TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 148, March 1970

EMBEDDING AS A DOUBLE COMMUTATOR IN A TYPE I AW^*-ALGEBRA(1)

BY
HERBERT HALPERN

1. **Introduction.** The purpose of this paper is the characterization of those C^*-algebras which can be written as their own double commutator in a type I AW^*-algebra. In a previous paper [5] the present author considered the module structure induced on a C^*-algebra $\mathscr{A}$ by its center $\mathscr{Z}$ which was taken to be a von Neumann algebra. It was shown that $\mathscr{A}$ is a von Neumann algebra if and only if it could be identified with the space of all bounded module homomorphisms into $\mathscr{Z}$ on a normed $\mathscr{Z}$-module. Here, an analogue of this theorem is obtained: a C^*-algebra $\mathscr{A}$ whose center is an AW^*-algebra $\mathscr{Z}$ can be isomorphically and isometrically embedded as a double commutator in a type I AW^*-algebra with center $\mathscr{Z}$ if and only if $\mathscr{A}$ can be written as the set of all bounded module homomorphisms into $\mathscr{Z}$ on a normed $\mathscr{Z}$-module M. The topology induced on the unit sphere of $\mathscr{A}$ by pointwise convergence on M will be the weak topology on the unit sphere of $\mathscr{A}$. This result can be regarded as a generalization of Sakai's theorem relating to von Neumann algebras [12] and in a certain sense it also illustrates that the generality of such an AW^*-algebra $\mathscr{A}$ as compared to a von Neumann algebra lies in its center.

The problem of embedding an AW^*-algebra $\mathscr{A}$ in a type I AW^*-algebra so as to preserve the sums of orthogonal projections was studied by H. Widom [18]. He found that such an embedding was possible if and only if $\mathscr{A}$ possesses a complete set $\{\phi_n\}$ of positive module homomorphisms into the center $\mathscr{Z}$ which mapped 1 into 1 and were completely additive on projections. He also studied those AW^*-algebras $\mathscr{A}$ which were embedded as double commutators in type I algebras and showed that a finite AW^*-subalgebra of a type I algebra $\mathscr{B}$ is its own double commutator in $\mathscr{B}$. T. Yen also studied the problem and showed that a type II_1 AW^*-algebra with finite trace is its own double commutator in a type I algebra [19].

2. **The weak topology.** Let H be an AW^*-module [10]. For each x and y in H let $w_{x,y}$ and w_x be the functions defined on the algebra $L(H)$ of all bounded linear operators on H by $w_{x,y}(A)=(Ax, y)$ and $w_x(A)=(Ax, x)$ respectively. The weak topology on a *-subalgebra $\mathscr{A}$ of $L(H)$ is the weakest topology on $\mathscr{A}$ in

Received by the editors May 8, 1969.

(1) This work was partially supported by the National Science Foundation.

which each function $A \to \|w_{x,y}(A)\|$ $(x, y \in H)$ or equivalently in which each function $A \to \|w_x(A)\|$ $(x \in H)$ is continuous on $\mathscr{A}$.

PROPOSITION 1. *Let H be an AW^*-module over the commutative AW^*-algebra $\mathscr{Z}$. Let $\mathscr{A}$ be a *-subalgebra of $L(H)$ which contains $\mathscr{Z}$ and let $\mathscr{A}_\sim$ be the algebraic $\mathscr{Z}$-module generated by the functions $w_{x,y}$ $(x, y \in H)$ restricted to $\mathscr{A}$. Then $\mathscr{A}_\sim$ is the set of weakly continuous $\mathscr{Z}$-module homomorphisms of the $\mathscr{Z}$-module $\mathscr{A}$ into $\mathscr{Z}$.*

Proof. It is sufficient to prove that $\mathscr{A}_\sim$ contains the set of weakly continuous module homomorphisms because clearly $\mathscr{A}_\sim$ is contained in the set of weakly continuous $\mathscr{Z}$-module homomorphisms. Let f be weakly continuous. There are elements x_i $(1 \leqq i \leqq n)$ in H such that $\|f(A)\| \leqq 1$ whenever $\|w_{x_i}(A)\| \leqq 1$ for every $i = 1, 2, \ldots, n$. If $A \in L(H)$, let $|A| = (A^*A)^{1/2}$. By setting $p(A) = \sum |w_{x_i}(A)|$ for $A \in L(H)$, we define a function of $L(H)$ into $\mathscr{Z}^+$ such that $p(A+B) \leqq p(A) + p(B)$ and $p(CA) = |C|p(A)$ for every A, B in $L(H)$ and C in $\mathscr{Z}$. We have that $|f(A)| \leqq 1$ whenever $p(A) \leqq 1$. Therefore, $|f(A)| \leqq p(A)$ for every A in $\mathscr{A}$. Setting $g(A) = (f(A) + f(A)^*)/2$, we obtain a function of $\mathscr{A}$ into the set of hermitian elements $H(\mathscr{Z})$ of $\mathscr{Z}$ which is a module homomorphism when $\mathscr{A}$ is considered to be an $H(\mathscr{Z})$-module. We still have that $g(A) \leqq p(A)$ for every A in $\mathscr{A}$. There is a module homomorphism h of the $H(\mathscr{Z})$-module $L(H)$ into $H(\mathscr{Z})$ such that $h(A) = g(A)$ for every A in $\mathscr{A}$ and $h(A) \leqq p(A)$ for every A in $L(H)$ [17]. Let $k(A) = h(A) - ih(iA)$. Then k is a module homomorphism of $L(H)$ into $\mathscr{Z}$. If $A \in L(H)$ and if U is a partial isometric operator in $\mathscr{Z}$ with $U|k(A)| = k(A)$ [19, Lemma 2.1], then

$$|k(A)| = k(U^*A) \leqq p(U^*A) \leqq p(A).$$

We also have that $k(A) + k(A^*) = f(A) + f(A^*)$ for every A in $\mathscr{A}$. However, this means that $k(A) = f(A)$ for every A in $\mathscr{A}$. This proves that k is a module homomorphism of $L(H)$ into $\mathscr{Z}$ which coincides with f on $\mathscr{A}$ and which satisfies $|k(A)| \leqq p(A)$.

