TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 148, March 1970

THE MODULE INDEX AND INVERTIBLE IDEALS

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Abstract. A. Fröhlich used the module index to classify the projective modules of an order in a finite dimensional commutative separable algebra over the quotient field of a Dedekind domain. This paper extends Fröhlich's results and classifies the invertible ideals of an order in a noncommutatives eparable algebra. Several properties of invertible ideals are considered, and examples are given.

1. **Introduction.** Let A be a Dedekind domain with quotient field K; let Σ be a finite dimensional separable K-algebra. A subring Λ of Σ is said to be an A-order if Λ contains A, Λ is A-torsion free and Λ contains a K-basis of Σ. There is always a maximal A-order Γ containing Λ. A finite dimensional torsion free A-module is called an A-lattice.

Let X be a finite dimensional torsion A-module. The order ideal, $\text{ord}_A X$, is the product of the A-annihilators of the composition factors of an A-composition series of X. Let M and N be two A-lattices which span the same vector space over K; i.e., $K \otimes_A M \cong K \otimes_A N$. Define the module index of M and N, $[M:N]$, by $[M:N] = (\text{ord}_A M/L) \cdot (\text{ord}_A N/L)^{-1}$ where L is any A-lattice such that $K \otimes_A L = K \otimes_A M$ and $L \subseteq M \cap N$.

In [5], A. Fröhlich gives criteria in terms of the module index to decide the projectivity of Λ-modules when Σ is commutative. More precisely, he shows that if M is a Λ-module such that $K \otimes_A M \cong \Sigma^{(r)}$, then M is Λ-projective if and only if $[\Gamma M : M] = [\Gamma : \Lambda]^r$, where Γ is a maximal order over Λ and ΓM denotes the smallest Γ module containing M. The object of this paper is to give analogues to Fröhlich's theorem when Σ is noncommutative. We will give examples, however, to show that in the fairly simple case of a finite dimensional matrix algebra over the quotient field of a discrete rank one valuation ring, no direct extensions of either the necessity or the sufficiency of Fröhlich's theorem can be given.

It is well known that for integral domains, an ideal is invertible if and only if it is projective. However, the statement that projectivity implies invertibility is not necessarily true for noncommutative rings. Nevertheless, since invertibility is a

Presented to the Society, January 25, 1970; received by the editors May 9, 1969 and, in revised form, September 9, 1969.

AMS Subject Classifications. Primary, 1620; Secondary, 1648.

Key Words and Phrases. Orders, module index, invertible ideals, equivalent idempotents, separable algebra, reduced orders.

[1] Portions of this paper were included in the author's Ph.D. dissertation at the University of Illinois supervised by Robert Fossum.

property of ideals that passes from the local case to the global and vice versa, we consider it in terms of Fröhlich type index conditions. We find that if the single index condition of Fröhlich's theorem is replaced by several index conditions in terms of idempotents and a multiplication condition on a two-sided full ideal, then we have necessary and sufficient conditions to insure the invertibility of the ideal. Further, it is not hard to see that for the case of full two-sided ideals and commutative K-algebras, these results imply Fröhlich's theorem.

2. **Preliminaries.** We first list a few basic properties of the module index. The proofs of these results follow directly from the definitions. Let M, N, R be A-lattices spanning the same K-space.

1. $[M{:}N]_p=[M_p{:}N_p]$ for every prime ideal p of A.
2. If $M\supseteq N$, then $[M{:}N]\subseteq A$.
3. $[M{:}N][N{:}R]=[M{:}R]$.
4. If $M=M_1\oplus M_2$, $N=N_1\oplus N_2$ with M_1 (resp. M_2) spanning the same space as N_1 (resp. N_2), then $[M_1\oplus M_2{:}N_1\oplus N_2]=[M_1{:}N_1][M_2{:}N_2]$.
5. Let $f\colon v\to v'$ be a K-isomorphism of finite dimensional vector spaces v, v'. Let M, N be two A-lattices spanning v. Then $[M{:}N]=[f(M){:}f(N)]$.

We will say that an ideal I of Λ is *full* if $I\otimes_A K\cong\Lambda\otimes_A K$.

We now let A denote a complete discrete rank one valuation ring with quotient field K. Let π be a generator of the maximal ideal of A, and let Σ be a finite dimensional separable K-algebra.

Let Λ denote an A-order in Σ, and let $\overline{\Lambda}=\Lambda/\pi\Lambda$. Then $\overline{\Lambda}$ is an algebra over the field $A/(\pi)$. Let $\bar{1}=\bar{e}_1+\bar{e}_2+\cdots+\bar{e}_r$ be a decomposition of $\bar{1}$ into primitive orthogonal idempotents. Then since A is a complete discrete valuation ring, there are primitive orthogonal idempotents $e_1,\ldots,e_r$ in Λ which map to $\bar{e}_1,\bar{e}_2,\ldots,\bar{e}_r$ by the natural map $\Lambda\to\overline{\Lambda}$ and such that $1=e_1+e_2+\cdots+e_r$.

We say that primitive orthogonal idempotents e_i and e_j in Λ are *equivalent* in Λ if $(e_i\Lambda e_j)(e_j\Lambda e_i)=e_i\Lambda e_i$. We will write $e_i\sim e_j$ to denote this equivalence.

Before we show that this is an equivalence relation, we will prove the following two lemmas.

Lemma 1. *Let e_i and e_j be primitive orthogonal idempotents in Λ. Then $e_i\sim e_j$ if and only if there are elements x_{ij} in $e_i\Lambda e_j$ and x_{ji} in $e_j\Lambda e_i$ such that $x_{ij}x_{ji}=e_i$ and $x_{ji}x_{ij}=e_j$.*

Proof. Assume that e_i is equivalent to e_j in Λ, and let x_{ij} in $e_i\Lambda e_j$ and x_{ji} in $e_j\Lambda e_i$ be such that $x_{ij}x_{ji}=e_i$. Then, $x_{ji}x_{ij}x_{ji}x_{ij}=x_{ji}e_ix_{ij}=x_{ji}x_{ij}$; so $x_{ji}x_{ij}$ is idempotent in $e_j\Lambda e_j$. Since e_j is a primitive idempotent, $x_{ji}x_{ij}$ is either zero or e_j. Because $x_{ij}x_{ji}x_{ij}=x_{ij}\neq 0$, $x_{ji}x_{ij}=e_j$.

The converse is clear. Q.E.D.

