi(\varphi) = Z_\xi$ for all $\xi < \mu$. Hence φ satisfies 4.1(a) and 4.1(b).

5. **Proof of 1.1.** By the remarks at the end of §3, we may suppose that spaces X and Y in $\mathfrak{S}$ are given, together with closed subsets $Z \subseteq X$ and $W \subseteq Y$ such that $X - Z \simeq Y - W \simeq \omega^\mu$, and that there is a mapping $\varphi \colon Z \to W$ such that

$$\varphi(Z \cap (X^{(\xi)} - Z)^-) = W \cap (Y^{(\xi)} - W)^-$$

for all ordinal numbers ξ. Our objective is to extend φ to a homeomorphism of X onto Y. As in the previous section, it is possible to deduce Theorem 1.1 from a result concerning the mappings of ω^μ into W.

LEMMA 5.1. *Let μ be a countable ordinal number larger than 0. Let d be a U.D. metric on the Boolean space W. Suppose that σ and τ are continuous mappings of ω^μ to W satisfying*

(i) $\Phi_\xi(\sigma) = \Phi_\xi(\tau)$ *for all $\xi < \mu$, and*

(ii)
$$\sup_{\xi < \mu} \sup_{\eta \in (\omega^\mu)^{(\xi)}} d(\sigma(\eta), \Phi_\xi(\sigma)) < \delta,$$
$$\sup_{\xi < \mu} \sup_{\eta \in (\omega^\mu)^{(\xi)}} d(\tau(\eta), \Phi_\xi(\tau)) < \delta.$$

Then there is a homeomorphism χ of ω^μ onto itself such that

(a) $\lim_{\eta \to \omega^\mu} d(\sigma(\eta), \tau(\chi(\eta))) = 0$,

(b) $d(\sigma(\eta), \tau(\chi(\eta))) < 2\delta$ *for all $\eta < \mu$.*

The conclusion (a) is the significant part of this proposition. However, property (b) is useful in carrying out the inductive proof. Note that (ii) can always be satisfied by taking $\delta > \text{diam } W$.

Before proving Lemma 5.1, let us see how it yields Theorem 1.1. If $\mu = 0$, the theorem is trivial. Therefore, we can assume that $\mu \geqq 1$. Choose any U.D. metrics on X and Y. It will cause no confusion to denote both metrics by the same letter d. Let π be the nearest point retraction of X onto Z and denote by ρ the nearest point retraction of Y onto W. The following property of π is needed.

(1) If $\{x_i \mid i < \omega\} \subseteq X - Z$ satisfies $\lim_{i \to \omega} d(x_i, Z) = 0$, then $\lim_{i \to \omega} \pi(x_i) = x$ if and only if $\lim_{i \to \omega} x_i = x$.

Proof. Suppose that $\lim_{i \to \omega} \pi(x_i) = x$. Then $x \in Z$ and $d(x_i, x) \leqq d(x_i, \pi(x_i)) + d(\pi(x_i), x) = d(x_i, Z) + d(\pi(x_i), x)$, which goes to zero by our hypotheses. Conversely, assume that $\lim_{i \to \omega} x_i = x$. Then $d(x, Z) = 0$, since $\lim_{i \to \omega} d(x_i, Z) = 0$.

Hence, $x \in Z$ (since Z is closed), and the continuity of π yields $\lim_{i \to \omega} \pi(x_i) = \pi(x) = x$.

It goes without saying that a similar result holds true for ρ.

By hypothesis, $X - Z \simeq \omega^\mu \simeq Y - W$. Let $\kappa: \omega^\mu \to X - Z$ and $\nu: \omega^\mu \to Y - W$ be homeomorphisms.

(2) For $\{\eta_i \mid i < \omega\} \subseteq \omega^\mu$, the conditions $\lim_{i \to \omega} \eta_i = \omega^\mu$ and $\lim_{i \to \omega} d(\kappa(\eta_i), Z) = 0$ (or $\lim_{i \to \omega} d(\nu(\eta_i), W) = 0$) are equivalent.

Proof. The two conditions are equivalent to the statement that no compact subset (of ω^μ or $X - Z$) contains more than a finite number of elements from the respective sequences. This property is clearly preserved by a homeomorphism.

Using (1) and (2), we can now prove

(3) $\Phi_\xi(\pi\kappa) = Z \cap (X^{(\xi)} - Z)^-$ and $\Phi_\xi(\rho\nu) = W \cap (Y^{(\xi)} - W)^-$ for all $\xi < \mu$.

Proof. By definition, $x \in \Phi_\xi(\pi\kappa)$ if and only if there is a sequence $\{\eta_i \mid i < \omega\} \subseteq (\omega^\mu)^{(\xi)}$ such that $\lim_{i \to \omega} \eta_i = \omega^\mu$, and $\lim_{i \to \omega} \pi\kappa(\eta_i) = x$. By virtue of (1) and (2), this in turn is equivalent to the existence of a sequence $\{x_i \mid i < \omega\} \subseteq X^{(\xi)} - Z$ which converges to x, together with the condition $x \in Z$. Thus, we see that $\Phi_\xi(\pi\kappa) = Z \cap (X^{(\xi)} - Z)^-$. Similarly, $\Phi_\xi(\rho\nu) = W \cap (X^{(\xi)} - W)^-$.

Define $\sigma = \varphi\pi\kappa: \omega^\mu \to W$ and $\tau = \rho\nu: \omega^\mu \to W$. Then

$$\Phi_\xi(\sigma) = \Phi_\xi(\varphi(\pi\kappa)) = \bigcap_{\eta < \omega^\mu} ((\varphi\pi\kappa)((\omega^\mu)^{(\xi)} \cap [\eta, \omega^\mu)))^-$$

$$= \varphi\left(\bigcap_{\eta < \omega^\mu} ((\pi\kappa)((\omega^\mu)^{(\xi)} \cap [\eta, \omega^\mu)))^-\right)$$

$$= \varphi(\Phi_\xi(\pi\kappa)) = \varphi(Z \cap (X^{(\xi)} - Z)^-)$$

by (3). By hypothesis, $(Z \cap (X^{(\xi)} - Z)^- = W \cap (Y^{(\xi)} - W)^-$ so that by (3) again we obtain $\Phi_\xi(\sigma) = W \cap (Y^{(\xi)} - W)^- = \Phi_\xi(\tau)$. Therefore, the hypothesis (i) of 5.1 is satisfied. Hence, there is a homeomorphism $\chi: \omega^\mu \to \omega^\mu$ such that

(4) $\lim_{\eta \to \omega^\mu} d(\sigma(\eta), \tau(\chi(\eta))) = 0$.

Finally, define $\psi: X \to Y$ by

$$\psi(x) = \nu(\chi(\kappa^{-1}(x))) \quad \text{if } x \in X - Z,$$
$$\psi(x) = \varphi(x) \qquad\qquad \text{if } x \in Z.$$

Plainly, ψ is bijective and $\psi|Z = \varphi$. Since X is compact, it suffices to show that ψ is continuous. Continuity for ψ easily reduces to proving the following fact.

(5) If $\{x_i \mid i < \omega\} \subseteq X - Z$, and $\lim_{i \to \omega} x_i = x \in Z$, then $\lim_{i \to \omega} \psi(x_i) = \varphi(x)$.

Proof. By (2), $\lim_{i \to \omega} \kappa^{-1}(x_i) = \omega^\mu$. By continuity,

$$\lim_{i \to \omega} \sigma(\kappa^{-1}(x_i)) = \lim_{i \to \omega} \varphi\pi\kappa(\kappa^{-1}(x_i)) = \varphi\pi\left(\lim_{i \to \omega} x_i\right) = \varphi(\pi(x)) = \varphi(x)$$

(since $x \in Z$). Combining these observations with (4), we obtain

$$\lim_{i \to \omega} \rho(\varphi(x_i)) = \lim_{i \to \omega} \rho\nu\chi\kappa^{-1}(x_i) = \lim_{i \to \omega} \tau\chi(\kappa^{-1}(x_i)) = \varphi(x).$$

Since χ is a homeomorphism, $\lim_{i \to \omega} \kappa^{-1}(x_i) = \omega^\mu$ implies $\lim_{i \to \omega} \chi(\kappa^{-1}(x_i)) = \omega^\mu$.

Thus, by (2), $\lim_{i \to \omega} d(\varphi(x_i), W) = \lim_{i \to \omega} d(\nu\chi\kappa^{-1}(x_i), W) = 0$. Hence, the analogue for ρ of (1), applied to $\lim_{i \to \omega} \rho(\varphi(x_i)) = \varphi(x)$, gives the desired result $\lim_{i \to \omega} \psi(x_i) = \varphi(x)$. This completes the deduction of 1.1 from 5.1.