Now for each x_i $(1 \leqq i \leqq n)$ there is a C_i in $\mathscr{Z}^+$ and a y_i in H such that $C_i y_i = x_i$ and such that $|y_i|$ is a projection in $\mathscr{Z}$. Let E_i be the abelian projection in $L(H)$ defined by $E_i x = (x, y_i) y_i$ [10, Lemma 13]. We have that

$$k(A(1-E)) = k((1-E)A) = 0,$$

where E is the least upper bound of $E_1, E_2, \ldots, E_n$. The projection E is in the closed two-sided ideal I_a of $L(H)$ generated by the abelian projections of $L(H)$ due to the relation

$$\operatorname{lub}\{E_1, E_2\} - E_1 \sim E_2 - \operatorname{glb}\{E_1, E_2\} \qquad [8, \text{Theorem } 5.4]$$

and to the fact that $E_2 - \operatorname{glb}\{E_1, E_2\}$ is abelian. There are orthogonal projections $P_1, P_2, \ldots, P_m$ in $\mathscr{Z}$ whose sum P is the central support of E such that each algebra $EL(H)EP_i$ is either zero or homogeneous of degree i (cf. [4, Theorem 2.1]). Since

$$f(A)(1-P) = k(A(1-P)) = 0$$

for every A in $\mathscr{A}$, it is sufficient to prove that each function $P_i f$ is in $\mathscr{A}_\sim$. So we may assume that $EL(H)E$ is homogeneous of degree m. There are equivalent orthogonal abelian projections $\{F_i \mid 1 \leqq i \leqq m\}$ of sum E and partial isometric operators $\{U_{ij} \mid 1 \leqq i, j \leqq m\}$ such that

(1) $U_{ij}U_{kl} = \delta_{il}U_{kj}$;

(2) $U_{ij} = U_{ji}^*$; and

(3) $U_{ii} = F_i$ for all i, j, k, l.

Thus $f(A) = k(A) = k(EAE) = \sum \tau_{F_j}(U_{ij}A)k(U_{ji})$. Here $\tau_{F_j}(B)$ denotes the unique element in $\mathscr{Z}P$ such that $\tau_{F_j}(B)F_j = F_jBF_j$ [8, Lemma 4.7]. Let z_j be an element in H such that $F_jx = (x, z_j)z_j$ [10, Lemma 13]. Then

$$\tau_{F_j}(U_{ij}A) = (U_{ij}Az_j, z_j) = (Az_j, U_{ji}z_j).$$

This proves that $f \in \mathscr{A}_\sim$. Q.E.D.

Let M be a normed vector space which is also an algebraic module over a commutative AW^*-algebra $\mathscr{Z}$; then M is said to be a normed $\mathscr{Z}$-module if $\|Ax\| \leqq \|A\| \|x\|$ for every $A \in \mathscr{Z}$ and $x \in M$. A bounded module homomorphism of M into $\mathscr{Z}$ will be called a functional of the module M. By defining operations in a pointwise fashion, we obtain an algebraic $\mathscr{Z}$-module structure on the set of all functionals of the module M. The function

$$\|\phi\| = \text{lub}\{\|\phi(x)\| \mid x \in M, \|x\| \leqq 1\}$$

defines a norm on the $\mathscr{Z}$-module of functionals. With this norm the module becomes a normed $\mathscr{Z}$-module. We call this module the dual of M and denote it by $M^\sim$.

THEOREM 2. *Let H be an AW^*-module over the commutative AW^*-algebra $\mathscr{Z}$ and $\mathscr{A}$ be a *-subalgebra of the algebra $L(H)$ of all bounded linear operators on H such that $\mathscr{A}$ is equal to its own second commutator in $L(H)$. For each A in $\mathscr{A}$ let F_A be the function defined on the $\mathscr{Z}$-module $\mathscr{A}_\sim$ (considered as a submodule of $\mathscr{A}^\sim$) of weakly continuous module homomorphisms of $\mathscr{A}$ into $\mathscr{Z}$ by $F_A(\phi) = \phi(A)$. Then $A \to F_A$ defines an isometric isomorphism of $\mathscr{A}$ onto the dual of $\mathscr{A}_\sim$.*

Proof. First let $\mathscr{A} = L(H)$. If $\Phi \in (A_\sim)^\sim$, then $\Phi(w_{x,y}) = \langle x, y\rangle$ defines a $\mathscr{Z}$-valued hermitian form on H such that

$$\|\langle x, y\rangle\| \leqq \|\Phi\| \|w_{x,y}\| \leqq \|\Phi\| \|x\| \|y\|.$$

The function $x \to \langle x, y\rangle$ is a bounded $\mathscr{Z}$-linear function of H into $\mathscr{Z}$. Therefore, there is a unique element A_y in H with $\langle x, y\rangle = (x, A_y)$ for every x in H [10, Theorem 5]. We have that $\|A_y\| \leqq \|\Phi\| \|y\|$. From the uniqueness of A_y we conclude that there is an A in $L(H)$ such that $Ay = A_y$ for every y in H. Thus, $\Phi(w_{x,y}) = w_{x,y}(A)$ for every $w_{x,y}$. Since functions of the form $w_{x,y}$ generate $\mathscr{A}_\sim$, we have that $\Phi(\phi) = \phi(A)$ for every $\phi \in \mathscr{A}_\sim$.

Now we have that $A \to F_A$ defines a $\mathscr{Z}$-linear function of $\mathscr{A}$ into $(\mathscr{A}_\sim)^\sim$. We have that $\|F_A\| \leqq \|A\|$ since $\|\phi(A)\| \leqq \|\phi\| \, \|A\|$ for every $\phi \in \mathscr{A}_\sim$. But

$$\|A\| = \text{lub}\,\{\|w_{x,y}(A)\| \mid \|w_{x,y}\| \leqq 1\}$$

and so $\|A\| = \|F_A\|$. Thus $A \to F_A$ is an isometric isomorphism of $\mathscr{A}$ into $(\mathscr{A}_\sim)^\sim$. The preceding paragraph allows us to conclude that $A \to F_A$ is onto $(\mathscr{A}_\sim)^\sim$.

