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ISOTOPISMS OF SEMIGROUPS OF FUNCTIONS

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1. **Introduction.** Let G and H be two groupoids. An isotopism [1, p. 57] from G onto H is an ordered triple (β, γ, δ) of bijections from G onto H such that $\beta(ab)=\gamma(a)\delta(b)$ for all $a, b \in G$. In this paper, we determine all isotopisms between certain semigroups of functions which were referred to in [2] as α-semigroups. This result is then applied to semigroups of continuous functions, semigroups of closed functions and semigroups of connected functions. For example, let X and Y be S^*-spaces [3] and let $S(X)$ and $S(Y)$ denote the semigroups, under composition, of all continuous selfmaps of X and Y respectively. Then for each isotopism (β, γ, δ) of $S(X)$ onto $S(Y)$, there exist three unique homeomorphisms h, j and k from X onto Y such that $\beta(f)=h\circ f\circ k^{-1}$, $\gamma(f)=h\circ f\circ j^{-1}$ and $\delta(f)=j\circ f\circ k^{-1}$ for each f in $S(X)$. Similar results are obtained for semigroups of closed functions and also for semigroups of connected functions.

2. **Isotopisms between α-semigroups.**

Definition (2.1). An α-semigroup is any semigroup, under composition, of selfmaps of a set which contains the identity function and all constant functions.

α-semigroups of functions occur quite naturally. For example, if X is any topological space, the semigroup of all continuous selfmaps of X and the semigroup of all connected selfmaps (the image of each connected set is connected) of X are α-semigroups and if X is a T_1 space, the semigroup of all closed selfmaps (the image of each closed set is closed) is an α-semigroup.

Theorem (2.2). *Let $\alpha(X)$ and $\alpha(Y)$ denote α-semigroups of functions on the sets X and Y respectively and suppose (β, γ, δ) is any isotopism from $\alpha(X)$ onto $\alpha(Y)$. Then there exist three unique bijections h, k and j from X onto Y such that the following conditions hold for each f in $\alpha(X)$:*

(2.2.1) $$\beta(f) = h\circ f\circ k^{-1},$$

(2.2.2) $$\gamma(f) = h\circ f\circ j^{-1},$$

(2.2.3) $$\delta(f) = j\circ f\circ k^{-1}.$$

We first prove a sequence of lemmas which will assist in proving the theorem. In what follows, $\langle x\rangle$ will denote the constant function in $\alpha(X)$ which maps each point of X into x and i_X will denote the identity of $\alpha(X)$. It is not difficult to show that the family of all constant functions of $\alpha(X)$ is precisely the set of left zeros of $\alpha(X)$. This fact will be used repeatedly.

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LEMMA (2.3). *For each left zero* $\langle x\rangle$ *in* $\alpha(X)$, $\beta\langle x\rangle=\gamma\langle x\rangle$.

Proof. Since δ is bijective, there is some $f\in\alpha(X)$ such that $\delta(f)=i_Y$. Then

$$\beta\langle x\rangle = \beta(\langle x\rangle\circ f) = \gamma\langle x\rangle\circ\delta(f) = \gamma\langle x\rangle\circ i_Y = \gamma\langle x\rangle.$$

LEMMA (2.4). $\beta\langle x\rangle$ *(and consequently,* $\gamma\langle x\rangle$*) is a left zero of* $\alpha(Y)$ *for each left zero* $\langle x\rangle$ *in* $\alpha(X)$.

Proof. For any $g\in\alpha(Y)$, there exists an $f\in\alpha(X)$ such that $\delta(f)=g$. Then we can use the previous lemma to get

$$\beta\langle x\rangle\circ g = \gamma\langle x\rangle\circ\delta(f) = \beta(\langle x\rangle\circ f) = \beta\langle x\rangle.$$

LEMMA (2.5). β *(and hence* γ*) maps the left zeros of* $\alpha(X)$ *bijectively onto the left zeros of* $\alpha(Y)$.