We turn now to the task of proving 5.1 by transfinite induction on μ. The following notation will be useful. Let F and G be subsets of a compact metric space, such that $F^- \supseteq G$. Denote $e(F, G) = \sup_{x \in F} d(x, G)$.

The following properties of e are easily obtained from this definition:

(a) $e(F, G) < \varepsilon$ implies that for every $x \in F$, there exists $y \in G$ such that $d(x, y) < \varepsilon$; if F is closed, the converse is true;

(b) $e(F, G) = e(F^-, G^-)$;

(c) $F^- \supseteq G$ and $G^- \supseteq H$ implies $e(F, G) \leq e(F, H) \leq e(F, G) + e(G, H)$;

(d) $e(F, F) = 0$;

(e) if V is an ordered set and $\{F_v \mid v \in V\}$ is a nested family of closed subsets of a compact metric space (i.e., $v < v'$ in V implies $F_v \supseteq F_{v'}$), then $\lim_{v \to \infty} e(F_v, \bigcap_{u \in V} F_u) = 0$.

Suppose now that φ is a continuous mapping of ω^μ into W. Denote

$$r_\xi(\varphi) = e(\varphi((\omega^\mu)^{(\xi)}), \Phi_\xi(\varphi)), \quad r(\varphi) = \sup_{\xi < \mu} r_\xi(\varphi).$$

More generally, if V is an open subset of ω^μ, denote

$$r_\xi(\varphi, V) = e(\varphi(V \cap (\omega^\mu)^{(\xi)}), \Phi_\xi(\varphi \mid V)), \quad r(\varphi, V) = \sup_{\xi < \mu} r_\xi(\varphi, V).$$

Using this notation, the condition (ii) of 5.1 can be reformulated more simply as $r(\sigma) < \delta$ and $r(\tau) < \delta$.

Case: $\mu = 1$. Since the order topology on ω is discrete, continuity is no restriction in this case. The first step of the argument is a proof that conclusion of 5.1 holds when the requirement that χ be surjective is dropped. The full statement of the proposition is then obtained by a modification of the usual method used to prove the Schröder-Bernstein theorem.

(6) Under the hypotheses of 5.1, with $\mu = 1$, there exists a strictly increasing mapping $\pi: \omega \to \omega$ such that

(a) $\lim_{\eta \to \omega} d(\sigma(\eta), \tau(\pi(\eta))) = 0$,

(b) $d(\sigma(\eta), \tau(\pi(\eta))) < 2\delta$ for all $\eta < \omega$, and

(c) $\pi(0) > 0$.

Proof. For $\eta < \omega$, choose x_η to be the unique point of $\Phi_0(\sigma)$ which is nearest to $\sigma(\eta)$. Note that $\lim_{\eta \to \omega} d(\sigma(\eta), x_\eta) = 0$, and $d(\sigma(\eta), x_\eta) < \delta$ for all $\eta < \omega$. Let $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots$ be a sequence of positive real numbers with $\varepsilon_0 < \delta$ and $\lim_{\eta \to \omega} \varepsilon_\eta = 0$. Define $\pi(\eta)$ by recursion so that it satisfies $0 < \pi(0) < \pi(1) < \cdots < \pi(\eta) < \pi(\eta + 1) < \cdots$, and

$$d(x_\eta, \tau(\pi(\eta))) \leq \varepsilon_\eta, \text{ for all } \eta < \omega.$$

These conditions can be satisfied, because $x_\eta \in \Phi_0(\sigma) = \Phi_0(\tau)$ implies that there is a subsequence $\{\eta_i \mid i < \omega\} \subseteq \omega$ such that $\lim_{i \to \omega} \tau(\eta_i) = x_\eta$. Since $d(\sigma(\eta), \tau(\pi(\eta))) \leq d(\sigma(\eta), x_\eta) + d(x_\eta, \tau(\pi(\eta))) \leq d(\sigma(\eta), x_\eta) + \varepsilon_\eta$, it is clear that π satisfies (a) and (b).

In order to complete the proof of 5.1 in the case $\mu = 1$, let $\pi : \omega \to \omega$ be strictly increasing, and satisfy conditions (a)–(c) of (6). Similarly, let $\rho : \omega \to \omega$ be strictly increasing, and such that $\lim_{\eta \to \omega} d(\tau(\eta), \sigma(\rho(\eta))) = 0$, $d(\tau(\eta), \sigma(\rho(\eta))) < 2\delta$ for all $\eta < \omega$, and $\rho(0) > 0$. By recursion, define

$$K_0 = L_0 = \omega, \qquad K_n = \rho(L_{n-1}), \qquad L_n = \pi(K_{n-1}), \quad \text{for } n \geq 1.$$

Note that $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$, $L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots$, and

$$K_{2n} = (\rho\pi)^n(\omega), \qquad L_{2n} = (\tau\rho)^n(\omega).$$

This last observation yields the conclusions

$$K_{2n} \subseteq \{k < \omega \mid k \geq 2n\} \quad \text{and} \quad L_{2n} \subseteq \{k < \omega \mid k \geq 2n\}.$$

In fact, since π and ρ are strictly increasing, and $\pi(0) > 0$, $\rho(0) > 0$, it follows that $\pi(k) > k$ and $\rho(k) > k$ for all k. Therefore, $(\rho\pi)^n(k) \geq k + 2n$ and $(\pi\rho)^n(k) \geq k + 2n$ by iteration. It follows that $\bigcap_{n < \omega} K_n = \bigcap_{n < \omega} L_n = \varnothing$. Hence, we have disjoint decompositions

$$\omega = \bigcup_{n < \omega} (K_n - K_{n+1}) = \bigcup_{n < \omega} (L_n - L_{n+1}).$$

Since π and ρ are one-to-one mappings, it follows that $\pi(K_n - K_{n+1}) = L_{n+1} - L_{n+2}$, and $\rho(L_n - L_{n+1}) = K_{n+1} - K_{n+2}$. Define $\chi : \omega \to \omega$ by

$$\chi(\eta) = \pi(\eta) \qquad \text{for } \eta \in \bigcup_{m < \omega} (K_{2m} - K_{2m+1}),$$

$$\chi(\eta) = (\rho^{-1})(\eta) \quad \text{for } \eta \in \bigcup_{m < \omega} (K_{2m+1} - K_{2m+2}).$$

Plainly, χ is a well-defined, bijective mapping of ω onto ω. If $\eta \in K_{2m} - K_{2m+1}$, then $d(\sigma(\eta), \tau(\chi(\eta))) = d(\sigma(\eta), \tau(\pi(\eta))) < 2\delta$. If $\eta \in K_{2m+1} - K_{2m+2}$, then $\eta = \rho(\zeta)$ for some $\zeta \in L_{2m} - L_{2m+1}$, so that $d(\sigma(\eta), \tau(\chi(\eta))) = d(\sigma(\rho(\xi)), \tau(\xi)) < 2\delta$. From these observations, it is also clear that $\lim_{\eta \to \omega} d(\sigma(\eta), \tau(\chi(\eta))) = 0$. Therefore, 5.1 holds when $\mu = 1$.

Induction step: $\mu = \alpha + 1$. By the results of §4 in [3], $(\omega^\mu)^{(\alpha)} = \{\omega^\alpha \cdot n \mid 0 < n < \omega\}$. As we have just shown, there is a one-to-one mapping[1] π of ω onto itself such that
  (7) $\lim_{n \to \omega} d(\sigma(\omega^\alpha \cdot n), \tau(\omega^\alpha \cdot \pi(n))) = 0$, and
  (8) $d(\sigma(\omega^\alpha \cdot n), \tau(\omega^\alpha \cdot \pi(n))) < 2\delta$ for $0 < n < \omega$.
From these two properties, it follows that
  (9) $\varepsilon = 2\delta - (\sup_{0 < n < \omega} d(\sigma(\omega^\alpha \cdot n), \tau(\omega^\alpha \cdot \pi(n)))) > 0$.
Let $\varepsilon_1 > \varepsilon_2 > \varepsilon > \cdots$ be a sequence of real numbers such that $\varepsilon_1 = \varepsilon/2$ (where ε is defined in (9)), and $\lim_{n \to \omega} \varepsilon_n = 0$. By the continuity of σ and τ, there exists an increasing sequence $\{\alpha_n \mid 0 \leq n < \omega\} \subseteq \omega^\alpha$ satisfying
  (10) $\lim_{n \to \omega} \alpha_n = \omega^\alpha$,

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[1] The mapping π in this part of the proof coincides with the χ that was constructed in the proof of the case $\mu = 1$. It should not be confused with the mapping π that was obtained in (6).