Now assume that $\mathscr{A}$ is an arbitrary *-subalgebra of $L(H)$ which is equal to its own double commutator. Let G be the bounded $\mathscr{Z}$-linear map which takes an element in $L(H)_\sim$ onto its restriction to $\mathscr{A}$. Then G is a map of $L(H)_\sim$ onto $\mathscr{A}_\sim$ (Proposition 1). If Φ is an element of $(\mathscr{A}_\sim)^\sim$, then $\Phi \cdot G$ defines an element of $(L(H)_\sim)^\sim$. By the first part of this proof we may find an A in $L(H)$ with $\Phi \cdot G(\phi) = \phi(A)$ for every ϕ in $L(H)_\sim$. If A is not in $\mathscr{A}$, there is a unitary operator U in the commutator of $\mathscr{A}$ such that $U^*AU \neq A$. Then there is an x in H with $w_x(A) - w_{Ux}(A) \neq 0$. But $\phi = w_x - w_{Ux}$ vanishes on $\mathscr{A}$ and so $\phi(A) = \Phi(G(\phi)) = 0$. This is a contradiction. Thus A is in $\mathscr{A}$. Since every ϕ in $\mathscr{A}_\sim$ has an extension to a function in $L(H)_\sim$, we conclude that $\Phi(\phi) = \phi(A)$ for every ϕ in $\mathscr{A}_\sim$. Thus we may apply the arguments of the preceding paragraph in order to show that $A \to F_A$ is an isometric isomorphism of $\mathscr{A}$ onto $(\mathscr{A}_\sim)^\sim$. Q.E.D.

REMARK. The algebra $\mathscr{A}$ in the preceding theorem is expressed as the dual of a module whose ring of multipliers is a subalgebra of the center of $\mathscr{A}$. This pathological feature can be removed by the following additional argument. Let $\mathscr{Z}_0$ be the center of $\mathscr{A}$. The commutator $\mathscr{Z}_0'$ of $\mathscr{Z}_0$ on H is a type I algebra by a proof that is entirely similar to the corresponding proof for von Neumann algebras (cf. [1, I, §2, Proposition 1 and §6, Problem 5]). The center of $\mathscr{Z}_0'$ is $\mathscr{Z}_0'' = \mathscr{Z}_0$. Since $\mathscr{Z}_0'$ is the algebra of all bounded linear operators on an AW^*-module over $\mathscr{Z}_0$ [10, Theorem 8] and since $\mathscr{A}$ is its own double commutator in $\mathscr{Z}_0'$, we may conclude that $\mathscr{A}$ is the dual of $\mathscr{Z}_0$-module by Theorem 2.

3. **The dual of a $\mathscr{Z}$-module.** Let $\mathscr{A}$ be a C^*-algebra whose center $\mathscr{Z}$ is an AW^*-algebra. Then $\mathscr{A}$ with its norm is a normed $\mathscr{Z}$-module. In this section whenever we talk about the module $\mathscr{A}$, we shall have this particular module structure in mind. If $\phi \in \mathscr{A}^\sim$ and $A \in \mathscr{A}$, the functional $(A \cdot \phi)(B) = \phi(AB)$ is in $\mathscr{A}^\sim$. This defines a right multiplication of elements of $\mathscr{A}^\sim$ by $\mathscr{A}$. Similarly, a left multiplication is defined by $(\phi \cdot A)(B) = \phi(BA)$. A functional ϕ in $\mathscr{A}^\sim$ is said to be positive if $\phi(A^*A) \geqq 0$ for every A in $\mathscr{A}$. Then ϕ is positive if $\phi(1) \geqq 0$ and $\|\phi(1)P\| = \|P \cdot \phi\|$ for every projection P in $\mathscr{Z}$. Indeed, if $\|\phi(1)P\| = \|P \cdot \phi\|$ for every projection P in $\mathscr{Z}$, then for every ζ in the spectrum of $\mathscr{Z}$ the relation $|\phi_\zeta(1)| = \|\phi_\zeta\|$ is seen to be true. Here $\phi_\zeta(A) = \phi(A)^\wedge(\zeta)$ where $B^\wedge$ denotes the Gelfand transform of $B \in \mathscr{Z}$. This means that $\phi_\zeta(A^*A) \geqq 0$ for every ζ [2, 2.1.9]. Therefore the functional ϕ is positive.

Suppose now that the module $\mathscr{A}$ is the dual of a normed $\mathscr{Z}$-module M. Since $\|A(\phi)\| \leqq \|A\| \, \|\phi\|$ for every $\phi \in M$ and $A \in \mathscr{A}$ and since $(C_1A_1 + C_2A_2)(\phi) = C_1A_1(\phi) + C_2A_2(\phi)$ for every C_1, C_2 in $\mathscr{Z}$ and A_1, A_2 in $\mathscr{A}$, the function $\phi \to \phi'$

of M into $\mathscr{A}^\sim$, where ϕ' is defined by $\phi'(A)=A(\phi)$, is a norm-decreasing $\mathscr{Z}$-module homomorphism of M into a submodule N of $\mathscr{A}^\sim$. We have that

$$\begin{aligned}\|A\| &= \text{lub}\,\{\|A(\phi)\| \mid \phi \in M,\ \|\phi\| \leqq 1\}\\ &\leqq \text{lub}\,\{\|\phi(A)\| \mid \phi \in N,\ \|\phi\| \leqq 1\} \leqq \|A\|\end{aligned}$$

and so we have that $\|A\| = \text{lub}\,\{\|\phi(A)\| \mid \phi \in N,\ \|\phi\| \leqq 1\}$. Actually, the module $\mathscr{A}$ is identified with the dual of N. Indeed, if $\Phi \in N^\sim$, then $\phi \to \Phi(\phi')$ defines an element of $M^\sim$. There is a unique element $A_\Phi = A$ in $\mathscr{A}$ such that $\Phi(\phi')=A(\phi)$ $=\phi'(A)$ for every $\phi \in M$. The function $\Phi \to A_\Phi$ of $N^\sim$ into $\mathscr{A}$ is easily seen to be an isometric isomorphism of the $\mathscr{Z}$-module $N^\sim$ onto the module $\mathscr{A}$. Since we are interested in the topology on $\mathscr{A}$ induced by pointwise convergence on M, we may assume that M is embedded in $\mathscr{A}^\sim$. We call this topology of pointwise convergence on M the $\sigma(\mathscr{A}, M)$-topology of $\mathscr{A}$.