Proof. The fact that β is injective follows from the definition of an isotopism. Furthermore, it is immediate that $(\beta^{-1},\gamma^{-1},\delta^{-1})$ is an isotopism from $\alpha(Y)$ onto $\alpha(X)$. Thus, Lemma (2.4) implies that for any $\langle y\rangle\in\alpha(Y)$, $\beta^{-1}\langle y\rangle$ is a left zero of $\alpha(X)$. Since $\beta(\beta^{-1}\langle y\rangle)=\langle y\rangle$, the conclusion follows.

LEMMA (2.6). δ *also maps the left zeros of* $\alpha(X)$ *bijectively onto the left zeros of* $\alpha(Y)$.

Proof. There exists a function f in $\alpha(X)$ such that $\gamma(f)=i_Y$. Then for any $\langle x\rangle\in\alpha(X)$,

$$\delta\langle x\rangle = i_Y\circ\delta\langle x\rangle = \gamma(f)\circ\delta\langle x\rangle = \beta\langle f(x)\rangle$$

which, by Lemma (2.4), is a left zero of $\alpha(Y)$. It is immediate that δ is injective and since $(\beta^{-1},\gamma^{-1},\delta^{-1})$ is an isotopism from $\alpha(Y)$ onto $\alpha(X)$, it follows from the first portion of this proof that $\delta^{-1}\langle y\rangle$ is a left zero of $\alpha(X)$ for each $\langle y\rangle\in\alpha(Y)$. Thus, the desired conclusion follows since $\delta(\delta^{-1}\langle y\rangle)=\langle y\rangle$.

Proof of Theorem (2.2). Let any $x\in X$ be given. By Lemma (2.4), there exists a $y\in Y$ such that $\beta\langle x\rangle=\gamma\langle x\rangle=\langle y\rangle$. We define a mapping h from X into Y by $h(x)=y$ and we note that

$$\beta\langle x\rangle = \gamma\langle x\rangle = \langle h(x)\rangle \tag{2.2.4}$$

for each $x\in X$. Furthermore, it follows from Lemma (2.5) that h maps X bijectively onto Y.

From Lemma (2.6), we conclude the existence of a $z\in Y$ such that $\delta\langle x\rangle=\langle z\rangle$. We define $j(x)=z$ and we note that

$$\delta\langle x\rangle = \langle j(x)\rangle \tag{2.2.5}$$

for each $x\in X$. It also follows from Lemma (2.6) that j maps X bijectively onto Y.

Finally, we define a function t from Y into X. Let any $y\in Y$ be given. Then $\delta(i_X)\circ\langle y\rangle$ is a left zero of $\alpha(Y)$ (the set of left zeros of $\alpha(Y)$ is the minimal two-sided ideal of $\alpha(Y)$) and, by Lemma (2.6), $\delta^{-1}(\delta(i_X)\circ\langle y\rangle)$ is a left zero of $\alpha(X)$

and consequently, is equal to $\langle x\rangle$ for some $x \in X$. We define $t(y)=x$ and note that $\langle t(y)\rangle=\delta^{-1}(\delta(i_X)\circ\langle y\rangle)$ or, equivalently

$$\delta\langle t(y)\rangle = \delta(i_X)\circ\langle y\rangle \tag{2.2.6}$$

for each $y \in Y$. We defer for the moment the verification that t maps Y bijectively onto X.

Now let f be any function in $\alpha(X)$ and let y be an arbitrary element from Y. Using both (2.2.4) and (2.2.6), we get

$$\begin{aligned}(h\circ f\circ t)(y) &= \langle h(f(t(y)))\rangle(y) = \beta\langle f(t(y))\rangle(y)\\ &= \beta(f\circ\langle t(y)\rangle)(y) = (\gamma(f)\circ\delta\langle t(y)\rangle)(y)\\ &= (\gamma(f)\circ\delta(i_X)\circ\langle y\rangle)(y) = (\beta(f\circ i_X)\circ\langle y\rangle)(y)\\ &= (\beta(f)\circ\langle y\rangle)(y) = \beta(f)(y).\end{aligned}$$