(11) if $\omega^\alpha \cdot (n-1) + \alpha_n \leq \eta \leq \omega^\alpha \cdot n$, then $d(\sigma(\eta), \sigma(\omega^\alpha \cdot n)) < \varepsilon_n$, and

(12) if $\omega^\alpha \cdot (\pi(n) - 1) + \alpha_n \leq \eta \leq \omega^\alpha \cdot \pi(n)$, then $d(\tau(\eta), \tau(\omega^\alpha \cdot \pi(n))) < \varepsilon_n$.

Next, choose an increasing sequence $\{\beta_n \mid 0 < n < \omega\} \subseteq \omega^\alpha$ satisfying either

(13) $\beta_n \geq \alpha_n + \omega^\gamma$ if $\alpha = \gamma + 1$,

or, when α is a limit ordinal,

(13') $\beta_n \geq \alpha_n + \omega^{\gamma_n}$ for some sequence $\gamma_1 < \gamma_2 < \cdots$, with $\lim_{n \to \omega} \gamma_n = \alpha$.

By (10), $\lim_{n \to \omega} \beta_n = \omega^\alpha$, and since $\beta_n < \omega^\alpha$ for all $n < \omega$, it follows that

(14) $\sum_{0 < n < \omega} \beta_n = \omega^\alpha$ and $\sum_{0 < n < \omega} \beta_{\pi(n)} = \omega^\alpha$.

Define subsets U_0, V_0, U_1, and V_1 of ω^α as follows:

$$U_0 = [0, \beta_1] \cup \bigcup_{0 < n < \omega} (\omega^\alpha \cdot n, \omega^\alpha \cdot n + \beta_{n+1}],$$

$$V_0 = [0, \beta_1] \cup \bigcup_{0 < n < \omega} (\omega^\alpha \cdot \pi(n), \omega^\alpha \cdot \pi(n) + \beta_{n+1}],$$

$$U_1 = \omega^\mu - U_0 = \bigcup_{0 < n < \omega} (\omega^\alpha \cdot (n-1) + \beta_n, \omega^\alpha \cdot n],$$

$$V_1 = \omega^\mu - V_0 = \bigcup_{0 < n < \omega} (\omega^\alpha \cdot (\pi(n) - 1) + \beta_n, \omega^\alpha \cdot \pi(n)].$$

Clearly, U_0, V_0, U_1, and V_1 are open and closed subsets of ω^μ. By virtue of (14), U_0 and V_0 are order isomorphic to ω^α. Also, because of (13), we have in the case that $\alpha = \gamma + 1$,

(15)
$$U_0 \cap (\omega^\mu)^{(\gamma)} \cap [\omega^\alpha \cdot n + \alpha_{n+1}, \omega^\alpha \cdot n + \beta_{n+1}] \neq \varnothing, \text{ and}$$
$$V_0 \cap (\omega^\mu)^{(\gamma)} \cap [\omega^\alpha \cdot \pi(n) + \alpha_{n+1}, \omega^\alpha \cdot \pi(n) + \beta_{n+1}] \neq \varnothing.$$

Similarly, (13') implies that if α is a limit ordinal, then

(15')
$$U_0 \cap (\omega^\mu)^{(\gamma_m)} \cap [\omega^\alpha \cdot n + \alpha_{n+1}, \omega^\alpha \cdot n + \beta_{n+1}] \neq \varnothing, \text{ and}$$
$$V_0 \cap (\omega^\mu)^{(\gamma_m)} \cap [\omega^\alpha \cdot \pi(n) + \alpha_{n+1}, \omega^\alpha \cdot \pi(n) + \beta_{n+1}] \neq \varnothing,$$

provided $m \leq n + 1$. Define a mapping $\chi_1 : U_1 \to V_1$ by

$$\chi_1(\omega^\alpha \cdot (n-1) + \zeta) = \omega^\alpha(\pi(n) - 1) + \zeta, \qquad \beta_n < \zeta \leq \omega^\alpha.$$

Plainly, χ_1 is a homeomorphism of U_1 onto V_1. Moreover, if $\beta_n < \zeta \leq \omega^\alpha$, then by (11), (12), and (13),

$$d(\sigma(\omega^\alpha \cdot (n-1) + \zeta), \tau(\chi_1(\omega^\alpha \cdot (n-1) + \zeta))) = d(\sigma(\omega^\alpha \cdot (n-1) + \zeta), \tau(\omega^\alpha(\pi(n) - 1) + \zeta))$$
$$< d(\sigma(\omega^\alpha \cdot n), \tau(\omega^\alpha \cdot \pi(n))) + 2\varepsilon_n.$$

Therefore, by (7) and (9), $\lim_{\eta \to \omega^\mu, \eta \in U_1} d(\sigma(\eta), \tau(\chi_1(\eta))) = 0$, and

$$d(\sigma(\eta), \tau(\chi_1(\eta))) < 2\delta \quad \text{for all } \eta \in U_1.$$

We complete the proof of this part of the induction by showing that the induction hypothesis can be applied to U_0 and V_0. It is enough to prove

(16) $\Phi_\xi(\sigma | U_0) = \Phi_\xi(\sigma)$, and $\Phi_\xi(\tau | V_0) = \Phi_\xi(\tau)$ for all $\xi < \alpha$.

Indeed, it will follow from (16) that $r(\sigma, U_0) \leq r(\sigma) < \delta$, and $r(\tau, V_0) < \delta$, so that since $U_0 \sim \omega^\alpha \sim V_0$, the induction hypothesis yields a homeomorphism χ_0 of U_0

onto V_0 such that $\lim_{\eta \to \omega^\mu, \eta \in U_0} d(\sigma(\eta), \tau(\chi_0(\eta))) = 0$ and $d(\sigma(\eta), \tau(\chi_0(\eta))) < 2\delta$ for all $\eta \in U_0$. Then $\chi = \chi_0 \cup \chi_1$ is a homeomorphism of ω^μ onto itself satisfying (a) and (b) of 5.1.

Proof of (16). Since U_0 is open and cofinal in ω^μ, it follows from Lemma 2.1 that $\Phi_\xi(\sigma | U_0) \subseteq \Phi_\xi(\sigma)$. Suppose that $x \in \Phi_\xi(\sigma)$. Let $\{\eta_n \mid n < \omega\}$ be a sequence in $(\omega^\mu)^{(\xi)}$ such that $\lim_{n \to \omega} \eta_n = \omega^\mu$ and $\lim_{n \to \omega} \sigma(\eta_n) = x$. Without loss of generality, we can assume $\eta_n \geqq \omega^\alpha \cdot (n-1) + \alpha_n$. Let $m < \omega$ be defined by $m = 0$ if α is a nonlimit ordinal, and if α is a limit ordinal, then take m to be the smallest number such that $\gamma_m \geqq \xi$, where $\gamma_1, \gamma_2, \ldots$ is the sequence which was chosen in (13'). Since $\xi < \alpha$, m is well defined. For $n > m$, define ξ_n by the conditions

$$\zeta_n = \eta_n \quad \text{if } \eta_n \in U_0,$$

$$\zeta_n = \min\,(U_0 \cap (\omega^\alpha)^{(\xi)} \cap [\omega^\alpha \cdot (k-1) + \alpha_k, \omega^\alpha \cdot (k-1) + \beta_k])$$

$$\text{if } \eta_n \in (\omega^\alpha \cdot (k-1) + \beta_k, \omega^\alpha \cdot k).$$

This definition makes sense by virtue of (15) or (15'), and the fact that $\eta_n > \omega^\alpha \cdot (n-1) + \alpha_n$ implies $k \geqq n > m$ and $\gamma_{k-1} \geqq \gamma_m \geqq \xi$ when $\eta_n \notin U_0$. By definition of ζ_n, we have $d(\sigma(\eta_n), \sigma(\zeta_n)) < 2\varepsilon_k \leqq 2\varepsilon_n$ in all cases. Also, $\zeta_n \in U_0 \cap (\omega^\mu)^{(\xi)} \cap [\omega^\alpha \cdot (n-1) + \alpha_n, \omega^\mu)$. Hence, $\lim_{n \to \omega} \zeta_n = \omega^\mu$ and $\lim_{n \to \omega} \sigma(\zeta_n) = \lim_{n \to \omega} \sigma(\eta_n) = x$. This proves that $x \in \Phi_\xi(\sigma | U_0)$. Consequently, $\Phi_\xi(\sigma | U_0) = \Phi_\xi(\sigma)$. The proof that $\Phi_\xi(\tau | V_0) = \Phi_\xi(\tau)$ is similar.