Let M be a submodule of $\mathscr{A}^\sim$. For each bounded subset $\{\phi_i\}$ of M and each set $\{P_i\}$ of mutually orthogonal projections in $\mathscr{Z}$ of sum 1, there is a unique $\phi = \sum P_i\phi_i$ in $\mathscr{A}^\sim$ satisfying the relation $P_i\phi = P_i\phi_i$ for each P_i. Let N be the smallest algebraic submodule of $\mathscr{A}^\sim$ which contains M and is closed under the formation of such sums. Then every element $\phi \in N$ is of the form $\phi = \sum P_i\phi_i$ where $\{\phi_i\}$ is a bounded subset of M and $\{P_i\}$ is a set of mutually orthogonal projections in $\mathscr{Z}$ of sum 1. The $\mathscr{Z}$-module N will be called the module generated by M in $\mathscr{A}^\sim$.

PROPOSITION 3. *Let $\mathscr{A}$ be a C^*-algebra whose center $\mathscr{Z}$ is an AW^*-algebra. Let M be a normed $\mathscr{Z}$-module whose dual is the module $\mathscr{A}$; let N be the module generated by M in $\mathscr{A}^\sim$. Then the dual of the module N is also equal to $\mathscr{A}$.*

Proof. Let Φ be a functional in $N^\sim$. Then the restriction Ψ of Φ to M is a bounded functional of the module M. There is an $A = A_\Phi$ in $\mathscr{A}$ such that $\Psi(\phi)$ $=\phi(A)$ for every $\phi \in M$. Let $\phi \in N$; there is a bounded subset $\{\phi_i\}$ of M and a set $\{P_i\}$ of mutually orthogonal projections in $\mathscr{Z}$ of sum 1 such that $P_i\phi = P_i\phi_i$ for each P_i. Then

$$P_i\Phi(\phi) = \Phi(P_i\phi_i) = \Psi(P_i\phi_i) = P_i\phi_i(A) = P_i\phi(A)$$

for each P_i. This means that $\Phi(\phi)=\phi(A)$. Suppose there is a second element A' in $\mathscr{A}$ such that $\Phi(\phi)=\phi(A')$ for every $\phi \in N$. Then every element of M vanishes on $A'-A$. Because $\mathscr{A}$ is the dual of M, we have that $A'=A$. This means that $\Phi \to A_\Phi$ is a module isomorphism of $N^\sim$ onto $\mathscr{A}$. We have that

$$\begin{aligned}\|\Phi\| &= \text{lub}\,\{\|\Phi(\phi)\| \mid \phi \in N,\ \|\phi\| \leqq 1\}\\ &\leqq \|A_\Phi\| = \text{lub}\,\{\|\phi(A_\Phi)\| \mid \phi \in M,\ \|\phi\| \leqq 1\} \leqq \|\Phi\|\end{aligned}$$

for every $\Phi \in N^\sim$. Therefore, the map $\Phi \to A_\Phi$ is an isometric isomorphism of the module $N^\sim$ onto the module $\mathscr{A}$. Q.E.D.

We need the following lemma which is known for σ-weakly continuous functionals on a von Neumann algebra (cf. [2, 12.2.3]).

LEMMA 4. *Let $\mathscr{A}$ be a C^*-algebra, E a projection in $\mathscr{A}$ and f a bounded linear functional on $\mathscr{A}$. If the norm of the function $g(A)=f(EA)$ on $\mathscr{A}$ is equal to that of f, then $g=f$.*

Proof. Let $\mathscr{B}$ be the enveloping von Neumann algebra of $\mathscr{A}$. We may consider $\mathscr{A}$ as a weakly dense subset of $\mathscr{B}$. The functionals f and g on $\mathscr{A}$ have unique extensions to weakly continuous functionals f' and g' respectively on $\mathscr{B}$. By the uniqueness of the extension we have that $g'(A)=f'(EA)$ for every A in $\mathscr{B}$. Since the unit sphere of $\mathscr{A}$ is weakly dense in that of $\mathscr{B}$ [7], we have that $\|f'\|=\|f\|=\|g\|=\|g'\|$. Therefore, $f'=g'$ and so $f=g$. Q.E.D.

We now prove the existence of a polar decomposition.

PROPOSITION 5. *Let $\mathscr{A}$ be a C^*-algebra whose center $\mathscr{Z}$ is an AW^*-algebra. Suppose that $\mathscr{A}$ is the dual of a normed $\mathscr{Z}$-module M. Then given ϕ in M, there is a partial isometric operator U in $\mathscr{A}$ such that $\theta=U\cdot\phi$ is a positive functional of the module $\mathscr{A}$ and such that the functional $U^*\cdot\theta$ is equal to ϕ.*

Proof. Let $\mathscr{A}_1$ be the unit sphere of $\mathscr{A}$. For each ϕ in M let

$$S(\phi) = \{|\phi(A)| \mid A \in \mathscr{A}_1\}.$$

If $|\phi(A_1)|$ and $|\phi(A_2)|$ are in $S(\phi)$, there are partial isometric operators V_1 and V_2 in $\mathscr{Z}$ such that $V_i\phi(A_i)=|\phi(A_i)|$ $(i=1,2)$. There is a projection P in $\mathscr{Z}$ such that

$$\begin{aligned}\operatorname{lub}\{|\phi(A_1)|, |\phi(A_2)|\} &= P|\phi(A_1)|+(1-P)|\phi(A_2)| \\ &= \phi(PV_1A_1+(1-P)V_2A_2) = |\phi(PV_1A_1+(1-P)V_2A_2)|.\end{aligned}$$

This proves that $S(\phi)$ is monotonely increasing in $\mathscr{Z}$. Since $\mathscr{Z}$ is an AW^*-algebra and since $S(\phi)$ is bounded above by $\|\phi\|$, the set $S(\phi)$ has a least upper bound $|\phi|$ such that $\| |\phi| \| \leqq \|\phi\|$. Actually, we have that $\|\phi\|=\| |\phi| \|$ for given $\varepsilon>0$, there is an A in $\mathscr{A}_1$ such that

$$\|\phi\|-\varepsilon \leqq \|\phi(A)\| \leqq \| |\phi(A)| \| \leqq \| |\phi| \|.$$

Since $\varepsilon>0$ is arbitrary we have that $\|\phi\|=\| |\phi| \|$. Now it is clear from the definition of $|\phi|$ that $|\phi|$ is a $\mathscr{Z}$-valued seminorm on M, i.e. $|\phi|$ is a map of M into $\mathscr{Z}^+$ such that

$$|\phi+\psi| \leqq |\phi|+|\psi| \quad \text{and} \quad |C\phi| = |C|\,|\phi|$$

for every ϕ, ψ in M and C in $\mathscr{Z}$.