Therefore,

$$\beta(f) = h\circ f\circ t \tag{2.2.7}$$

for each f in $\alpha(X)$. We use (2.2.4) and (2.2.5) to get

$$\begin{aligned}(h\circ f\circ j^{-1})(y) &= \langle h(f(j^{-1}(y)))\rangle(y) = \beta\langle f(j^{-1}(y))\rangle(y)\\ &= \beta(f\circ\langle j^{-1}(y)\rangle)(y) = (\gamma(f)\circ\delta\langle j^{-1}(y)\rangle)(y)\\ &= (\gamma(f)\circ\langle y\rangle)(y) = \gamma(f)(y).\end{aligned}$$

Hence, we have

$$\gamma(f) = h\circ f\circ j^{-1} \tag{2.2.8}$$

for each f in $\alpha(X)$.

Now choose g in $\alpha(X)$ such that $\gamma(g)=i_Y$ and use (2.2.7) and (2.2.8) to get

$$\begin{aligned}\delta(f) &= i_Y\circ\delta(f) = \gamma(g)\circ\delta(f) = \beta(g\circ f)\\ &= h\circ g\circ f\circ t = (h\circ g\circ j^{-1})\circ(j\circ f\circ t)\\ &= \gamma(g)\circ(j\circ f\circ t) = i_Y\circ(j\circ f\circ t) = j\circ f\circ t.\end{aligned}$$

Therefore,

$$\delta(f) = j\circ f\circ t \tag{2.2.9}$$

for each f in $\alpha(X)$.

Next, we show that t is bijective. Among other things, we have shown thus far that if (β, γ, δ) is an isotopism from $\alpha(X)$ onto $\alpha(Y)$ then there exists a bijection j from X onto Y and a function t from Y into X such that the third element δ of the triple (β, γ, δ) has the form (2.2.9). Applying this to the isotopism $(\beta^{-1}, \gamma^{-1}, \delta^{-1})$, we conclude the existence of a bijection v from Y onto X and a function w from X into Y such that

$$\delta^{-1}(g) = v\circ g\circ w \tag{2.2.10}$$

for each $g \in \alpha(Y)$. Then (2.2.9) and (2.2.10) together yield $i_X = \delta^{-1}(\delta(i_X)) = v \circ j \circ t \circ w$ and $i_Y = \delta(\delta^{-1}(i_Y)) = j \circ v \circ w \circ t$. Therefore $t \circ w = (v \circ j)^{-1}$ and $w \circ t = (j \circ v)^{-1}$ and since both v and j are bijections from Y onto X and X onto Y respectively, it follows that $t \circ w$ is a bijection from X onto X and $w \circ t$ is a bijection from Y onto Y. Consequently, t must be a bijection from Y onto X. We can now define $k = t^{-1}$ and conditions (2.2.1), (2.2.2) and (2.2.3) follow from (2.2.7), (2.2.8) and (2.2.9) respectively.

To complete the proof, we need only show that the bijections h, j and k are unique. Suppose conditions (2.2.1), (2.2.2) and (2.2.3) hold while at the same time

(2.2.11) $$\beta(f) = u \circ f \circ w^{-1},$$

(2.2.12) $$\gamma(f) = u \circ f \circ v^{-1},$$

(2.2.13) $$\delta(f) = v \circ f \circ w^{-1}$$

for three bijections u, v, and w which map X onto Y. Then for any $x \in X$ (2.2.1) and (2.2.11) imply that $\langle h(x)\rangle = \beta\langle x\rangle = \langle u(x)\rangle$ and (2.2.3) and (2.2.12) imply that

$$\langle j(x)\rangle = \delta\langle x\rangle = \langle v(x)\rangle$$

from which it readily follows that $h = u$ and $j = v$. Using this information together with (2.2.1) and (2.2.11), we get $u \circ k^{-1} = h \circ k^{-1} = \beta(i_X) = u \circ w^{-1}$ which implies that $k = w$. This concludes the proof of Theorem (2.1).

3. **Isotopisms of semigroups of continuous functions.** We recall a definition from [3].

DEFINITION (3.1). A topological space X is an S^*-space if it is T_1 and for each closed subset F of X and each point p in $X - F$, there exists a point $y \in X$ and a continuous selfmap f of X such that $f(x) = y$ for x in F and $f(p) \neq y$.