Induction step: $\mu = $ *limit ordinal*. Since the basic idea of the proof in this case may be obscured by technical details, it seems worthwhile to begin with an outline of the argument. The essential idea is to reduce the proof to the special case of 5.1 in which there exists an ordinal number $\alpha < \mu$ such that $\Phi_\xi(\sigma) = \Phi_\xi(\tau) = \{x_0\}$ for all $\xi > \alpha$. This reduction will be accomplished by the following lemma.

LEMMA 5.2. *Assume that the hypotheses of 5.1 are satisfied with μ a limit ordinal. Let $\beta_0 < \beta_1 < \beta_2 < \cdots$ be a sequence of ordinal numbers such that $\lim_{n \to \omega} \beta_n = \mu$, and let $\delta_0 \geqq \delta_1 \geqq \delta_2 \geqq \cdots$ be a sequence of positive real numbers such that $\delta_0 \leqq \delta$ and $\lim_{n \to \omega} \delta_n = 0$. Choose any $x_0 \in \bigcap_{\xi < \mu} \Phi_\xi(\sigma)$. Then there exists a strictly increasing sequence $\{\alpha_n \mid n < \omega\}$ of ordinal numbers, and families $\{U_n \mid n < \omega\}$, $\{V_n \mid n < \omega\}$ of open and closed subsets of ω^μ such that*

(a) $\beta_n < \alpha_n < \mu$,

(b) $U_n \cup V_n \subseteq (\omega^{\beta_{n-1}}, \omega^\mu)$ *if $n > 0$*,

(c) $U_n \cap U_m = V_n \cap V_m = \varnothing$ *if $m \neq n$*,

(d) $\bigcup_{n < \omega} U_n = \bigcup_{n < \omega} V_n = \omega^\mu$,

(e) $U_n \cap (\omega^\mu)^{(\xi)}$ *and* $V_n \cap (\omega^\mu)^{(\xi)}$ *are cofinal in ω^μ for every $\xi < \mu$*,

(f) $r(\sigma, U_n) < \delta_n, r(\tau, V_n) < \delta_n$ *for all $n < \omega$*,

(g) $\Phi_\xi(\sigma | U_n) = \Phi_\xi(\tau | V_n) = \{x_0\}$ *if $\xi > \alpha_n$*,

(h) $\Phi_\xi(\sigma | U_n) = \Phi_\xi(\tau | V_n) = \Phi_\xi(\sigma)$ *if $n > 0$ and $\alpha_{n-1} < \xi \leqq \alpha_n$, or $n = 0$ and $\xi \leqq \alpha_0$*,

(i) $\Phi_\xi(\sigma | U_n) = \Phi_\xi(\tau | V_n) = \Phi_{\alpha_{n-1}+1}(\sigma)$ *if $n > 0$ and $\xi \leqq \alpha_{n-1}$*.

It is easy to deduce from (e) that $U_n \sim \omega^\mu$ and $V_n \sim \omega^\mu$. Thus, the special case of 5.1 can be used to define a homeomorphism $\chi_n \colon U_n \to V_n$ such that

$$\lim_{\eta \to \omega^\mu, \, \eta \in U_n} d(\sigma(\eta), \tau(\chi_n(\eta))) = 0$$

and $d(\sigma(\eta), \tau(\chi_n(\eta))) < \delta_n$ for all $\eta \in U_n$. It then follows without difficulty that $\chi = \bigcup_{n < \omega} \chi_n$ satisfies the conditions of 5.1.

We begin with some lemmas which lead to a proof of the special case of 5.1. The first two of these results do not require the hypothesis that μ is a limit ordinal.

LEMMA 5.3. *Let $0 < \xi < \mu$, and suppose that for each $\eta \in (\omega^\mu)^{(\xi)}$, there is given $\zeta_\eta < \eta$. Define $V = \bigcup_{\eta \in (\omega^\mu)^{(\xi)}} (\zeta_\eta, \eta]$.*
Then V is open and closed in ω^μ, and $V \supseteq (\omega^\mu)^{(\xi)}$.

Proof. Plainly, V is open and $V \supseteq (\omega^\mu)^{(\xi)}$. Suppose that $\rho \in \omega^\mu - V$. Let η be the smallest member of $(\omega^\mu)^{(\xi)} \cap (\rho, \omega^\mu]$. Note that $\eta \notin (\omega^\mu)^{(\xi+1)}$, since otherwise there exists $\eta' \in (\omega^\mu)^{(\xi)}$ such that $\rho < \eta' < \eta$. Thus, either η is the smallest element of $(\omega^\mu)^{(\xi)}$, or else η has a predecessor η_1 in $(\omega^\mu)^{(\xi)}$. Plainly, $\eta_1 < \rho$. Since $\rho \leq \zeta_\eta$ (otherwise, $\rho \in (\zeta_\eta, \eta] \subseteq V$), it follows that either $[0, \zeta_\eta]$, or $(\eta_1, \zeta_\eta]$ is a neighborhood of ρ, contained in $\omega^\mu - V$. Hence, V is closed.

LEMMA 5.4. *Let V be a subset of ω^μ which is open and closed. Suppose that $\alpha \leq \mu$, and V satisfies $V \cap (\omega^\mu)^{(\alpha)} = \varnothing$, and $V \cap (\omega^\mu)^{(\xi)}$ is cofinal in ω^μ for every $\xi < \alpha$. Then V is order isomorphic to ω^α.*

Proof. Since V is well ordered, there is an order isomorphism of V onto an ordinal number ζ. Because V is closed, its order topology coincides with its relative topology. Hence, by Lemma 2.1 and our hypotheses, ζ (considered as a topological space with its order topology) satisfies: $\zeta^{(\alpha)} = \varnothing$, and if $\xi < \alpha$, then $\zeta^{(\xi)} \neq \varnothing$, and $\zeta^{(\xi)}$ has no last element.

Let $\zeta = \omega^{\gamma_1} \cdot n_1 + \omega^{\gamma_2} \cdot n_2 + \cdots + \omega^{\gamma_m} \cdot n_m$, where $\zeta \geq \gamma_1 > \gamma_2 > \cdots > \gamma_m$ and $0 < n_i < \omega$ for all i (see [2, p. 67]). If either $n_m > 1$, or $m > 1$, then it follows from the results of §4 in [3] that $\zeta^{(\gamma_m)}$ has a largest element, namely $\omega^{\gamma_1} \cdot n_1 + \omega^{\gamma_2} \cdot n_2 + \cdots + \omega^{\gamma_m} \cdot (n_m - 1)$. Thus, $\zeta = \omega^{\gamma_1}$. The conditions $\zeta^{(\xi)} \neq \varnothing$ for $\xi < \alpha$ and $\zeta^{(\alpha)} = \varnothing$ imply that $\gamma_1 = \alpha$.

It will be convenient to call a subset V of a well ordered set U full in U if V is open and closed in U, and $V \cap U^{(\xi)} = V^{(\xi)}$ is cofinal in U for every $\xi < \lambda(U)$. By 5.4, if V is full in ω^μ, then $V \sim \omega^\mu$.

LEMMA 5.5. *Let μ be a countable limit ordinal, $\alpha < \mu$, $\sigma \colon \omega^\mu \to W$ a continuous mapping, and U a full subset of ω^μ. Then there exists a strictly increasing, cofinal sequence $\{\eta_n \mid n < \omega\} \subseteq ((\omega^\mu)^{(\alpha)} \cap U) - (\omega^\mu)^{(\alpha+1)}$ such that*

$$\Phi_\alpha(\sigma \mid U) = \bigcap_{m < \omega} (\sigma(\{\eta_n \mid n \geq m\}))^-.$$

Proof. Let $\{x_k \mid k < \omega\}$ be a countable, dense subset of $\Phi_\alpha(\sigma \mid U)$. Choose a sequence $\{\varepsilon_m \mid m < \omega\}$ of positive real numbers such that $\lim_{m \to \omega} \varepsilon_m = 0$. Let

$\{\beta_n \mid n < \omega\}$ be a strictly increasing cofinal subsequence of μ. Finally, take π to be any mapping of ω onto $\omega \times \omega$. Define η_n recursively by the conditions

(i) if $\pi(n) = (k, m)$, then $d(\sigma(\eta_n), x_k) < \varepsilon_m$,

(ii) $\eta_n > \omega^{\beta_n}$,

(iii) if $n > 0$, then $\eta_n > \eta_{n-1}$,

(iv) η_n is the smallest ordinal number in $(\omega^\mu)^{(\alpha)} \cap U$ which satisfies (i), (ii), and (iii).