Let ϕ be a given element in M. By considering M as a module over the hermitian elements $H(\mathscr{Z})$ of $\mathscr{Z}$, we can construct, by using the generalized Hahn-Banach Theorem [17], an $H(\mathscr{Z})$-module homomorphism F of M into $H(\mathscr{Z})$ such that

(1) $F(\phi)=|\phi|$,

(2) $F(\psi)\leqq|\psi|$ for every ψ in M, and such that

(3) $\alpha F_1+(1-\alpha)F_2=F$ implies $F_1=F_2=F$

whenever F_1 and F_2 are $H(\mathscr{Z})$-module homomorphisms satisfying (1) and (2) and α is a real number between 0 and 1. Setting $G(\psi)=F(\psi)-iF(i\psi)$ for every ψ

in M, we obtain a $\mathscr{Z}$-module homomorphism of M into $\mathscr{Z}$. For every ψ in M there is a partial isometric operator V in $\mathscr{Z}$ such that $VG(\psi)=|G(\psi)|$. Thus we have that $|G(\psi)|=G(V\psi)=F(V\psi)\leqq|V\psi|\leqq|\psi|$. Since $\| \, |\psi| \, \|=\|\psi\|$ for every ψ in M, the functional G is an element of $M^\sim$; and consequently there is an element U in $\mathscr{A}$ such that $G(\psi)=\psi(U)$ for every ψ in M. In particular we have that $\phi(U)=|\phi|$. Since $\|G\|\leqq 1$, we have that $\|U\|\leqq 1$. Let θ be the functional in $\mathscr{A}^\sim$ defined by $\theta(A)=\phi(UA)$ for every A in $\mathscr{A}$. The functional θ is positive since

$$\|P\theta\| \leqq \|P\phi\| = \| \, |P\phi| \, \| = \|P\phi(U)\| = \|P\theta(1)\| \leqq \|P\theta\|$$

for every projection P in $\mathscr{Z}$. We show that U is an extreme point of $\mathscr{A}_1$. Indeed, if there are A_1 and A_2 in $\mathscr{A}_1$ and $0<\alpha<1$ that satisfy $\alpha A_1+(1-\alpha)A_2=U$, then

$$\alpha\psi(A_1)+(1-\alpha)\psi(A_2) = \psi(U)$$

for every ψ in M. Setting $F_j(\psi)=(\psi(A_j)+\psi(A_j)^*)/2$ $(j=1,2)$, we obtain an $H(\mathscr{Z})$-module homomorphism of M into $H(\mathscr{Z})$. We have that $F_j(\psi)\leqq|\psi(A_j)|\leqq|\psi|$ for each ψ in M. Also

$$\alpha F_1(\phi)+(1-\alpha)F_2(\phi) = F(\phi) = |\phi|.$$

So $F_1(\phi)=F_2(\phi)=|\phi|$. Since F is an extreme point (relation (3)), we have that $F_1=F_2=F$. Then

$$(\psi(A_j)+\psi(A_j)^*)/2 = F(\psi)$$

and

$$(i\psi(A_j)+(i\psi(A_j))^*)/2 = F(i\psi)$$

for every ψ implies $\psi(A_j)=F(\psi)-iF(i\psi)=\psi(U)$ for every ψ in M. This means that $A_1=A_2=U$. Hence U is an extreme point of $\mathscr{A}_1$. Therefore, U is a partial isometric operator in $\mathscr{A}$ [6].

We complete the proof by showing that $\theta(U^*A)=\phi(A)$ for every A in $\mathscr{A}$. For ζ in the spectrum of $\mathscr{Z}$ and ψ in $\mathscr{A}^\sim$ let ψ_ζ be the bounded linear functional on $\mathscr{A}$ defined by $\psi_\zeta(A)=\psi(A)^\wedge(\zeta)$; for B in $\mathscr{A}$ let $B\cdot\psi_\zeta$ be defined by $B\cdot\psi_\zeta(A)=\psi_\zeta(BA)$. Notice that $\|B\cdot\psi_\zeta\|\leqq\|B\| \, \|\psi_\zeta\|$. We have that $\|\phi_\zeta\|\leqq \text{glb}\,\{\|P\phi\| \mid P \text{ a projection in } \mathscr{Z}$ with $P^\wedge(\zeta)=1\}=\text{glb}\,\{\|P\phi(U)\| \mid P$ a projection in $\mathscr{Z}$ with $P^\wedge(\zeta)=1\}\leqq|\phi(U)^\wedge(\zeta)|$ $=\|\theta_\zeta\|\leqq\|\phi_\zeta\|$. Indeed, given $\varepsilon>0$ there is a projection P in $\mathscr{Z}$ with $P^\wedge(\zeta)=1$ and $\|P\phi(U)\|\leqq|\phi(U)^\wedge(\zeta)|+\varepsilon$. Thus $\|\theta_\zeta\|=\|\phi_\zeta\|$. However, we also have that

$$\|\theta_\zeta\| = \|(UU^*U)\cdot\phi_\zeta\| \leqq \|(UU^*)\cdot\phi_\zeta\| \leqq \|\phi_\zeta\| = \|\theta_\zeta\|.$$

By Lemma 4 we conclude that $(UU^*)\cdot\phi_\zeta=\phi_\zeta$. Since ζ is arbitrary we have that $\phi(A)=\theta(U^*A)$ for all A in $\mathscr{A}$. Q.E.D.