The class of all S^*-spaces is rather extensive. For example, [3, p. 296, Theorems 2 and 3] every 0-dimensional Hausdorff space and every completely regular Hausdorff space which contains an arc is an S^*-space. For two such spaces X and Y, we can apply Theorem (2.1) to get the following result:

THEOREM (3.2). *Let $S(X)$ and $S(Y)$ denote the semigroups under composition, of all continuous selfmaps on the S^*-spaces X and Y respectively and suppose (β, γ, δ) is an isotopism from $S(X)$ onto $S(Y)$. Then there exist three unique homeomorphisms h, j and k from X onto Y such that for each f in $S(X)$, the following conditions hold*:

(3.2.1) $$\beta(f) = h \circ f \circ k^{-1},$$

(3.2.2) $$\gamma(f) = h \circ f \circ j^{-1},$$

(3.2.3) $$\delta(f) = j \circ f \circ k^{-1}.$$

Proof. It follows immediately that there exist three unique bijections h, j and k

mapping X onto Y such that conditions (3.2.1), (3.2.2) and (3.2.3) hold. Our task here is to show that they are, in fact, homeomorphisms.

First, we note that $\gamma(i_X) \circ \delta(\beta^{-1}(i_Y)) = \beta(i_X \circ \beta^{-1}(i_Y)) = i_Y$ and also that

$$\delta(\beta^{-1}(i_Y)) \circ \gamma(i_X) = j \circ (h^{-1} \circ k) \circ k^{-1} \circ (h \circ j^{-1}) = i_Y.$$

That is, $\gamma(i_X)$ is a unit of the semigroup $S(Y)$ or, equivalently, $\gamma(i_X)$ is a homeomorphism mapping Y onto Y.

Now, it is a straightforward matter to show that for any $p \in X$ and f in $S(X)$, $h[f^{-1}(p)] = \gamma(i_X)[(\gamma(f))^{-1}(h(p))]$ which is a closed subset of Y since $\gamma(i_X)$ is a homeomorphism. One readily shows that $\{f^{-1}(p) : p \in X, f \in S(X)\}$ is a basis for the closed subsets of X since X is an S^*-space. Consequently, the image under h of each closed subset of X is a closed subset of Y. Since everything we have done thus far applies equally well to the isotopism $(\beta^{-1}, \gamma^{-1}, \delta^{-1})$ we may conclude also that h^{-1} takes closed subsets of Y into closed subsets of X. Thus, h is a homeomorphism.

Similar techniques yield the fact that j is also a homeomorphism. First one shows that $j[f^{-1}(p)] = (\gamma(f))^{-1}(h(p))$. This, coupled with the fact that X is an S^*-space allows us to conclude that j is a closed function. If one uses γ^{-1} in place of γ, one shows that j^{-1} is also a closed function and hence that j is a homeomorphism.

Finally, one shows that $k^{-1}[K] = h^{-1}[\beta(i_X)[K]]$ for each closed subset K of Y. Since $\beta(i_X)$ is a homeomorphism from Y onto Y and h is a homeomorphism from X onto Y, it follows that $k^{-1}[K]$ is a closed subset of X. For similar reasons, $k[H]$ is a closed subset of Y for each closed subset H of X. Thus, k is a homeomorphism and the proof is complete.

By taking $\beta = \gamma = \delta$ in the previous result we immediately get the following corollary which is Theorem 1 of [3].

COROLLARY (3.3). *Let X and Y be S^*-spaces. Then for each isomorphism β from $S(X)$ onto $S(Y)$, there exists a unique homeomorphism h from X onto Y such that $\beta(f) = h \circ f \circ h^{-1}$ for each f in $S(X)$.*

4. **Isotopisms of semigroups of selfmaps on $\mathfrak{R}$-spaces.** In this section, we investigate isotopisms of semigroups which include semigroups of closed functions and semigroups of connected functions as special cases. First we recall some definitions and results from [4]. In what follows, $\mathscr{P}(X)$ will denote the family of all subsets of a nonempty set X.