Lemma 2. *Let e_i and e_j be primitive orthogonal idempotents in Λ. Then $e_i\sim e_j$ if and only if $e_i\Lambda\cong e_j\Lambda$ (as right Λ-modules) if and only if $\Lambda e_i\cong\Lambda e_j$ (as left Λ-modules).*

Proof. Let $\sigma: e_i\Lambda \to e_j\Lambda$, $\tau: e_j\Lambda \to e_i\Lambda$ be inverse Λ-isomorphisms. Then $\sigma(e_i) = e_jx$ for x in Λ, so $\sigma(e_i^2)=\sigma(e_i)e_i=e_jxe_i$. Thus, we can take x in $e_j\Lambda e_i$. Likewise, we have $\tau(e_j)=e_iy$ for y in $e_i\Lambda e_j$. Now $e_i=\tau(\sigma(e_i))=\tau(e_jx)=\tau(e_j)x=e_iyx$, $e_j=\sigma(\tau(e_j))=e_jxy$. But $e_iy=y$, $e_jx=x$, so $yx=e_i$, $xy=e_j$. Therefore, by Lemma 1, $e_i\sim e_j$.

If $e_i\sim e_j$, then by Lemma 1, there are elements x in $e_j\Lambda e_i$, y in $e_i\Lambda e_j$ such that $yx=e_i$, $xy=e_j$. Let $\sigma(e_i)=e_jx$, define a Λ-map from $e_i\Lambda$ to $e_j\Lambda$. Let $e_j\lambda$ be in $e_j\Lambda$. Then, $\sigma(e_iy)=e_jxy\lambda=e_j\lambda$, so σ is onto. If $\sigma(e_i\lambda)=0$, then $x\lambda=0$ in Λ, so $yx\lambda=\lambda=0$. Hence σ is an isomorphism.

The proofs for left modules are similar. Q.E.D.

Now it follows from Lemma 2 that "$\sim$" is an equivalence relation.

Lemma 3. *If e_i, e_i', e_j and e_j' are idempotents in Λ such that $e_i\sim e_i'$ and $e_j\sim e_j'$, then $e_i\Lambda e_j\cong e_i'\Lambda e_j'$ (as A-modules).*

Proof. First, assume that e_i is equivalent to e_i'. Then there is an x_i in $e_i\Lambda e_i'$ and y_i in $e_i'\Lambda e_i$ such that $x_iy_i=e_i$ and $y_ix_i=e_i'$. Define $\eta: e_i\Lambda e_j \to e_i'\Lambda e_j$ by $\eta(e_ize_j) = y_ize_j$. If $y_ize_j=0$, then $x_iy_ixe_j=e_ize_j=0$. So η is one-to-one. For $e_i'we_j$ in $e_i'\Lambda e_j$, $\eta(e_ix_iwe_j)=y_ix_iwe_j=e_i'we_j$, so η is onto. Thus η is an isomorphism.

The proof for the case where e_j is equivalent to e_j' is symmetric. Q.E.D.

An A-order Λ in Σ is said to be reduced if its identity has a decomposition into primitive orthogonal idempotents $1=e_1+\cdots+e_n$, such that no distinct e_i and e_j are equivalent [6, Appendix].

For the A-order Λ in Σ, let $f_1,\ldots,f_k$ denote representatives of the distinct equivalence classes of equivalent idempotents. Let $f=f_1+\cdots+f_k$. Let $\tilde{\Lambda} = f\Lambda f$; then $\tilde{\Lambda}$ is reduced in $f\Sigma f$.

Lemma 4. *The map $I\to\tilde{I}=fIf=I\cap\tilde{\Lambda}$ of two-sided Λ-modules to two-sided $\tilde{\Lambda}$-modules is one-to-one and preserves products if they are defined. Further, the map preserves sums and intersections.*

Proof. Let I and J denote two-sided Λ-modules. The proofs that $(fIf)\cap(fJf) =f(I\cap J)f$ and that $fIf+fJf=f(I+J)f$ are ordinary set inclusion arguments and are omitted.

We note that $\Lambda f\Lambda=\Lambda$. For, if e_i is one of the idempotents of Λ, there is an f_j such that e_i is equivalent to f_j; i.e., $e_i\Lambda e_i=(e_i\Lambda f_j)(f_j\Lambda e_i)$. Hence, $e_i=e_ixf_jye_i$ for elements x and y in Λ. Thus, in particular, e_i is in $\Lambda f\Lambda$, so $1=\sum_{i=1}^{r} e_i$ is in $\Lambda f\Lambda$. Therefore, $\Lambda f\Lambda=\Lambda$.

Since $fIJf=fI\Lambda Jf=fI\Lambda f\Lambda Jf=(fIf)(fJf)$, it is clear that products are preserved when defined.

Finally, if $fIf=fJf$, then $\Lambda f\Lambda I\Lambda f\Lambda=\Lambda f\Lambda J\Lambda f\Lambda$. So, $\Lambda I\Lambda=I=J=\Lambda J\Lambda$. Hence, the map $I\to\tilde{I}$ is one-to-one. Q.E.D.

We will say that a two-sided (fractional) ideal I of Λ is *invertible* if there is a two-sided Λ-module J such that $IJ=JI=\Lambda$.

3. **Index conditions in reduced orders.** In this section, let A be a complete discrete rank one valuation ring, let Λ be a reduced A-order in Σ, and let Γ be a maximal A-order containing Λ. Let $f_1, \ldots, f_k$ denote the nonequivalent primitive orthogonal idempotents of Λ, and assume they are primitive in Γ.

THEOREM 5. *Let I be an invertible two-sided ideal in reduced A-order Λ. Then I is Λ-cyclic, Λ-free, and there is a permutation σ of the set $\{1, 2, \ldots, k\}$ such that for all i and j, $i, j=1, 2, \ldots, k$,*

(a) $[f_i\Gamma f_j : f_i\Lambda f_j] = [f_{\sigma^{-1}(i)} I\Gamma f_j : f_{\sigma^{-1}(i)} If_j]$,

(b) $[f_i\Gamma f_j : f_i\Lambda f_j] = [f_i\Gamma If_{\sigma(j)} : f_i If_{\sigma(j)}]$,

(c) $(f_{\sigma^{-1}(i)} If_i)(f_i\Lambda f_j) = f_{\sigma^{-1}(i)} If_j$,

(d) $(f_j\Lambda f_i)(f_i If_{\sigma(i)}) = f_j If_{\sigma(i)}$.

Further, $[\Gamma : \Lambda] = [I\Gamma : I] = [\Gamma I : I]$.