It is evident from the definition of $\Phi_\alpha(\sigma \mid U)$ that the set of $\eta \in (\omega^\mu)^{(\alpha)} \cap U$ satisfying (i), (ii), and (iii) is nonempty, so that the definition of η_n is well posed. We must first show that $\eta_n \notin (\omega^\mu)^{(\alpha+1)}$. Suppose otherwise. Then using the continuity of σ and the fact that U is open, it would be possible to find $\eta \in (\omega^\mu)^{(\alpha)} \cap U$ such that

$$d(\sigma(\eta_n), \sigma(\eta)) < \varepsilon_m - d(\sigma(\eta_n), x_k) \quad \text{and} \quad \eta_n > \eta > \max\{\omega^{\beta_n}, \eta_{n-1}\}.$$

However, this would plainly contradict the minimality of η_n. It follows from (iii) and (ii) that the sequence $\{\eta_n \mid n < \omega\}$ is strictly increasing and cofinal in ω^μ. From (i) and the fact that π maps ω onto $\omega \times \omega$, we conclude that for any $m < \omega$,

$$\{\sigma(\eta_n) \mid n \geq m\}^- \supseteq \{x_k \mid k < \omega\}^- = \Phi_\alpha(\sigma \mid U).$$

On the other hand,

$$\bigcap_{n < \omega} (\sigma((\omega^\mu)^{(\alpha)} \cap U \cap [\omega^{\beta_n}, \omega^\mu)))^- = \Phi_\alpha(\sigma \mid U),$$

so that by compactness, if $\varepsilon > 0$, then $\{n \mid d(\sigma(\eta_n), \Phi_\alpha(\sigma \mid U)) \geq \varepsilon\}$ is finite. Thus, $\bigcap_{m < \omega} (\sigma(\{\eta_n \mid n \geq m\}))^- = \Phi_\alpha(\sigma \mid U)$.

We will now use Lemmas 5.2–5.5 to complete the inductive proof of 5.1. The somewhat tedious proof of 5.2 is postponed to the end of the section.

Our standing assumptions are that μ is a countable limit ordinal, and Proposition 5.1 holds for all smaller ordinals. For the moment, assume also that there exists $\alpha < \mu$ such that $\Phi_\xi(\sigma) = \Phi_\xi(\tau) = \{x_0\}$ for all ξ satisfying $\alpha < \xi < \mu$. The starting point for the induction step in this case is the following fact.

(17) There is an open and closed set V in ω^μ such that

(a) $V \cap (\omega^\mu)^{(\alpha+1)} = \varnothing$ and $V \cap (\omega^\mu)^{(\alpha)}$ is cofinal in ω^μ,

(b) $\Phi_\xi(\sigma \mid V) = \Phi_\xi(\tau \mid V) = \Phi_\xi(\sigma)$ for all $\xi \leq \alpha$,

(c) $\lim_{\eta \to \omega^\mu, \eta \in \omega^\mu - V} d(\sigma(\eta), \tau(\eta)) = 0$, and

(d) $d(\sigma(\eta), \tau(\eta)) < 2\delta$ for all $\eta \in \omega^\mu - V$.

Proof. By 5.5, there exist strictly increasing, sequences $\{\eta'_n \mid n < \omega\}$ and $\{\eta''_n \mid n < \omega\}$ which are cofinal in ω^μ, contained in $(\omega^\mu)^{(\alpha)} - (\omega^\mu)^{(\alpha+1)}$, and satisfy $\Phi_\alpha(\sigma) = \bigcap_{m < \omega} (\sigma(\{\eta'_n \mid n \geq m\}))^-$ and $\Phi_\alpha(\tau) = \bigcap_{m < \omega} (\tau(\{\eta''_n \mid n \geq m\}))^-$. Since $\Phi_\alpha(\sigma) = \Phi_\alpha(\tau)$, it is easy to see that the strictly increasing sequence $\{\eta_n \mid n < \omega\}$ obtained by arranging the set union $\{\eta'_n \mid n < \omega\} \cup \{\eta''_n \mid n < \omega\}$ in ascending order also satisfies

(i) $\Phi_\alpha(\sigma) = \bigcap_{m < \omega} (\sigma(\{\eta_n \mid n \geq m\}))^-$, and

(ii) $\Phi_\alpha(\tau) = \bigcap_{m < \omega} (\tau(\{\eta_n \mid n \geq m\}))^-$.

(The proof of this statement is essentially the same as the last step in the proof of 5.5.) Let

$$\varepsilon = \min\{\delta - r(\sigma),\, \delta - r(\tau)\}.$$

Choose a decreasing sequence $\{\varepsilon_n \mid n < \omega\}$ of positive real numbers with $\varepsilon_0 \leqq \varepsilon$, and $\lim_{n\to\omega} \varepsilon_n = 0$. If $\rho \in (\omega^\mu)^{(\alpha+1)}$, then there is a unique $n < \omega$ such that $\eta_{n-1} < \rho < \eta_n$ (or $\rho < \eta_0$ if $n = 0$). Choose ζ_ρ such that

  (iii) $\eta_{n-1} \leqq \zeta_\rho < \rho$, and

  (iv) $\zeta_\rho \leqq \eta \leqq \rho$ implies $d(\sigma(\eta), \sigma(\rho)) < \varepsilon_n$, $d(\tau(\eta), \tau(\rho)) < \varepsilon_n$.

It is possible to select such a ζ_ρ by the continuity of σ and τ. Define

$$U = \bigcup_{\rho \in (\omega^\mu)^{(\alpha+1)}} (\zeta_\rho, \rho], \qquad V = \omega^\mu - U.$$

By 5.3, U and V are open and closed, and $U \supseteq (\omega^\mu)^{(\alpha+1)}$, so that $V \cap (\omega^\mu)^{(\alpha+1)} = \varnothing$. On the other hand, by (iii), $V \supseteq \{\eta_n \mid n < \omega\}$, so that $V \cap (\omega^\mu)^{(\alpha)}$ is cofinal in ω^μ. Suppose that $\eta \in U$. Then there exists $\rho \in (\omega^\mu)^{(\alpha+1)}$, and $n < \omega$ such that $\eta_{n-1} \leqq \zeta_\rho < \eta \leqq \rho < \eta_n$.

Consequently,

$$d(\sigma(\eta), \tau(\eta)) \leqq d(\sigma(\eta), \sigma(\rho)) + d(\sigma(\rho), x_0) + d(\tau(\rho), x_0) + d(\tau(\eta), \tau(\rho))$$
$$< d(\sigma(\rho), x_0) + d(\tau(\rho), x_0) + 2\varepsilon_n.$$

Since $\rho \in (\omega^\mu)^{(\alpha+1)}$ and $\Phi_{\alpha+1}(\sigma) = \Phi_{\alpha+1}(\tau) = \{x_0\}$, it follows that

  (v) $\lim_{\rho \to \omega^\mu, \rho \in (\omega^\mu)^{(\alpha+1)}} \sigma(\rho) = x_0 = \lim_{\rho \to \omega^\mu, \rho \in (\omega^\mu)^{(\alpha+1)}} \tau(\rho)$, and $d(\sigma(\rho), x_0) \leqq r(\sigma)$, $d(\tau(\rho), x_0) \leqq r(\tau)$ for all $\rho \in (\omega^\mu)^{(\alpha+1)}$. Thus, $\lim_{\eta \to \omega^\mu, \eta \in V} d(\sigma(\eta), \tau(\eta)) = 0$ and $d(\sigma(\eta), \tau(\eta)) < r(\sigma) + r(\tau) + (\delta - r(\sigma)) + (\delta - r(\tau)) = 2\delta$. It is an obvious consequence of (i) and (ii) that $\Phi_\alpha(\sigma|V) \supseteq \Phi_\alpha(\sigma)$ and $\Phi_\alpha(\tau|V) \supseteq \Phi_\alpha(\tau)$. The reverse inclusions are clear from 4.2, so that $\Phi_\alpha(\sigma|V) = \Phi_\alpha(\tau|V) = \Phi_\alpha(\sigma)$. If $\xi < \alpha$, then $x_0 \in \Phi_\alpha(\sigma) = \Phi_\alpha(\sigma|V) \subseteq \Phi_\xi(\sigma|V)$. Similarly, $x_0 \in \Phi_\xi(\tau|V)$. By (iv) and (v), $\lim_{\eta \to \omega^\mu, \eta \in U} \sigma(\eta) = x_0$ and $\lim_{\eta \to \omega^\mu, \eta \in U} \tau(\eta) = x_0$. Hence, $\Phi_\xi(\sigma|U) = \Phi_\xi(\tau|U) = \{x_0\}$ for all $\xi < \mu$. Therefore, by 4.2, $\Phi_\xi(\sigma) = \Phi_\xi(\sigma|U) \cup \Phi_\xi(\sigma|V) = \{x_0\} \cup \Phi_\xi(\sigma|V) = \Phi_\xi(\sigma|V)$, and similarly, $\Phi_\xi(\tau) = \Phi_\xi(\tau|V)$ for all $\xi < \alpha$. This completes the proof of the statement (17).