DEFINITION (4.1). Let $\mathscr{F}$ be any subfamily of $\mathscr{P}(X)$. A selfmap f of X is said to be $\mathscr{F}$-invariant if $f[H] \in \mathscr{F}$ for each $H \in \mathscr{F}$.

DEFINITION (4.2). A subfamily $\mathscr{F}$ of $\mathscr{P}(X)$ is an $\mathfrak{R}$-family if it contains X and all singletons of X and for each set H in $\mathscr{F}$, there exists an $\mathscr{F}$-invariant selfmap f on X such that $f[X] = H$.

For a discussion of $\mathfrak{R}$-families, see §2 in [4, pp. 526–530]. We quote without proof several results from these pages.

THEOREM (4.3). *Let X be a nonempty set and let $\mathscr{F}$ be a family of subsets of X which satisfies the following conditions*:

(4.3.1) $\mathscr{F}$ *contains X and all singletons of X*,

(4.3.2) $\mathscr{F}$ *is closed under finite intersections*,

(4.3.3) *if $H \in \mathscr{F}$, then $H \cup \{p\} \in \mathscr{F}$ for each $p \in X$.*

Then $\mathscr{F}$ is an $\mathfrak{R}$-family.

From this latter theorem, one immediately gets

COROLLARY (4.4). *The family of all closed subsets of a T_1 topological space is an $\mathfrak{R}$-family.*

THEOREM (4.5). *Any family of connected subsets of a completely regular Hausdorff space with cardinality c (the cardinality of the continuum) together with all singletons and the space itself is an $\mathfrak{R}$-family.*

Actually, this latter result was stated with less generality in [4] as Theorem (2.8). However, the proof given there can easily be modified to produce the stronger result stated here.

DEFINITION (4.6). An $\mathfrak{R}$-space is a pair $(X, \mathscr{F}_X)$ where X is a nonempty set and $\mathscr{F}_X$ is an $\mathfrak{R}$-family of subsets of X.

DEFINITION (4.7). Let $(X, \mathscr{F}_X)$ and $(Y, \mathscr{F}_Y)$ be two $\mathfrak{R}$-spaces. A function f from X into Y is an $\mathfrak{R}$-function if $f[H] \in \mathscr{F}_Y$ for each H in $\mathscr{F}_X$. If f is a bijection from X onto Y and both f and f^{-1} are $\mathfrak{R}$-functions then f is referred to as an $\mathfrak{R}$-bijection.

DEFINITION (4.8). Let $(X, \mathscr{F}_X)$ be any $\mathfrak{R}$-space. The semigroup, under composition, of all $\mathfrak{R}$-functions mapping X into X will be denoted by $\mathfrak{R}(X, \mathscr{F}_X)$.

Now we are in a position to determine all isotopisms between semigroups of the form $\mathfrak{R}(X, \mathscr{F}_X)$.

THEOREM (4.9). *Let $(X, \mathscr{F}_X)$ and $(Y, \mathscr{F}_Y)$ be $\mathfrak{R}$-spaces and let (β, γ, δ) be an isotopism from $\mathfrak{R}(X, \mathscr{F}_X)$ onto $\mathfrak{R}(Y, \mathscr{F}_Y)$. Then there exist three unique $\mathfrak{R}$-bijections h, j and k from X onto Y such that the following conditions hold for each f in $\mathfrak{R}(X)$*:

(4.9.1) $$\beta(f) = h \circ f \circ k^{-1},$$

(4.9.2) $$\gamma(f) = h \circ f \circ j^{-1},$$

(4.9.3) $$\delta(f) = j \circ f \circ k^{-1}.$$

Proof. Theorem (2.2) guarantees us the existence of three unique bijections h, j and k which satisfy conditions (4.9.1), (4.9.2) and (4.9.3). We must show that these three functions are $\mathfrak{R}$-bijections. Suppose $H \in \mathscr{F}_X$. Then $f[X] = H$ for some $f \in \mathfrak{R}(X, \mathscr{F}_X)$ and we use (4.9.1) to get