Proof. Since I is invertible, there is a two-sided Λ-module J such that $IJ = JI = \Lambda$. Thus

$$f_i\Lambda f_j = \sum_{s=1}^{k} (f_i If_s)(f_s Jf_j).$$

In particular,

$$f_i\Lambda f_i = \sum_{s=1}^{k} (f_i If_s)(f_s Jf_i). \tag{1}$$

Since f_i is a primitive idempotent, $f_i\Lambda f_i$ has a single maximal ideal M_i, and the factor algebra $f_i\Lambda f_i/M_i$ is a division ring. Further, since $f_i\Lambda f_i$ is an A-algebra, every ideal is regular, so the Jacobson radical of $f_i\Lambda f_i$, being the intersection of all maximal right (or left) regular ideals, must be M_i [7]. Consider

$$f_i\Lambda f_i/M_i = \sum_{s=1}^{k} (f_i If_s)(f_s Jf_i)/M_i. \tag{2}$$

Then, since not all of the products $(f_i If_s)(f_s Jf_i)$ are in M_i and since $f_i\Lambda f_i/M_i$ has no nontrivial ideals, there is an $s = \sigma(i)$, σ a function of $\{1, 2, \ldots, k\}$ into itself, such that

$$f_i\Lambda f_i = (f_i If_{\sigma(i)})(f_{\sigma(i)} Jf_i) + M_i.$$

Then by [9, Theorem 4.1, p. 12],

$$(f_i If_{\sigma(i)})(f_{\sigma(i)} Jf_i) = f_i\Lambda f_i.$$

Therefore, there is an x_i in $f_i If_{\sigma(i)}$ and a y_i in $f_{\sigma(i)} Jf_i$ such that $x_i y_i = f_i$. Since $y_i x_i y_i x_i = y_i f_i x_i = y_i x_i$ is in $f_{\sigma(i)}\Lambda f_{\sigma(i)}$, and since $f_{\sigma(i)}$ is a primitive idempotent, then $y_i x_i$ is either zero or $f_{\sigma(i)}$. But $x_i y_i x_i = x_i \neq 0$, so $y_i x_i = f_{\sigma(i)}$. Such x_i and y_i can be found for all i, $i = 1, 2, \ldots, k$.

We claim that σ is a permutation. Consider

$$x_1\Lambda + \cdots + x_k\Lambda \subseteq I. \tag{3}$$

The left-hand side of (3) contains all of the x_i, so

$$(x_1\Lambda + \cdots + x_k\Lambda)J = \Lambda, \tag{4}$$

since the left-hand side of this expression contains all of the $x_i y_i$; therefore, it contains 1. Hence, multiplying equation (4) on the right by I, we have

$$I = (x_1\Lambda + \cdots + x_k\Lambda)JI,$$

or

$$I = x_1\Lambda + \cdots + x_k\Lambda. \tag{5}$$

Similarly,

$$J = \Lambda y_1 + \cdots + \Lambda y_k. \tag{6}$$

On noting that $y_j x_i = 0$ if $i \neq j$, we have that

$$\Lambda = JI = \sum_{j=1}^{k} \Lambda y_j x_j \Lambda = \sum_{j=1}^{k} \Lambda f_{\sigma(j)} \Lambda.$$

Now, if f_α is a member of the set $\{f_1, \ldots, f_k\}$, consider the equation

$$f_\alpha \Lambda f_\alpha = \sum_{j=1}^{k} (f_\alpha \Lambda f_{\sigma(j)})(f_{\sigma(j)} \Lambda f_\alpha).$$

As before, there must be a β in $\{1, 2, \ldots, k\}$ such that

$$f_\alpha \Lambda f_\alpha = (f_\alpha \Lambda f_{\sigma(\beta)})(f_{\sigma(\beta)} \Lambda f_\alpha). \tag{7}$$

But equation (7) implies that $f_\alpha \sim f_{\sigma(\beta)}$ which implies $\alpha = \sigma(\beta)$.

Thus, since σ is a permutation, and $x_i y_j = 0$ for $i \neq j$, $(x_1 + \cdots + x_k)\Lambda J$ has all of the elements $x_i y_i$, and so it has 1. Hence,

$$(x_1 + \cdots + x_k)\Lambda J = \Lambda. \tag{8}$$

Multiplying equation (8) on the right by I, we have

$$I = (x_1 + \cdots + x_k)\Lambda JI = (x_1 + \cdots + x_k)\Lambda,$$

so I is cyclic as a right module.

Since

$$I = \sum_{i,j=1}^{k} (x_1 + \cdots + x_k) f_i \Lambda f_j,$$

then

$$f_{\sigma^{-1}(\alpha)} I f_\beta = x_{\sigma^{-1}(\alpha)} f_\alpha \Lambda f_\beta,$$

and so

$$\begin{aligned}(f_{\sigma^{-1}(i)} I f_i)(f_i \Lambda f_j) &= x_{\sigma^{-1}(i)}(f_i \Lambda f_i)(f_i \Lambda f_j)\\ &= x_{\sigma^{-1}(i)} f_i \Lambda f_j\\ &= f_{\sigma^{-1}(i)} I f_j,\end{aligned}$$

which establishes (c).

We claim that I is isomorphic to Λ and is therefore free. Set $x = x_1 + \cdots + x_k$. Define $\eta\colon \Lambda \to I$ by $\eta(\lambda) = x$. $IJ = JI = \Lambda$, so there is a y in J such that $xy = yx = 1$;

hence x is a unit in Σ, and η is one-to-one. η is obviously onto, so it is an isomorphism.

By methods symmetric to the preceding, we have

$$I = \Lambda(x_1+\cdots+x_k) = \Lambda x, \qquad J = (y_1+\cdots+y_k)\Lambda.$$

Therefore,

$$I\Gamma = (x_1+\cdots+x_k)\Gamma = x\Gamma, \qquad \Gamma I = \Gamma(x_1+\cdots+x_k) = \Gamma x,$$
$$\Gamma J = \Gamma(y_1+\cdots+y_k), \qquad J\Gamma = (y_1+\cdots+y_k)\Gamma.$$

Since x is a unit in Σ, we have

$$[\Gamma:\Lambda] = [x\Gamma:x\Lambda] = [I\Gamma:I], \quad \text{and} \quad [\Gamma:\Lambda] = [\Gamma x:\Lambda x] = [\Gamma I:I].$$

Define maps $\theta_i: f_i\Lambda f_j \to f_{\sigma^{-1}(i)}If_j$ by $\theta_i(f_i\lambda f_j) = x_{\sigma^{-1}(i)}f_i\lambda f_j$. Since we have that $x_{\sigma^{-1}(i)}f_i\Lambda f_j = f_{\sigma^{-1}(i)}If_j$, θ_i is onto. If $x_{\sigma^{-1}(i)}f_i\lambda f_j = 0$, then $y_{\sigma^{-1}(i)}x_{\sigma^{-1}(i)}f_i\lambda f_j = f_i\lambda f_j = 0$, so the θ_i are one-to-one and are isomorphisms. The θ_i extend to isomorphisms

$$\theta_i': f_i\Gamma f_j \to f_{\sigma^{-1}(i)}I\Gamma f_j$$

by $\theta_i'(f_i\gamma f_j) = x_{\sigma^{-1}(i)}f_i\gamma f_j$. Hence, for all i and j, $i, j = 1, \ldots, k$,

$$[f_i\Gamma f_j : f_i\Lambda f_j] = [f_{\sigma^{-1}(i)}I\Gamma f_j : f_{\sigma^{-1}(i)}If_j].$$