The induction hypothesis can now be used to carry out the induction step in our special case $\Phi_\xi(\sigma) = \Phi_\xi(\tau) = \{x_0\}$ for $\alpha < \xi < \mu$. Let U and V be as in (17). By 5.4, $V \sim \omega^{\alpha+1}$. From (17b), we have $r(\sigma, V) \leqq r(\sigma) < \delta$ and $r(\tau, V) \leqq r(\tau) < \delta$. Thus, by (17b) again, the induction hypothesis is applicable: there exists a homeomorphism χ_0 of V onto itself such that

$$\lim_{\eta \to \omega^\mu,\, \eta \in V} d(\sigma(\eta), \tau(\chi_0(\eta))) = 0,$$

and

$$d(\sigma(\eta), \tau(\chi_0(\eta))) < 2\delta \quad \text{for all } \eta \in V.$$

Define $\chi : \omega^\mu \to \omega^\mu$ by $\chi(\eta) = \chi_0(\eta)$ for $\eta \in V$, $\chi(\eta) = \eta$ for $\eta \in U$. Clearly χ is a homeomorphism of ω^μ onto itself. By (17c), (17b) and the properties of χ_0 listed above, it is clear that χ satisfies 5.1(a) and (b).

We will now show that the general case of the induction step can be deduced from the special case that has just been treated, using Lemma 5.2. All notation will be as in 5.2. By 5.2(e) and 5.4, each of the sets U_n and V_n is order isomorphic to ω^μ. Thus, all of the hypotheses of the special case are satisfied when U_n and V_n are identified with ω^μ. Consequently, for each $n < \omega$, there is a homeomorphism χ_n of U_n to V_n such that

$$\lim_{\eta \to \omega^\mu, \eta \in U_n} d(\sigma(\eta), \tau(\chi_n)) = 0,$$

and

$$d(\sigma(\eta), \tau(\chi_n(\eta))) < 2\delta_n \quad \text{for all } \eta \in U_n.$$

Define $\chi = \bigcup_{n < \omega} \chi_n$. Plainly, χ is a one-to-one mapping of ω^μ onto ω^μ such that $d(\sigma(\eta), \tau(\chi(\eta))) < 2\delta$ for all $\eta \in \omega^\mu$. Let $\varepsilon > 0$. Since $\lim_{n \to \omega} \delta_n = 0$, there exists $m < \omega$ such that $\delta_n < \varepsilon/2$ for all $n \geq m$. Consequently, $\eta \in U_n$, $n \geq m$ implies $d(\sigma(\eta), \tau(\chi(\eta))) < \varepsilon$. Since

$$\lim_{\eta \to \omega^\mu, \eta \in U_n} d(\sigma(\eta), \tau(\chi_n(\eta))) = 0 \quad \text{for all } n,$$

there exists $\zeta < \omega^\mu$ such that if $\eta \geq \zeta$ and $\eta \in \bigcup_{n < m} U_n$, then $d(\sigma(\eta), \tau(\chi(\eta))) < \varepsilon$. Therefore, by the choice of m, this inequality holds for every $\eta \geq \zeta$. This argument proves that $\lim_{\eta \to \omega^\mu} d(\sigma(\eta), \tau(\chi(\eta))) = 0$. It remains to show that χ is a homeomorphism. Let $W \subseteq \omega^\mu$ be open. Then $W \cap V_n$ is open for each n. Consequently, $\chi^{-1}(W \cap V_n) = \chi_n^{-1}(W \cap V_n)$ is open. Finally, $\chi^{-1}(W) = \bigcup_{n < \omega} \chi^{-1}(W \cap V_n)$ is open in ω^μ. This shows that χ is continuous. Similarly, χ^{-1} is continuous. Thus, χ is a homeomorphism.

The final job is to prove Lemma 5.2. We will approach this result through a sequence of technical lemmas.

LEMMA 5.6. *Let* $\sigma: \omega^\mu \to W$ *be continuous. Then* $\lim_{\xi \to \mu} r_\xi(\sigma) = 0$.

Proof. Denote

$$T = \bigcap_{\xi < \mu} \Phi_\xi(\sigma) = \bigcap_{\eta < \omega^\mu} \bigcap_{\xi < \mu} (\sigma((\omega^\mu)^{(\xi)} \cap [\eta, \omega^\mu)))^-.$$

Let $\varepsilon > 0$ be arbitrary. Using various properties of e which were listed earlier, we conclude

(i) there exists $\eta_0 < \omega^\mu$ such that

$$e\left(\bigcap_{\xi < \mu} (\sigma((\omega^\mu)^{(\xi)} \cap [\eta_0, \omega^\mu)))^-, T\right) < \varepsilon/2,$$

(ii) there exists $\xi_0 < \mu$ such that $\omega^{\xi_0} > \eta_0$ and for all $\xi \geq \xi_0$,

$$e\left((\sigma((\omega^\mu)^{(\xi)} \cap [\eta_0, \omega^\mu)))^-, \bigcap_{\xi < \mu} (\sigma((\omega^\mu)^{(\xi)} \cap [\eta_0, \omega^\mu)))^-\right) < \varepsilon/2.$$

By (i) and (ii), it follows that

(iii) $\xi \geq \xi_0$ implies $e((\sigma((\omega^\mu)^{(\xi)} \cap [\eta_0, \omega^\mu)))^-, T) < \varepsilon$. Clearly, if $\xi \geq \xi_0$, then $\omega^\xi \geq \omega^{\xi_0} > \eta_0$ so that $(\omega^\mu)^{(\xi)} \subseteq [\eta_0, \omega^\mu)$. Therefore, $\sigma((\omega^\mu)^{(\xi)} \cap [\eta_0, \omega^\mu)) = \sigma((\omega^\mu)^{(\xi)})$.

Finally, note that $(\sigma((\omega^\mu)^{(\xi)}))^- \supseteq \Phi_\xi(\sigma) \supseteq T$ implies

$$r_\xi(\sigma) = e(\sigma((\omega^\mu)^{(\xi)}), \Phi_\xi(\sigma)) = e((\sigma(\omega^\mu)^{(\xi)})^-, \Phi_\xi(\sigma)) \leqq e((\sigma((\omega^\mu)^{(\xi)}))^-, T) < \varepsilon$$

for all $\xi \geqq \xi_0$. Since $\varepsilon > 0$ was arbitrary, this completes the proof.

In the next four lemmas, the following hypotheses are assumed to be in effect:

U is a full subset of ω^μ, that is, U is open and closed, and $U \cap (\omega^\mu)^{(\xi)}$ is cofinal in ω^μ for all $\xi < \mu$;

σ is a continuous mapping of ω^μ to W.

LEMMA 5.7. *Let $\varepsilon > 0$. Choose any x_0 in $\bigcap_{\xi < \mu} \Phi_\xi(\sigma | U)$. Then there is a set $V \subseteq U$ such that*

(a) *V and $M = U - V$ are full in ω^μ,*

(b) *$r(\sigma, V) < \varepsilon$,*

(c) *$\Phi_\xi(\sigma | V) = \{x_0\}$ for all $\xi < \mu$,*

(d) *$\Phi_\xi(\sigma | M) = \Phi_\xi(\sigma | U)$ for all $\xi < \mu$.*

Proof. Let $\{\alpha_n \mid n < \omega\}$ be an increasing sequence of nonlimit ordinals such that $\lim_{n \to \omega} \alpha_n = \mu$. Let $\{\varepsilon_n \mid n < \omega\}$ be a decreasing sequence of positive real numbers with $\varepsilon_0 < \varepsilon/3$, and $\lim_{n \to \omega} \varepsilon_n = 0$. Choose a strictly increasing sequence $\{\eta_n \mid n < \omega\}$ such that $\eta_n \in U^{(\alpha_n)}$, $d(\sigma(\eta_n), x_0) < \varepsilon_n$.