Let $\mathscr{A}$ be a C^*-algebra whose center $\mathscr{Z}$ is an AW^*-algebra. Let ϕ be a positive functional in $\mathscr{A}^\sim$. There is a set $\{C_i\}$ of positive elements in $\mathscr{Z}$ and a set $\{P_i\}$ of mutually orthogonal projections in $\mathscr{Z}$ of sum P such that

$$P_iC_i\phi(1) = P_i \quad \text{and} \quad (1-P)\phi(1) = 0.$$

Then setting $\psi(A)=\sum C_iP_i\phi(A)$ for A in $\mathscr{A}$, we obtain a positive functional ψ of the module $\mathscr{A}$ such that $\phi(1)\psi=\phi$. Due to the general Hahn-Banach theorem there is a positive functional ϕ_0 of the module $\mathscr{A}$ such that $\phi_0(1)=1$. So every positive functional in $\mathscr{A}^\sim$ can be decomposed into the product of a state of the module $\mathscr{A}$ (i.e. a positive functional taking 1 into 1) and an element in $\mathscr{Z}^+$ [11], [15].

Let ϕ be a positive functional in $\mathscr{A}^\sim$ such that $\phi(1)$ is a projection. Let L_ϕ be the left ideal defined by $L_\phi=\{A\in\mathscr{A}\mid\phi(A^*A)=0\}$ and let $\mathscr{A}-L_\phi$ be the $\mathscr{A}$-module $\mathscr{A}$ reduced modulo L_ϕ. Setting $(A-L_\phi, B-L_\phi)=\phi(B^*A)$ for A and B in $\mathscr{A}$, we introduce a $\mathscr{Z}$-valued hermitian form on $\mathscr{A}-L_\phi$ and then using this form and the norm of $\mathscr{Z}$, we introduce a norm on $\mathscr{A}-L_\phi$. Let H'_ϕ be the set of all pairs $(\{x_i\},\{P_i\})=x$ where $\{x_i\}$ is a bounded subset of $\mathscr{A}-L_\phi$ and $\{P_i\}$ is a set of mutually orthogonal projections in $\mathscr{Z}$ of sum 1. If $y=(\{y_j\},\{Q_j\})$ is in H'_ϕ, then set $y=x$ if and only if $y_jQ_jP_i=x_iQ_jP_i$ for all i and j. The hermitian form on $\mathscr{A}-L_\phi$ has a unique extension to H'_ϕ. The completion H_ϕ of H'_ϕ in the norm introduced by the hermitian form is an AW^*-module over $\mathscr{Z}$ with inner product induced by the hermitian form on H'_ϕ. Actually, the module H_ϕ is not faithful over $\mathscr{Z}$ but it is faithful over $\mathscr{Z}\phi(1)$. The representation of $\mathscr{A}$ on $\mathscr{A}-L_\phi$ by left multiplication has a unique extension to a representation π_ϕ of $\mathscr{A}$ as bounded linear operators on H_ϕ. The map π_ϕ is seen to be a module homomorphism as well as a *-algebra homomorphism. This map is called the canonical representation induced by ϕ of $\mathscr{A}$ on H_ϕ [18, §§2–3].

Now let $\mathscr{A}$ be an AW^*-algebra with center $\mathscr{Z}$. Suppose $\mathscr{A}$ is a subalgebra of the algebra $L(H)$ of all bounded linear operators on an AW^*-module H over $\mathscr{Z}$. Let $\{A_i\}$ be a bounded subset of $\mathscr{A}$ and let $\{P_i\}$ be a set of orthogonal projections in $\mathscr{Z}$ of least upper bound 1. It is immaterial whether $\mathscr{Z}$ is considered as a subalgebra of $\mathscr{A}$ or of $L(H)$ in order to evaluate this least upper bound. Then there is a unique A in $\mathscr{A}$ (respectively B in $L(H)$) such that $P_iA=A_iP_i$ (respectively, $P_iB=A_iP_i$) for each P_i. This means that $A=B$. This remark plus I. Kaplansky's matrix method for passing from the hermitian to the nonhermitian case ([7]; also cf. [1, I, §3, Theorem 3]) gives the following version of H. Widom's lemma [18, Lemma 4.2].

LEMMA. *Let H be an AW^*-module over the commutative AW^*-algebra $\mathscr{Z}$. Let $\mathscr{A}$ be an AW^*-algebra with center $\mathscr{Z}$ and let $\mathscr{A}$ be a subalgebra of $L(H)$. Given any B in the double commutator of $\mathscr{A}$ on H, any $x_1, x_2, \ldots, x_n$ in H, and any $\varepsilon>0$, then there is an A in $\mathscr{A}$ whose norm is majorized by that of B such that $\|(A-B)x_i\|<\varepsilon$ for every $i=1, 2, \ldots, n$.*

PROPOSITION 6. *Let $\mathscr{A}$ be a C^*-algebra whose center $\mathscr{Z}$ is an AW^*-algebra. Suppose $\mathscr{A}$ is the dual of a normed $\mathscr{Z}$-module M. Then $\mathscr{A}$ is an AW^*-algebra. Furthermore, let N be the smallest $\mathscr{Z}$-module in $\mathscr{A}^\sim$ which contains M and is closed under* *left and right multiplication by elements of $\mathscr{A}$. Then the module $\mathscr{A}$ is the dual of the module N.*

Proof. Let S be the set of all states in $\mathscr{A}^\sim$. For each $\phi \in S$, let π_ϕ be the canonical representation of $\mathscr{A}$ on the AW^*-module H_ϕ over $\mathscr{Z}$ which is induced by ϕ. Let $H=\sum \oplus H_\phi$ and let $\pi=\sum \oplus \pi_\phi$ [10, §5]. Then π is a $\mathscr{Z}$-linear, norm-decreasing, $*$-homomorphism of the algebra $\mathscr{A}$ into $L(H)$. Now, we have that

$$\|A\| = \operatorname{lub}\{\|\phi(A)\| \mid \phi \in M, \|\phi\| \leqq 1\}.$$

Let $\varepsilon>0$ be given; there is a ϕ in the unit sphere of M such that $\|\phi(A)\| \geqq \|A\|-\varepsilon$. There is a partial isometry V in $\mathscr{A}$ such that $V\cdot\phi$ is a positive functional on the module $\mathscr{A}$ and $(VV^*)\cdot\phi=\phi$ (Proposition 5). Then we have that $\|V\cdot\phi\|=\|\phi\|$. There is a C in $\mathscr{Z}^+$ and a state ψ on the module $\mathscr{A}$ such that $C\psi=V\cdot\phi$. Then $\|C\| =\|\phi(V)\|=\|\phi\|\leqq 1$. We have that $V-L_\psi$ has norm not exceeding one in H_ψ. Thus

$$\|(\pi_\psi(A)(1-L_\psi),\, V-L_\psi)\| = \|\psi(V^*A)\| \geqq \|\phi(A)\| \geqq \|A\|-\varepsilon.$$

Since $\varepsilon>0$ is arbitrary we have that $\|\pi(A)\|=\|A\|$. So π is an isometric isomorphism of $\mathscr{A}$ into a $*$-subalgebra of $L(H)$.