In order to obtain the other index conditions, we proceed symmetrically. Thus we show that

$$(f_i\Lambda f_i)(f_iIf_{\sigma(i)}) = f_jIf_{\sigma(i)},$$

and show that the maps $\mu_i: f_i\Lambda f_j \to f_iIf_{\sigma(j)}$ defined by $\mu_i(f_i\lambda f_j) = f_i\lambda f_j x_{\sigma(j)}$ are isomorphisms which extend to isomorphisms $\mu_i': f_i\Gamma f_j \to f_i\Gamma If_{\sigma(j)}$. From this, the index condition (b) is immediate. Q.E.D.

In Example 14 we shall see that $[\tilde{\Gamma}:\tilde{\Lambda}] = [\tilde{\Gamma}\tilde{I}:\tilde{I}] = [\tilde{I}\tilde{\Gamma}:\tilde{I}]$ may be true in Λ but that $[\Gamma:\Lambda] \neq [\Gamma I:I]$ and $[\Gamma:\Lambda] \neq [I\Gamma:I]$ in Λ. Further, this example will show that if Λ is not reduced, then an invertible Λ-ideal I may not be a cyclic Λ-ideal. Thus the condition that Λ is reduced is essential in Theorem 5.

Now we will prove the converse to Theorem 5. Define

$$(\Lambda:I)_r = \{x \in \Sigma : Ix \subseteq \Lambda\}, \quad (\Lambda:I)_l = \{x \in \Sigma : xI \subseteq \Lambda\}.$$

THEOREM 6. *Let I be a full two-sided ideal of the reduced A-order Λ in Σ. Let $f_1, \ldots, f_k$ be the nonequivalent primitive orthogonal idempotents of Λ. If there is a permutation τ of the set $\{1, 2, \ldots, k\}$ such that*

(a) $[f_i\Gamma f_i : f_i\Lambda f_i] = [f_{\tau(i)}I\Gamma f_i : f_{\tau(i)}If_i]$,

(b) $[f_{\tau(i)}\Gamma f_{\tau(i)} : f_{\tau(i)}\Lambda f_{\tau(i)}] = [f_{\tau(i)}\Gamma If_i : f_{\tau(i)}If_i]$,

(c) $(f_{\tau(i)}If_i)(f_i\Lambda f_j) = f_{\tau(i)}If_j$,

(d) $(f_j\Lambda f_i)(f_iIf_{\tau^{-1}(i)}) = f_jIf_{\tau^{-1}(i)}$,

for all i and j, $i, j = 1, 2, \ldots, k$, then I is invertible by the two-sided Λ-module $J = (\Lambda:I)_r = (\Lambda:I)_l$.

Proof. We consider $f_{\tau(i)}If_i$ as a right $f_i\Lambda f_i$-module and $f_{\tau(i)}I\Gamma f_i$ as a right $f_i\Gamma f_i$-module.

Since A is a complete discrete rank one valuation ring and since $f_i\Gamma f_i$ has no idempotents other than f_i or 0, we can apply [14, Theorem 77.12, p. 548] and [1, Proposition 2.7, p. 8] to see that $f_{\tau(i)}I\Gamma f_i$ is a free Γ-module of rank 1. Since f_i is a primitive idempotent, $f_i\Gamma f_i$ has a single maximal ideal N_i which must be the Jacobson radical of $f_i\Gamma f_i$. Further, $f_i\Gamma f_i/N_i$ is a division algebra over $A/(\pi)$.

We claim that $f_{\tau(i)}I\Gamma f_i/f_{\tau(i)}I\Gamma f_iN_i$ is a free $f_i\Gamma f_i/N_i$-module of rank one. For, if $\bar{x}_1, \ldots, \bar{x}_r$ is a basis of $f_{\tau(i)}I\Gamma f_i/f_{\tau(i)}I\Gamma f_iN_i$ over $f_i\Gamma f_i/N_i$ with preimages $x_1, \ldots, x_r$ in $f_{\tau(i)}I\Gamma f_i$ (by the natural map $f_{\tau(i)}I\Gamma f_i \to f_{\tau(i)}I\Gamma f_i/f_{\tau(i)}I\Gamma f_iN_i$), then let B be the $f_i\Gamma f_i$-submodule generated by the set $\{x_1, \ldots, x_r\}$. Then $B+f_{\tau(i)}I\Gamma f_iN_i = f_{\tau(i)}I\Gamma f_i$. Hence, by Nakayama's Lemma, $B = f_{\tau(i)}I\Gamma f_i$, and $r=1$. Thus

$$(f_{\tau(i)}I\Gamma f_i/f_{\tau(i)}I\Gamma f_iN_i : f_i\Gamma f_i/N_i) = 1.$$

Consider the composed map

$$\theta_i : f_{\tau(i)}If_i \to f_{\tau(i)}I\Gamma f_i \to f_{\tau(i)}I\Gamma f_i/f_{\tau(i)}I\Gamma f_iN_i.$$

Let z_i be an element of $f_{\tau(i)}If_i$ which does not lie in a kernel of θ_i. The images from $f_{\tau(i)}If_i$ generate $f_{\tau(i)}I\Gamma f_i/f_{\tau(i)}I\Gamma f_iN_i$ over $f_i\Gamma f_i/N_i$, and so z_i freely generates $f_{\tau(i)}I\Gamma f_i/f_{\tau(i)}I\Gamma f_iN_i$ over $f_i\Gamma f_i/N_i$. Hence

$$z_i + f_{\tau(i)}I\Gamma f_iN_i = f_{\tau(i)}I\Gamma f_i,$$

and by Nakayama's Lemma,

$$z_if_i\Gamma f_i = f_{\tau(i)}I\Gamma f_i.$$

Therefore, we have shown that $f_{\tau(i)}I\Gamma f_i$ is freely generated over $f_i\Gamma f_i$ by an element z_i of $f_{\tau(i)}If_i$. Certainly, we can find such z_i for all i.

From the sequence

$$f_i\Gamma f_i \cong z_if_i\Gamma f_i = f_{\tau(i)}I\Gamma f_i \supseteq f_{\tau(i)}If_i \supseteq z_if_i\Lambda f_i \cong f_i\Lambda f_i$$

and by hypothesis (a), we have that

$$f_{\tau(i)}If_i = z_if_i\Lambda f_i$$

for all i, $i=1, \ldots, k$.