For each $n < \omega$, choose $\zeta_n \in U^{(\alpha_n - 1)}$ such that $\eta_{n-1} < \zeta_n < \eta_n$ ($\zeta_0 < \eta_0$), and $\zeta_n \leqq \zeta \leqq \eta_n$ implies $d(\sigma(\zeta), \sigma(\eta_n)) < \varepsilon_n$. Let $V = \bigcup_{n < \omega} (\zeta_n, \eta_n]$. Then V is full. Since $\zeta_n \in M = U - V$, it follows that M is also full. By construction, $d(\sigma(\zeta), x_0) < (2/3)\varepsilon$ for all $\zeta \in V$. Therefore, $r(\sigma, V) < \varepsilon$. Also, it is evident that $\lim_{\eta \to \omega^\mu, \eta \in V} \sigma(\eta) = x_0$. Hence, $\Phi_\xi(\sigma | V) = \{x_0\}$ for all $\xi < \mu$. Since $\lim_{n \to \omega} \sigma(\zeta_n) = x_0$, we have $x_0 \in \Phi_\xi(\sigma | M)$. By 4.2, $\Phi_\xi(\sigma | U) = \Phi_\xi(\sigma | V) \cup \Phi_\xi(\sigma | M) = \{x_0\} \cup \Phi_\xi(\sigma | M) = \Phi_\xi(\sigma | M)$ for all $\xi < \mu$.

LEMMA 5.8. *Let δ be a positive real number, and $\alpha < \mu$ an ordinal number such that if $\alpha \leqq \xi < \mu$, then $e(\sigma(U \cap (\omega^\mu)^{(\xi)}), \Phi_\xi(\sigma | U)) < \delta$. Then there exists $V \subseteq U$ such that*

(a) *V is full in ω^μ,*

(b) *$V \supseteq U \cap (\omega^\mu)^{(\alpha)}$,*

(c) *$r(\sigma, V) < 2\delta$,*

(d) *$\Phi_\xi(\sigma | V) = \Phi_\alpha(\sigma | U)$ for $\xi \leqq \alpha$,*

(e) *$\Phi_\xi(\sigma | V) = \Phi_\xi(\sigma | U)$ for $\xi \geqq \alpha$.*

Proof. Since $e((\sigma(U \cap (\omega^\mu)^{(\alpha)}))^-, \Phi_\alpha(\sigma | U)) = e(\sigma(U \cap (\omega^\mu)^{(\alpha)}), \Phi_\alpha(\sigma | U)) < \delta$, it follows that

$$\sigma(U \cap (\omega^\mu)^{(\alpha)})^- \subseteq \{x \in W \mid d(x, \Phi_\alpha(\sigma | U)) < \delta\}.$$

By compactness and the zero dimensionality of W, there is an open and closed set N such that

$$\sigma(U \cap (\omega^\mu)^{(\alpha)}) \subseteq N \subseteq \{x \in W \mid d(x, \Phi_\alpha(\sigma | U)) < \delta\}.$$

Let $M = \sigma^{-1}(N) \cap U$. Then M is open and closed, and $M \supseteq U \cap (\omega^\mu)^{(\alpha)}$. Let $\{\alpha_n \mid n < \omega\}$ be a strictly increasing sequence of ordinals such that $\lim_{n \to \omega} \alpha_n = \mu$.

Choose a decreasing sequence $\{\varepsilon_n \mid n < \omega\}$ of real numbers with $\varepsilon_0 < \delta$ and $\lim_{n \to \omega} \varepsilon_n = 0$. By the continuity of σ, there exist ordinal numbers ζ_n associated with each $\eta \in U \cap (\omega^\mu)^{(\alpha)}$ such that $(\zeta_n, \eta] \subseteq M$, if $\omega^{\alpha_n} < \eta < \omega^{\alpha_{n+1}}$, then $\zeta_n \geqq \omega^{\alpha_n}$, and $d(\sigma(\rho), \sigma(\eta)) < \varepsilon_n$ for all $\rho \in (\zeta_n, \eta]$.

Define $V = \bigcup_{\eta \in U \cap (\omega^\mu)^{(\alpha)}} (\zeta_n, \eta]$. By 5.3, V is open and closed. Clearly, $U \supseteq V \supseteq U \cap (\omega^\mu)^{(\alpha)}$, so that $V \cap (\omega^\mu)^{(\xi)} = U \cap (\omega^\mu)^{(\xi)}$ for all $\xi \geqq \alpha$. In particular, V is full in ω^μ, (e) holds, and

$$e(\sigma(V \cap (\omega^\mu)^{(\xi)}), \Phi_\xi(\sigma \mid V)) \leqq e(\sigma(U \cap (\omega^\mu)^{(\xi)}), \Phi_\xi(\sigma \mid U)) < \delta$$

for $\xi \geqq \alpha$. If $\rho \in V \cap (\omega^{\alpha_n}, \omega^{\alpha_{n+1}}]$, then

$$d(\sigma(\rho), \Phi_\alpha(\sigma \mid U)) \leqq \varepsilon_n + e(\sigma(U \cap (\omega^\mu)^{(\alpha)} \cap (\omega^{\alpha_n}, \omega^\mu)), \Phi_\alpha(\sigma \mid U))$$
$$\leqq \varepsilon_n + e(\sigma(U \cap (\omega^\mu)^{(\alpha)}), \Phi_\alpha(\sigma \mid U))$$
$$\leqq \varepsilon_0 + \delta.$$

It follows that $\Phi_\xi(\sigma \mid V) \subseteq \Phi_\alpha(\sigma \mid U) = \Phi_\alpha(\sigma \mid V) \subseteq \Phi_\xi(\sigma \mid V)$ for all $\xi \leqq \alpha$, and $r(\sigma, V) \leqq \varepsilon_0 + \delta < 2\delta$.

LEMMA 5.9. *Let $\alpha < \mu$. Then there is an open and closed subset V of U such that*
(a) $V \cap (\omega^\mu)^{(\alpha+1)} = \varnothing$,
(b) $V \cap (\omega^\mu)^{(\alpha)}$ *is cofinal in ω^μ, and*
(c) $\Phi_\alpha(\sigma \mid V) = \Phi_\alpha(\sigma \mid U)$.

Proof. Choose a decreasing sequence $\{\varepsilon_n \mid n < \omega\}$ of positive real numbers with $\varepsilon_0 < r(\sigma, U)$, and $\lim_{n \to \omega} \varepsilon_n = 0$. Let $\{\eta_n \mid n < \omega\}$ be a sequence of ordinals which satisfies the conditions of 5.5. Since $\eta_n \notin (\omega^\mu)^{(\alpha+1)}$, either η_n is the least element of $(\omega^\mu)^{(\alpha)}$, or else η_n has a predecessor ζ_n in $(\omega^\mu)^{(\alpha)}$. We can assume that the latter condition holds for all n. (Otherwise, delete η_0.) Define $V = \bigcup_{n < \omega} (\zeta_n, \eta_n]$. Since $\{\eta_n \mid n < \omega\}$ is cofinal in ω^μ, it is clear that V is open and closed. Since $(\zeta_n, \eta_n] \cap (\omega^\mu)^{(\alpha+1)} = \varnothing$ for all n, it follows that $V \cap (\omega^\mu)^{(\alpha+1)} = \varnothing$. Plainly, $V \cap (\omega^\mu)^{(\alpha)} = \{\eta_n \mid n < \omega\}$ is cofinal in ω^μ, and

$$\Phi_\alpha(\sigma \mid V) = \bigcap_{\eta < \omega^\mu} (\sigma(V \cap (\omega^\mu)^{(\alpha)} \cap (\eta, \omega^\mu)))^- = \bigcap_{m < \omega} (\sigma(\{\eta_n \mid n \geqq m\}))^- = \Phi_\alpha(\sigma \mid U)$$

(by 5.5).