We show that the double commutator $\mathscr{B}$ of $\pi(\mathscr{A})$ on H is isometric isomorphic to the second dual $\mathscr{A}^{\sim\sim}$ of the module $\mathscr{A}$. Let $\phi \in \mathscr{A}^\sim$; then ϕ may be written as a linear combination of four positive functionals ϕ_i $(1 \leqq i \leqq 4)$ of the module $\mathscr{A}$ [11], [15]. There are positive elements C_i $(1\leqq i\leqq 4)$ in $\mathscr{Z}$ and states ψ_i $(1\leqq i\leqq 4)$ of the module $\mathscr{A}$ such that $C_i\psi_i=\phi_i$ $(1\leqq i\leqq 4)$. There are x_i $(1\leqq i\leqq 4)$ in H such that

$$\psi_i(A) = (\pi(A)x_i, x_i) \qquad (1 \leqq i \leqq 4)$$

for every A in $\mathscr{A}$. Thus there is an element ϕ' in $\mathscr{B}_\sim$ such that $\phi'(\pi(A))=\phi(A)$ for every A in $\mathscr{A}$. Because $\pi(\mathscr{A})$ is weakly dense in $\mathscr{B}$ (H. Widom's lemma), there is only one functional ϕ' in $\mathscr{B}_\sim$ such that $\phi'(\pi(A))=\phi(A)$ for every A in $\mathscr{A}$. This proves that the relation $\phi \to \phi'$ is a function of $\mathscr{A}^\sim$ into $\mathscr{B}_\sim$. It is easily seen to be $\mathscr{Z}$-linear. For each ψ in $\mathscr{B}_\sim$ the relation $\phi(A)=\psi(\pi(A))$ defines a bounded functional ϕ of the module $\mathscr{A}$ such that $\phi'=\psi$. So the map $\phi \to \phi'$ is onto $\mathscr{B}_\sim$. Furthermore, for each ϕ in $\mathscr{A}^\sim$ we have that

$$\|\phi\| = \operatorname{lub}\{\|\phi(A)\| \mid \|A\| \leqq 1\} = \operatorname{lub}\{\|\phi'(A)\| \mid A \in \pi(\mathscr{A}), \|A\| \leqq 1\} = \|\phi'\|$$

since the unit sphere of $\pi(\mathscr{A})$ is weakly dense in the unit spheres of $\mathscr{B}$ (H. Widom's lemma). This proves that $\mathscr{A}^\sim$ is isometrically isomorphic with $\mathscr{B}_\sim$ and thus that $\mathscr{A}^{\sim\sim}$ is isometrically isomorphic with $\mathscr{B}$ (Remark, Theorem 2).

Let ρ be the transpose of the identity map of M into $\mathscr{A}^\sim$, i.e. let ρ be the map of $\mathscr{B}$ into $\mathscr{A}$ given by $\phi(\rho(A))=\phi'(A)$ for every A in $\mathscr{B}$ and ϕ in M. Then we have that

$$\phi(A) = \phi'(\pi(A)) = \phi(\rho(\pi(A)))$$

for every ϕ in M and A in $\mathscr{A}$. This means that $\rho(\pi(A))=A$ and that $\pi\cdot\rho(\pi(A))$

$=\pi(A)$ for every A in $\mathscr{A}$. Therefore the map $\eta=\pi\cdot\rho$ is a projection of $\mathscr{B}$ onto $\pi(\mathscr{A})$. We have that

$$\begin{aligned}\|\eta(A)\| &= \|\rho(A)\| = \operatorname{lub}\{\|\phi(\rho(A))\| \mid \phi\in M,\ \|\phi\| \leqq 1\}\\ &\leqq \operatorname{lub}\{\|\phi'(A)\| \mid \phi\in\mathscr{A}^{\sim},\ \|\phi'\| \leqq 1\} \leqq \|A\|\end{aligned}$$

for every A in $\mathscr{B}$. Thus the function η is a projection of norm 1. This proves that $\mathscr{A}$ is an AW^*-algebra due to a result of Tomiyama [16, Theorem 5]. Also following Tomiyama, we can show that the kernel K of η is an ideal in $\mathscr{B}$. Indeed, if A and C are in $\pi(\mathscr{A})$ and if $\eta(B)=0$, then $\eta(ABC)=A\eta(B)C=0$. Now if A is in $\mathscr{B}$, then A is the weak limit of a net $\{A_n\}$ in $\pi(\mathscr{A})$. This means that

$$\phi(\rho(AB)) = \phi'(AB) = \lim\phi'(A_nB) = \lim\phi'(\eta(A_nB)) = \lim\phi'(A_n\eta(B)) = 0$$

for every $\phi\in M$. This proves that $\rho(AB)=0$ and that $\eta(AB)=0$, and therefore that K is a left ideal. Similarly, we obtain that K is a right ideal and therefore that K is a two-sided ideal. By the same reasoning we see that K is weakly closed. Let $\{E_n \mid n\in D\}$ be a maximal set of mutually orthogonal nonzero projections in K. Let F be the family of finite subsets of D. For each s in F let $E_s=\sum\{E_n \mid n\in s\}$. Let E be the least upper bound of $\{E_s\}$ in $\mathscr{B}$. Now given an element ϕ in M, a nonzero projection P in $\mathscr{Z}$, and an $\varepsilon>0$, there is an s_0 in F and a nonzero projection Q in $\mathscr{Z}P$ such that $\|\phi'(E_s-E)Q\|\leqq\varepsilon$ whenever $s\supset s_0$ ([3, Lemma 4.2] and [18, Lemma 1.4]). Since