Set $z=z_1+\cdots+z_k$, and we will show that $z\Lambda = I$. Note that $(f_i\Lambda f_i)(f_i\Lambda f_j) = f_i\Lambda f_j$ for all i and j. Thus,

$$\begin{aligned} z\Lambda &= \sum_{i=1}^{k} z_i\Lambda = \sum_{i,j} z_i\Lambda f_j = \sum_{i,j} z_if_i\Lambda f_j = \sum_{i,j} z_i(f_i\Lambda f_i)(f_i\Lambda f_j) \\ &= \sum_{i,j} (f_{\tau(i)}If_i)(f_i\Lambda f_j) = \sum_{i,j} f_{\tau(i)}If_j = I \end{aligned}$$

(using hypothesis (c) and the fact that τ is a permutation).

Now we consider $f_{\tau(i)}If_i$ as a left $f_{\tau(i)}\Lambda f_{\tau(i)}$-module and $f_{\tau(i)}\Gamma If_i$ as a left $f_{\tau(i)}\Gamma f_{\tau(i)}$-module. Hence, by a proof symmetric to the above, we can pick an element z'_i in $f_{\tau(i)}If_i$ which does not lie in the kernel of the composed map

$$\theta_i^1 : f_{\tau(i)}If_i \to f_{\tau(i)}\Gamma If_i \to f_{\tau(i)}\Gamma If_i/N_{\tau(i)}f_{\tau(i)}\Gamma If_i.$$

We claim that z_i can be chosen to be z_i'. Certainly, this is true if $f_{\tau(i)}If_i$ is not contained in the union of $f_{\tau(i)}I\Gamma f_i N_i$ and $N_{\tau(i)}f_{\tau(i)}\Gamma If_i$ (considered as subsets of Γ). Assume that $f_{\tau(i)}If_i$ is contained in this union. It is known that z_i is not in $f_{\tau(i)}I\Gamma f_i N_i$, so it must be in $N_{\tau(i)}f_{\tau(i)}\Gamma If_i$. Also, z_i' is not in $N_{\tau(i)}f_{\tau(i)}\Gamma If_i$, so it must be in $f_{\tau(i)}I\Gamma f_i N_i$. Consider z_i+z_i' in $f_{\tau(i)}If_i$. If z_i+z_i' is in $f_{\tau(i)}I\Gamma f_i N_i$, then z_i must be in $f_{\tau(i)}I\Gamma f_i N_i$, a contradiction. Assuming z_i+z_i' is in $N_{\tau(i)}f_{\tau(i)}\Gamma If_i$ implies that z_i' is in $N_{\tau(i)}f_{\tau(i)}\Gamma If_i$, again a contradiction. Therefore, $f_{\tau(i)}If_i$ is not contained in the above union.

Hence, by proofs completely symmetric to the previous ones,

$$f_{\tau(i)}If_i = f_{\tau(i)}\Lambda f_{\tau(i)}z_i$$

(using index condition (b)), and $\Lambda z = I$ (using condition (d)).

Since I is full, $z\Sigma=\Sigma$ and $\Sigma z=\Sigma$. Thus, let w be in Σ such that $zw=1=wz$. Set $J=\Lambda w\Lambda$. Then

$$JI = \Lambda w\Lambda z\Lambda = \Lambda wz\Lambda = \Lambda, \quad \text{and} \quad IJ = \Lambda z\Lambda w\Lambda = \Lambda zw\Lambda = \Lambda.$$

Thus I is invertible.

Since $J\subseteq(\Lambda:I)_r$ and $(\Lambda:I)_r=JI(\Lambda:I)_r\subseteq J\Lambda=J$, we have that $J=(\Lambda:I)_r$. Similarly, $J=(\Lambda:I)_l$. Q.E.D.

It is appropriate to note here that in the commutative case every order is reduced, so Theorems 5 and 6 generalize Fröhlich's Theorem to the noncommutative case.

THEOREM 7. *Let I be a full two-sided ideal in the reduced A-order Λ. Let $f_1,\dots,f_k$ be a set of reduced idempotents for Λ. Then I is Λ-invertible if and only if the following conditions are satisfied*:

(a) $[f_i\Gamma I^k f_i : f_i I^k f_i]=[f_i\Gamma I^{2k} f_i : f_i I^{2k} f_i]$,
(b) $[f_i I^k\Gamma f_i : f_i I^k f_i] = [f_i I^{2k}\Gamma f_i : f_i I^{2k} f_i]$,
(c) $[f_i\Lambda f_i : f_i I^{2k} f_i]=[f_i\Lambda f_i : f_i I^k f_i]^2$,
(d) $(f_i I^k f_i)(f_i\Lambda f_j)=f_i I^k f_j$,
(e) $(f_j\Lambda f_i)(f_i I^k f_i)=f_j I^k f_i$.

Proof. Assume that I is Λ-invertible. In Theorem 5, we showed that there was a permutation σ of $1,2,\dots,k$ and that there were elements x_i in $f_i If_{\sigma(i)}$ such that

$$x_{\sigma^{-1}(i)}f_i\Lambda f_j = f_{\sigma^{-1}(i)}If_j,$$

and such that for $x=x_1+\cdots+x_k$, we have $\Lambda x=x\Lambda=I$. Now,

$$x^2 = (x_1+\cdots+x_k)^2 = \sum_{i=1}^{k} x_i x_{\sigma(i)},$$

$$x^3 = (x_1+\cdots+x_k)^3 = \sum_{i=1}^{k} x_i x_{\sigma(i)} x_{\sigma^2(i)},$$

$$x^n = (x_1+\cdots+x_k)^n = \sum_{i=1}^{k} x_i x_{\sigma(i)}\cdots x_{\sigma^n(i)}.$$

So since $\sigma^k=1$, $x^k=z_1+\cdots+z_k$ with z_i an element of $f_iI^kf_i$. Set $z=z_1+\cdots+z_k$. Then $I^k=z\Lambda=\Lambda z$. Hence we have

$$I^k = (z_1+\cdots+z_k)\Lambda = (z_1f_1\Lambda+\cdots+z_kf_k\Lambda).$$

So $f_iI^kf_i=z_if_i\Lambda f_i$. Similarly, $f_iI^kf_i=f_i\Lambda f_iz_i$. We note further that

$$\begin{aligned} z_if_iI^kf_i &= z_if_i\Lambda f_if_iI^kf_i \\ &= z_if_i\Lambda f_iz_if_i\Lambda f_i \\ &= z_if_i\Lambda f_i. \end{aligned}$$

Since we have $f_iI^{2k}f_i=z_i^2f_i\Lambda f_i=f_i\Lambda f_iz_i^2$, then $f_iI^kf_i$ is Λ-isomorphic to $f_iI^{2k}f_i$ by an isomorphism which extends to an isomorphism from $f_iI^k\Gamma f_i$ to $f_iI^{2k}\Gamma f_i$. Thus the index condition (b) holds. By similar arguments, it is clear that the index condition (a) holds.