$$h[H] = h[f[X]] = h[f[k^{-1}[Y]]] = (\beta(f))[Y].$$

Therefore, $h[H] \in \mathscr{F}_Y$. In a similar manner, one shows that $j[H]=(\delta(f))[Y]$ which implies that $j[H]$ also belongs to $\mathscr{F}_Y$. Analogous arguments for the isotropism $(\beta^{-1}, \gamma^{-1}, \delta^{-1})$ result in the conclusion that $h^{-1}[K]$ and $j^{-1}[K]$ belong to $\mathscr{F}_X$ whenever K belongs to $\mathscr{F}_Y$. Thus, both h and j are $\mathfrak{R}$-bijections. Finally, one observes that for any $H \in \mathscr{F}_X$ and $K \in \mathscr{F}_Y$, $k[H]=h[(\beta^{-1}(i_Y))[H]]$ and $k^{-1}[K] = h^{-1}[(\beta(i_X))[K]]$. Since h is an $\mathfrak{R}$-bijection, it follows that $k[H] \in \mathscr{F}_Y$ and $k^{-1}[K] \in \mathscr{F}_X$. Thus, k is also an $\mathfrak{R}$-bijection.

We recall that by a closed selfmap of a topological space X, we mean a function f with the property that $f[H]$ is closed whenever H is closed. If $f[H]$ is connected whenever H is connected, we refer to f as a connected function. Connected functions have been studied by Tanaka [6] and Pervin and Levine [5]. We investigated semigroups of connected functions in [2]. In keeping with the notation there, we denote the semigroup, under composition of all connected selfmaps of a topological space X by $T(X)$ and we denote the semigroup, under composition, of all closed selfmaps by $\Gamma(X)$. The following result is an immediate consequence of Corollary (4.4) and Theorem (4.9).

THEOREM (4.10). *Let X and Y be T_1 topological spaces and let (β, γ, δ) be an isotopism from $\Gamma(X)$ onto $\Gamma(Y)$. Then there exist three unique homeomorphisms h, j and k from X onto Y such that the following conditions hold for each f in $\Gamma(X)$:*

(4.10.1) $$\beta(f) = h \circ f \circ k^{-1},$$

(4.10.2) $$\gamma(f) = h \circ f \circ j^{-1},$$

(4.10.3) $$\delta(f) = j \circ f \circ k^{-1}.$$

A biconnected mapping from a topological space X onto a topological space Y is any bijection h with the property that both h and h^{-1} are connected mappings. Theorems (4.5) and (4.9) immediately result in

THEOREM (4.11). *Let X and Y be completely regular connected Hausdorff spaces with cardinality c and let (β, γ, δ) be an isotopism from $T(X)$ onto $T(Y)$. Then there exist three unique biconnected mappings from X onto Y, such that the following conditions hold for each f in $T(X)$:*

(4.11.1) $$\beta(f) = h \circ f \circ k^{-1},$$

(4.11.2) $$\gamma(f) = h \circ f \circ j^{-1},$$

(4.11.3) $$\delta(f) = j \circ f \circ k^{-1}.$$

It is shown in [5, p. 495, Theorem 3.10] that if X and Y are locally connected compact Hausdorff spaces, then any biconnected mapping from X onto Y is, in fact, a homeomorphism. This fact and Theorem (4.11) immediately imply

THEOREM (4.12). *Let X and Y be connected locally connected compact Hausdorff spaces with cardinality c and suppose (β, γ, δ) is an isotopism from $T(X)$ onto $T(Y)$.*

Then there exist three unique homeomorphisms h, j and k from X onto Y such that the following conditions hold for each f in $T(X)$:

$$\beta(f) = h \circ f \circ k^{-1}, \tag{4.12.1}$$

$$\gamma(f) = h \circ f \circ j^{-1}, \tag{4.12.2}$$

$$\delta(f) = j \circ f \circ k^{-1}. \tag{4.12.3}$$

In conclusion, we remark that if one takes $\beta=\gamma=\delta$ in Theorem (4.10) of this paper, one gets Theorem (2.10) of [2].

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