LEMMA 5.10. *In addition to the standing hypotheses, assume that $x_0 \in \bigcap_{\xi < \mu} \Phi_\xi(\sigma \mid U)$, δ is any positive real number, ε is a positive real number which satisfies $\varepsilon \geqq r(\sigma, U)$, $\beta < \mu$, and $\alpha \in (\beta, \mu)$ is such that $r_\xi(\sigma, U) < \delta$ for all $\xi \in (\alpha, \mu)$. Then there exists $V \subseteq U$ such that*
(a) V *and $M = U - V$ are full subsets of ω^μ,*
(b) $V \supseteq U \cap [0, \omega^\beta]$,
(c) $r(\sigma, V) \leqq \varepsilon$, $r(\sigma, M) < 2\delta$,
(d) $\Phi_\xi(\sigma \mid V) = \Phi_\xi(\sigma \mid U)$ *for $\xi \leqq \alpha$, $\Phi_\xi(\sigma \mid V) = \{x_0\}$ for $\xi > \alpha$, and*
(e) $\Phi_\xi(\sigma \mid M) = \Phi_\xi(\sigma \mid U)$ *for $\xi > \alpha$, $\Phi_\xi(\sigma \mid M) = \Phi_{\alpha+1}(\sigma \mid U)$ for $\xi < \alpha$.*

Proof. By 5.8, there exists $U_1 \subseteq U$ such that U_1 is full in ω^μ

$$U_1 \supseteq U \cap (\omega^\mu)^{(\alpha+1)}, \qquad r(\sigma, U_1) < 2\delta$$

$$\Phi_\xi(\sigma|U_1) = \Phi_{\alpha+1}(\sigma|U) \quad \text{for } \xi \leq \alpha+1,$$

and

$$\Phi_\xi(\sigma|U_1) = \Phi_\xi(\sigma|U) \qquad \text{for } \xi \geq \alpha+1.$$

By 5.7, there exists $V_1 \subseteq U_1$ such that V_1 and $U_2 = U_1 - V_1$ are full in ω^μ and satisfy

$$r(\sigma, V_1) < \varepsilon,$$

$$\Phi_\xi(\sigma|V_1) = \{x_0\} \quad \text{for all } \xi < \mu, \text{ and}$$

(i) $\Phi_\xi(\sigma|U_2) = \Phi_\xi(\sigma|U_1)$ for all $\xi < \mu$.

By 5.9, there exists $V_2 \subseteq U_2$ such that V_2 is open and closed in ω^μ,

(ii) $V_2 \cap (\omega^\mu)^{(\alpha+1)} = \varnothing$ and $V_2 \cap (\omega^\mu)^{(\alpha)}$ is cofinal in ω^μ, and

(iii) $\Phi_\alpha(\sigma|V_2) = \Phi_\alpha(\sigma|U_2)$.

Define

$$M = (U_2 - V_2) \cap (\omega^\beta, \omega^\mu), \qquad V = U - M.$$

Note that

(iv) $V = (U - V_1) \cup V_1 \cup V_2 \cup (U \cap [0, \omega^\beta])$.

Clearly, V and M are open and closed, and since V_1 is full, so is V. It follows from (ii) and the fact that U_2 is full that $U_2 - V_1$, and hence also M, is full. Using (i), (ii), and 4.2, we obtain $\Phi_\xi(\sigma|M) = \Phi_\xi(\sigma|(U_2-V_2)) = \Phi_\xi(\sigma|U_2) = \Phi_\xi(\sigma|U_1) = \Phi_\xi(\sigma|U)$ for all $\xi \geq \alpha+1$. If $\xi \leq \alpha$, then

$$\Phi_{\alpha+1}(\sigma|U) = \Phi_{\alpha+1}(\sigma|(U_2-V_2)) \subseteq \Phi_\xi(\sigma|(U_2-V_2)) \subseteq \Phi_\xi(\sigma|U_2) \subseteq \Phi_\xi(\sigma|U_1)$$
$$= \Phi_{\alpha+1}(\sigma|U).$$

This proves (e) and shows that $\Phi_\xi(\sigma|M) = \Phi_\xi(\sigma|U_1)$ for all $\xi < \mu$. Therefore, since $M \subseteq U_2 \subseteq U_1$, it follows that $r(\sigma, M) \leq r(\sigma, U_1) < 2\delta$. We next compute $\Phi_\xi(\sigma|V)$. Suppose first that $\xi \geq \alpha+1$. Then by (ii)

(v) $V \cap (\omega^\mu)^{(\xi)} = V_1 \cap (\omega^\mu)^{(\xi)}$.

Consequently, $\Phi_\xi(\sigma|V) = \Phi_\xi(\sigma|V_1) = \{x_0\}$ for $\xi \geq \alpha+1$. If $\xi \leq \alpha$, then by (i) and (iii) we obtain

$$\Phi_\xi(\sigma|V) \supseteq \Phi_\alpha(\sigma|V) \supseteq \Phi_\alpha(\sigma|V_2) = \Phi_\alpha(\sigma|U_2) = \Phi_\alpha(\sigma|U_1) = \Phi_{\alpha+1}(\sigma|U).$$

Hence, by 4.2,

$$\Phi_\xi(\sigma|U) = \Phi_\xi(\sigma \mid V) \cup \Phi_\xi(\sigma|M) = \Phi_\xi(\sigma|V) \cup \Phi_{\alpha+1}(\sigma|U) = \Phi_\xi(\sigma|V).$$

This finishes the proof of (d). If $\xi \geq \alpha+1$, then by (v) we have $r_\xi(\sigma, V) = r_\xi(\sigma, V_1)$ $\leq r(\sigma, V_1) < \varepsilon$. On the other hand, for $\xi \leq \alpha$, $\Phi_\xi(\sigma|V) = \Phi_\xi(\sigma|U)$ and $V \subseteq U$, so that $r_\xi(\sigma, V) \leq r_\xi(\sigma, U) \leq r(\sigma, U) \leq \varepsilon$. Consequently, $r(\sigma, V) \leq \varepsilon$.

Proof of Lemma 5.2. We construct the α_n, U_n, and V_n recursively. Assume that α_j, U_j, and V_j have been chosen for $j < n$ satisfying

(i) $\beta_j < \alpha_j < \mu$,

(ii) each U_j and V_j is full in ω^μ,

(iii) $\bigcup_{j<n} U_j \supseteq [0, \omega^{\beta_{n-1}}], \bigcup_{j<n} V_j \supseteq [0, \omega^{\beta_{n-1}}]$,

(iv) $U_i \cap U_j = V_i \cap V_j = \varnothing$ if $i \ne j$,

(v) $r(\sigma, U_j) < \delta_j, r(\tau, V_j) < \delta_j$ for all $j < n$,

(vi) $\Phi_\xi(\sigma | U_j) = \Phi_\xi(\tau | V_j) = \{x_0\}$ if $\xi > \alpha_j$,

(vii) $\Phi_\xi(\sigma | U_j) = \Phi_\xi(\tau | V_j) = \Phi_\xi(\sigma)$ if $j > 0$ and $\alpha_{j-1} < \xi \le \alpha_j$, or $j = 0$ and $\xi \le \alpha_0$,

(viii) $\Phi_\xi(\sigma | U_j) = \Phi_\xi(\tau | V_j) = \Phi_{\alpha_{j-1}+1}(\sigma)$ if $j > 0$ and $\xi \le \alpha_{j-1}$.

If $X_n = \omega^\mu \bigcup_{j<n} U_n$, $Y_n = \omega^\mu - \bigcup_{j<n} V_j$, then

(ix) $\Phi_\xi(\sigma | X_n) = \Phi_\xi(\tau | Y_n) = \Phi_\xi(\sigma)$ for $\xi > \alpha_{n-1}$,

(x) $\Phi_\xi(\sigma | X_n) = \Phi_\xi(\tau | Y_n) = \Phi_{\alpha_{n-1}+1}(\sigma)$ for $\xi \le \alpha_{n-1}$, and

(xi) $r(\sigma, X_n) < \delta_n, r(\tau, Y_n) < \delta_n$.

If $n = 0$, these conditions are satisfied vacuously, except for the last two, which are valid by hypothesis. By 5.6, it is possible to choose $\alpha_n < \mu$ such that $\alpha_n > \beta_n$, $\alpha_n > \alpha_{n-1}$, and $e(\sigma(X_n \cap (\omega^\mu)^{(\xi)}), \Phi_\xi(\sigma | X_n)) < (1/2)\delta_{n+1}$, $e(\tau(Y_n \cap (\omega^\mu)^{(\xi)}), \Phi_\xi(\tau | Y_n)) < (1/2)\delta_{n+1}$ for all ξ satisfying $\alpha_n < \xi < \mu$. The desired $U_n \subseteq X_n$, $V_n \subseteq Y_n$ are now obtained by applying 5.10 to X_n and σ, and to Y_n and τ (with $\alpha = \alpha_n$, $\beta = \beta_n$, $\delta = (1/2)\delta_{n+1}$, and $\varepsilon = \delta_n$). The conditions (i)–(xi) are easily verified. It is clear that $\{\alpha_n \mid n < \omega\}$, $\{U_n \mid n < \omega\}$, $\{V_n \mid n < \omega\}$ satisfy all of the conditions of 5.2.

This completes the proof of 5.2, so that we are done.

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