Now,

$$\begin{aligned} [f_i\Lambda f_i:f_iI^{2k}f_i] &= [f_i\Lambda f_i:f_iI^kf_i][f_iI^kf_i:f_iI^{2k}f_i] \\ &= [f_i\Lambda f_i:f_iI^kf_i][f_i\Lambda f_iz_i:f_i\Lambda f_iz_i^2] \\ &= [f_i\Lambda f_i:f_iI^kf_i][f_i\Lambda f_i:f_i\Lambda f_iz_i] \\ &= [f_i\Lambda f_i:f_iI^kf_i][f_i\Lambda f_i:f_iI^kf_i] \\ &= [f_i\Lambda f_i:f_iI^kf_i]^2. \end{aligned}$$

Hence index condition (c) is satified.

Finally,

$$\begin{aligned} (f_iI^kf_i)(f_i\Lambda f_i) &= z_if_i\Lambda f_if_i\Lambda f_i \\ &= z_if_i\Lambda f_i \\ &= z_iI^kf_i. \end{aligned}$$

So the condition (d) is true, and by similar arguments condition (e) is true.

Now we will assume the conditions (a) through (e). In Theorem 6, we saw that we could find elements z_i in $f_iI^kf_i$ which freely generate $f_iI^k\Gamma f_i$ over $f_i\Gamma f_i$ as right $f_i\Gamma f_i$-modules and which freely generate $f_i\Gamma I^kf_i$ over $f_i\Gamma f_i$ as left $f_i\Gamma f_i$-modules.

Then

$$\begin{aligned} [f_i\Lambda f_i:f_iI^{2k}f_i] &= [f_i\Lambda f_i:f_iI^kf_i][f_iI^kf_i:f_iI^{2k}f_i] \\ &= [f_i\Lambda f_i:f_iI^kf_i][f_i\Gamma I^kf_i:f_i\Gamma I^{2k}f_i] \\ &= [f_i\Lambda f_i:f_iI^kf_i][f_i\Gamma f_iz_i:f_i\Gamma f_iz_i^2] \\ &= [f_i\Lambda f_i:f_iI^kf_i][f_i\Gamma f_i:f_i\Gamma f_iz_i] \\ &= [f_i\Lambda f_i:f_iI^kf_i][f_i\Gamma f_i:f_i\Gamma I^kf_i], \end{aligned}$$

since conditions (c) hold for all i. We must have

$$[f_i\Lambda f_i:f_iI^kf_i] = [f_i\Gamma f_i:f_i\Gamma I^kf_i],$$

or

$$[f_i\Gamma f_i:f_i\Lambda f_i] = [f_i\Gamma I^kf_i:f_iI^kf_i].$$

In a similar fashion we have

$$[f_i\Gamma f_i:f_i\Lambda f_i] = [f_iI^k\Gamma f_i:f_iI^kf_i].$$

Thus in view of conditions (d) and (e), we have from Theorem 6 that I^k is Λ-invertible, i.e., I is Λ-invertible. Q.E.D.

4. **Reduction from a Dedekind domain.** In this section, we will let D be a Dedekind domain with quotient field K. Assume Σ is a finite dimensional separable K-algebra, Λ a D-order in Σ and Γ a maximal D-order in Σ containing Λ. Let P be a maximal prime ideal in D. Let $K_{(P)}$ denote the completion of K at P, and let $D_{(P)}$ be the complete discrete valuation ring in $K_{(P)}$. Let $\Sigma_{(P)}=\Sigma\otimes_D D_{(P)}$, $\Lambda_{(P)}=\Lambda\otimes_D D_{(P)}$, $I_{(P)}=I\otimes_D D_{(P)}$, etc. There are well-known canonical embeddings of $\Gamma_{(P)}$, $\Lambda_{(P)}$, $I_{(P)}$ into $\Sigma_{(P)}$ [5]. For the D-order Λ and a maximal prime ideal P of D, we will let $e_1, e_2, \ldots, e_{r_P}$ be the primitive orthogonal idempotents of $\Lambda_{(P)}$ and assume that they are primitive in $\Gamma_{(P)}$. Let $f_1, f_2, \ldots, f_{k_P}$ be representatives of the distinct equivalence classes of idempotents in $\Lambda_{(P)}$, $f=f_1+\cdots+f_{k_P}$ and let $\tilde{\Lambda}_{(P)}=f\Lambda_{(P)}f$, $\tilde{I}_{(P)}=fI_{(P)}f$ for I a two-sided Λ-module.

We will now give "global" versions of the theorems of the preceding section.

THEOREM 8. *Let I be an invertible two-sided ideal in the D-order Λ. Let $\{P\}$ denote the set of all maximal prime ideals of D. Then there are permutations σ of the sets $\{1, 2, \ldots, k_P\}$ such that for i and j and all primes P of D,*

(a) $[f_i\Gamma_{(P)}f_j:f_i\Lambda_{(P)}f_j]=[f_{\sigma^{-1}(i)}I\Gamma_{(P)}f_j:f_{\sigma^{-1}(i)}I_{(P)}f_j]$,

(b) $[f_i\Gamma_{(P)}f_j:f_i\Lambda_{(P)}f_j]=[f_i\Gamma I_{(P)}f_{\sigma(j)}:f_iI_{(P)}f_{\sigma(j)}]$,

(c) $(f_{\sigma^{-1}(i)}I_{(P)}f_i)(f_i\Lambda_{(P)}f_j)=f_{\sigma^{-1}(i)}I_{(P)}f_j$,

(d) $(f_j\Lambda_{(P)}f_i)(f_iI_{(P)}f_{\sigma(i)})=f_jI_{(P)}f_{\sigma(i)}$,

where $f_1, \ldots, f_{k_P}$ are the reduced idempotents of $\tilde{\Lambda}_{(P)}$.

Proof. Since D is a Dedekind domain, I is reflexive and $I=\bigcap_P I_{(P)}$. Also, $I_{(P)}=\Lambda_{(P)}$ for almost all P. So if each $I_{(P)}$ is invertible by $J_{(P)}$ in $\Lambda_{(P)}$, set $J=\bigcap_P J_{(P)}$, and consider IJ. $(IJ)_{(P)}=\Lambda_{(P)}$ for all P, so $IJ=\Lambda$ since $\Lambda=\bigcap_P \Lambda_{(P)}=IJ$. Certainly if I is invertible by J in Λ, then $(IJ)_{(P)}=I_{(P)}J_{(P)}=\Lambda_{(P)}$, so $I_{(P)}$ is invertible in $\Lambda_{(P)}$ for all P. Hence by [5, Theorem 2, p. 204], we need only prove the theorem for each $D_{(P)}$.

Let $I_{(P)}$ be a two-sided $\Lambda_{(P)}$-ideal which is invertible in $\Lambda_{(P)}$ by $J_{(P)}$. Setting $\tilde{I}_{(P)}=fI_{(P)}f$ and $\tilde{J}_{(P)}=fJ_{(P)}f$, we have by Lemma 4 that $\tilde{I}_{(P)}$ is a two-sided $\tilde{\Lambda}_{(P)}$-ideal which is invertible in $\tilde{\Lambda}_{(P)}$ by $\tilde{J}_{(P)}(\tilde{I}_{(P)}\tilde{J}_{(P)}=(fI_{(P)}f)(fJ_{(P)}f)=fI_{(P)}J_{(P)}f=f\Lambda_{(P)}f=\tilde{\Lambda}_{(P)})$. Since $\tilde{\Lambda}_{(P)}$ is reduced, we can apply Theorem 5 to obtain a permutation σ of the set $\{1, 2, \ldots, k_P\}$ such that the conclusions (a), (b), (c) and (d) are true. Q.E.D.

THEOREM 9. *If I is a full two-sided ideal of the D-order Λ such that for all P there is a permutation σ of the $\{1, 2, \ldots, k_P\}$ having the properties*

(a) $[f_j\Gamma_{(P_a)}f_j:f_j\Lambda_{(P)}f_j]=[f_{\sigma^{-1}(j)}I\Gamma_{(P)}f_j:f_{\sigma^{-1}(j)}I_{(P_a)}f_j]$,

(b) $[f_{\sigma^{-1}(j)}\Gamma_{(P)}f_{\sigma^{-1}(j)}:f_{\sigma^{-1}(j)}\Lambda_{(P)}f_{\sigma^{-1}(j)}]=[f_{\sigma^{-1}(j)}\Gamma I_{(P)}f_j:f_{\sigma^{-1}(j)}I_{(P)}f_j]$,

(c) $(f_{\sigma^{-1}(i)}I_{(P)}f_i)(f_i\Lambda_{(P)}f_j)=f_{\sigma^{-1}(i)}I_{(P)}f_j$,

(d) $(f_j\Lambda_{(P)}f_i)(f_iI_{(P)}f_{\sigma(i)})=(f_jI_{(P)}f_{\sigma(i)})$,

for all i and j, where $f_1,\ldots,f_{k_P}$ denote the reduced idempotents of $\tilde{\Lambda}_{(P)}$. Then I is invertible in Λ.

Proof. Just as in the proof of Theorem 8, to prove that I is invertible in Λ, it is sufficient to prove that for all P, $I_{(P)}$ is invertible in $\Lambda_{(P)}$. Further, in Theorem 8, we showed that if $I_{(P)}$ was $\Lambda_{(P)}$-invertible, then $\tilde{I}_{(P)}$ was $\tilde{\Lambda}_{(P)}$-invertible. Note that if $\tilde{I}_{(P)}$ is $\tilde{\Lambda}_{(P)}$-invertible by $\tilde{J}_{(P)}$, then

$$\begin{aligned}\tilde{\Lambda}_{(P)} &= \tilde{I}_{(P)}\tilde{J}_{(P)}\\ &= (fI_{(P)}f)(fJ_{(P)}f)\\ &= fI_{(P)}fJ_{(P)}f.\end{aligned}$$

Thus,

$$\begin{aligned}\tilde{\Lambda}_{(P)} &= fI_{(P)}\Lambda_{(P)}f\Lambda_{(P)}J_{(P)}f\\ &= fI_{(P)}\Lambda_{(P)}J_{(P)}f\\ &= fI_{(P)}J_{(P)}f.\end{aligned}$$

So $I_{(P)}J_{(P)}$ maps to $\tilde{\Lambda}_{(P)}$ in the map of $\Lambda_{(P)}$-modules defined in Lemma 4. Hence $I_{(P)}J_{(P)}=\Lambda_{(P)}$. Thus, $I_{(P)}$ is $\Lambda_{(P)}$-invertible if and only if $\tilde{I}_{(P)}$ is $\tilde{\Lambda}_{(P)}$-invertible. Therefore, it suffices to prove the result for reduced orders over complete discrete valuation rings, and by Theorem 6, the proof is completed. Q.E.D.

Finally we give the "global version" of Theorem 7.

THEOREM 10. *Let I be a full two-sided ideal in the D-order Λ. Let P denote a maximal prime in D and let $f_1,\ldots,f_{k_P}$ then I is Λ-invertible if and only if the following conditions hold for every maximal prime P and for all i and j:*

(a) $[f_i\Gamma I_{(P)}^{k_P}f_i:f_iI_{(P)}^{k_P}f_i]=[f_i\Gamma I_{(P)}^{2k_P}f_i:f_iI_{(P)}^{2k_P}f_i]$,

(b) $[f_iI^{k_P}\Gamma_{(P)}f_i:f_iI_{(P)}^{k_P}f_i]=[f_iI^{2k_P}\Gamma_{(P)}f_i:f_iI_{(P)}^{2k_P}f_i]$,

(c) $[f_i\Lambda_{(P)}f_i:f_iI_{(P)}^{2k_P}f_i]=[f_i\Lambda_{(P)}f_i:f_iI_{(P)}^{k_P}f_i]^2$,

(d) $(f_iI_{(P)}^{k_P}f_i)(f_i\Lambda_{(P)}f_j)=(f_iI_{(P)}^{k_P}f_j)$,

(e) $(f_j\Lambda_{(P)}f_i)(f_iI_{(P)}^{k_P}f_i)=(f_jI_{(P)}^{k_P}f_i)$.

Proof. It was proved in Theorems 8 and 9 that I is Λ-invertible if and only if $I_{(P)}$ is $\Lambda_{(P)}$-invertible for every maximal prime P in D, and $I_{(P)}$ is $\Lambda_{(P)}$-invertible if and only if $\tilde{I}_{(P)}$ is $\tilde{\Lambda}_{(P)}$-invertible. Hence this result follows from Theorem 7. Q.E.D.

5. **Examples.** This section will give examples and counterexamples relevant to the preceding sections. Let A be a discrete valuation ring with quotient field K and prime element π. Let Σ_n be the K-algebra of $n\times n$ matrices. Let $\Lambda=[\pi^{r_{ij}}A]$ denote an A-order in Σ. This notation means that the (i,j) positions of Λ has elements in the form $\pi^{r_{ij}}a$ with r_{ij} an integer and a any element of A.

We note the following lemma.

LEMMA 11. *Let* $\Lambda=[\pi^{r_{ij}}A]$, $I=[\pi^{s_{ij}}A]$ *and* $(\Lambda:I)=\{x : Ix\subseteq\Lambda\}=[\pi^{m_{ij}}A]$. *Then I is left* Λ*-projective if and only if for every P*, $P=1,\dots,n$, *there is a* k_P, $1\leqq k_P\leqq n$ *such that* $m_{P,k_P}=-s_{k_PP}$.

The lemma is actually a restatement in the language of matrices of a classical result of Cartan and Eilenberg [3, p. 132].

EXAMPLE 12. *In* Σ_2, *the fact that a full left ideal is* Λ*-projective does not necessarily imply that* $[\Gamma I:I]=[\Gamma:\Lambda]$. Note that this example shows that one part of Fröhlich's Theorem does not extend to all finite dimensional central simple algebras.

Set $\Gamma=\Lambda_2$, and let Λ be the A-order contained in Γ of the form

$$\Lambda=\begin{bmatrix} A & \pi^2 A \\ \pi A & A \end{bmatrix}.$$

Let I be the left ideal of the form

$$I=\begin{bmatrix} \pi^4 A & \pi^3 A \\ \pi^2 A & \pi A \end{bmatrix}.$$

Then ΓI has the form

$$\Gamma I=\begin{bmatrix} \pi^2 A & \pi A \\ \pi^2 A & \pi A \end{bmatrix}.$$

Then $[\Gamma:\Lambda]=\pi^3 A$ and $[\Gamma I:I]=\pi^4 A$, and $[\Gamma I:I]\neq[\Gamma:\Lambda]$. Using Lemma 11, it is easy to check that I is left Λ-projective.

EXAMPLE 13. *In* Σ_2, *the fact that* $[\Gamma I:I]=[\Gamma:\Lambda]$ *does not imply that a full left ideal is* Λ*-projective*. Therefore, this example along with Example 11 shows that Fröhlich's Theorem has no direct extension to central simple algebras.

Set $\Gamma=\Lambda_2$ and let Λ be as in Example 11. Let I be the left Λ-ideal of the form

$$\begin{bmatrix} \pi^3 A & \pi^3 A \\ \pi^2 A & \pi A \end{bmatrix}.$$

Then ΓI is of the form

$$\begin{bmatrix} \pi^2 A & \pi A \\ \pi^2 A & \pi A \end{bmatrix}.$$

Then $[\Gamma:\Lambda]=\pi^3 A=[\Gamma I:I]$. It is easy to check that I is not Λ-projective by using Lemma 11.

One of the conclusions of Theorem 5 is that $[\Gamma:\Lambda]=[I\Gamma:I]=[\Gamma I:I]$ for I an invertible two-sided Λ-ideal. Therefore it is reasonable to ask if this condition is strong enough to imply invertibility. The following example shows that the condition is not strong enough even if I is assumed to be both right and left Λ-projective.

EXAMPLE 14. *In* Σ_2 *there is a two-sided* Λ*-deal which is right and left projective and such that* $[I\Gamma:I]=[\Gamma I:I]=[\Gamma:\Lambda]$ *but is not invertible in* Λ.

Let

$$\Gamma = \Lambda_2, \quad \Lambda = \begin{bmatrix} A & \pi A \\ \pi A & A \end{bmatrix},$$

and

$$I = \begin{bmatrix} \pi^2 A & \pi A \\ \pi A & A \end{bmatrix}.$$

Then I is a two-sided Λ-ideal and

$$[\Gamma I : I] = [I\Gamma : I] = [\Gamma : \Lambda] = \pi^2 A.$$

It is seen by Lemma 11 and its symmetric counterpart for right ideals that I is both right and left Λ-projective. I is not Λ-invertible since it is not cyclic and Λ is reduced.

EXAMPLE 15. This example shows that if Λ is not a reduced A-order, then *an invertible ideal I may not be cyclic*. Further, $[\Gamma I : I] \neq [\Gamma : \Lambda]$ and $[I\Gamma : I] \neq [\Gamma : \Lambda]$, but $[\tilde{I}\tilde{\Gamma} : \tilde{I}] = [\tilde{\Gamma} : \tilde{\Lambda}] = [\tilde{\Gamma}\tilde{I} : \tilde{I}]$. So, in particular, the condition that Λ be a reduced A-order is essential in Theorem 5.

Let

$$\Lambda = \begin{bmatrix} A & A & A \\ \pi A & A & \pi A \\ A & A & A \end{bmatrix}$$

and

$$I = \begin{bmatrix} \pi A & A & \pi A \\ \pi A & \pi A & \pi A \\ \pi A & A & \pi A \end{bmatrix}.$$

Λ is not reduced since $e_{11} \sim e_{33}$. I is Λ-invertible by

$$J = \begin{bmatrix} A & \pi^{-1} A & A \\ A & A & A \\ A & \pi^{-1} A & A \end{bmatrix}.$$

If $\Gamma = \Lambda_3$, then

$$[\Gamma : \Lambda] = \pi^2 A, \qquad [\Gamma I : I] = \pi^6 A, \qquad [I\Gamma : I] = \pi^3 A.$$

To see that I is not cyclic, assume the contrary. On noting that $I^2 = \pi\Lambda$, we see that there are elements a, b in Λ such that $x^2 = \pi a$, $xbx = \pi$. Then $(\det a)(\det b) = 1$, and $(\det x)^2 = \pi^3(\det b)$. Let $\det x = \pi^s$. Then $2s = 3$, a contradiction.

Now set $f_1 = e_{11}$, $f_2 = e_{22}$, and $f = f_1 + f_2$. Then

$$\tilde{\Lambda} = f\Lambda f = \begin{bmatrix} A & A \\ \pi A & A \end{bmatrix}, \quad \tilde{I} = fIf = \begin{bmatrix} \pi A & A \\ \pi A & \pi A \end{bmatrix}.$$

Set

$$x = \begin{bmatrix} 0 & 1 \\ \pi & 0 \end{bmatrix}.$$

Then, $\tilde{I}=\tilde{\Lambda}\tilde{x}=\tilde{x}\tilde{\Lambda}$, and $\Lambda=\Lambda_2$, so

$$[\tilde{\Gamma}:\tilde{\Lambda}] = [\tilde{\Gamma}\tilde{I}:\tilde{I}] = [\tilde{I}\tilde{\Gamma}:\tilde{I}] = \pi A.$$

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