$$\phi'(E_s) = \sum\{\phi'(\eta(E_n)) \mid n\in s\} = 0,$$

we have that $\|\phi'(E)Q\|\leqq\varepsilon$. Let $\{Q_n\}$ be a maximal set of mutually orthogonal nonzero projections in $\mathscr{Z}$ such that $\|\phi'(E)Q_n\|\leqq\varepsilon$ for every Q_n. It is evident that $\sum Q_n=1$ and hence that $\|\phi'(E)\| \leqq \varepsilon$. Since $\varepsilon>0$ is arbitrary we see that $\phi(\rho(E))=\phi'(E)=0$. Since ϕ is arbitrary, we have that $\rho(E)=0$; and therefore, we have that $E\in K$. Because K is generated in the uniform topology by its projections, we have that $AE=EA=A$ for every A in K. This means that E is a projection in the center of $\mathscr{B}$ and that $\mathscr{B}E=K$. This proves that η is an isomorphism of the algebra $\mathscr{B}(1-E)$ onto the algebra $\pi(\mathscr{A})$. The map η is also a module isomorphism.

Let N be the smallest $\mathscr{Z}$-module in $\mathscr{A}^{\sim}$ which contains M and is closed under right and left multiplication by elements of $\mathscr{A}$. We show that $N^{\sim}$ is isometric isomorphic to $\mathscr{A}$. Let Φ be a bounded functional of the module N. There is a functional Ψ of the module $\mathscr{A}^{\sim}$ such that $\Psi(\phi)=\Phi(\phi)$ for all ϕ in N and such that $\|\Psi\|=\|\Phi\|$ [11], [15]. There is an element B in $\mathscr{B}$ such that $\Psi(\phi)=\phi'(B)$ for all $\phi\in\mathscr{A}^{\sim}$. If $\phi\in M$ and if $A\in\mathscr{A}$, we have that

$$(A\cdot\phi)'(\pi(C)) = A\cdot\phi(C) = \phi(AC) = \phi'(\pi(AC)) = \pi(A)\cdot\phi'(\pi(C))$$

for every C in $\mathscr{A}$. Since $(A\cdot\phi)'$ and $\pi(A)\cdot\phi'$ are weakly continuous on $\mathscr{B}$ and since $\pi(\mathscr{A})$ is weakly dense in $\mathscr{B}$, we have that $(A\cdot\phi)'=\pi(A)\cdot\phi'$. Therefore, we have that

$$\Phi(A\cdot\phi) = \phi'(\pi(A)B) = \phi'(\pi(A)\eta(B)) = \phi'(\pi(A\rho(B))) = \phi(A\rho(B)) = (A\cdot\phi)(\rho(B)).$$

Similarly, we have that $\Phi(\phi\cdot A)=(\phi\cdot A)(\rho(B))$. So there is a B_Φ in $\mathscr{A}$ such that $\Phi(\phi)=\phi(B_\Phi)$ for every ϕ in N. If $B'_\Phi\in\mathscr{A}$ and if $\phi(B_\Phi)=\phi(B'_\Phi)$ for every ϕ in N, then B_Φ is equal to B'_Φ. Hence, there is a unique B_Φ in $\mathscr{A}$ such that $\Phi(\phi)=\phi(B_\Phi)$ for every $\phi\in N$. The function $\Phi\to B_\Phi$ is obviously a module isomorphism of the module $N^\sim$ onto the module $\mathscr{A}$. Finally we have that

$$\begin{aligned}\|B_\Phi\| &= \text{lub}\,\{\|\phi(B_\Phi)\| \mid \phi\in M,\quad \|\phi\|\leqq 1\}\\ &\leqq \text{lub}\,\{\|\phi(B_\Phi)\| \mid \phi\in N,\quad \|\phi\|\leqq 1\} = \|\Phi\| \leqq \|B_\Phi\|.\end{aligned}$$

So $\Phi\to B_\Phi$ is an isometric isomorphism of $N^\sim$ onto $\mathscr{A}$. Q.E.D.

THEOREM 7. *Let $\mathscr{A}$ be a C^*-algebra whose center is an AW^*-algebra $\mathscr{Z}$. Suppose $\mathscr{A}$ is the dual of a $\mathscr{Z}$-module M. Let N' be the smallest $\mathscr{Z}$-module in the dual of the module $\mathscr{A}$ which contains M and is closed under left and right multiplication by elements of $\mathscr{A}$, and let N be the module generated by N' in $\mathscr{A}^\sim$. Then $\mathscr{A}$ may be embedded as a double commutator in the algebra of all bounded linear operators on an AW^*-module over $\mathscr{Z}$ so that the weak topology and the $\sigma(\mathscr{A}, N)$-topology coincide on the unit sphere of $\mathscr{A}$.*

Proof. Let S be the set of all positive functionals ϕ in N such that $\phi(1)$ is a projection in $\mathscr{Z}$. For each ϕ in S let π_ϕ be the canonical representation of $\mathscr{A}$ on the AW^*-module H_ϕ over $\mathscr{Z}\phi(1)$ which is induced by ϕ. Then H_ϕ may be considered as an AW^*-module over $\mathscr{Z}$. Let $H=\sum\oplus\{H_\phi \mid \phi\in S\}$ and let $\pi=\sum\oplus\{\pi_\phi \mid \phi\in S\}$. The AW^*-module H is a faithful AW^*-module over $\mathscr{Z}$. Indeed, if P is a nonzero projection in $\mathscr{Z}$, then

$$\text{lub}\,\{\|\phi(P)\| \mid \phi\in M,\ \|\phi\|\leqq 1\} = 1.$$

So $\|\phi(P)\|\neq 0$ for some ϕ in the unit sphere of M. Let V be a partial isometry in $\mathscr{A}$ such that $V\cdot\phi$ is a positive functional and such that $VV^*\cdot\phi=\phi$ (Proposition 5). Then $(V\cdot\phi)(P)\neq 0$ because $|\phi(P)|^2=|\phi(VV^*P)|^2\leqq\phi(V)\phi(VP)$. There is a C in $\mathscr{Z}^+$ such that $\phi(CV)$ is a nonzero projection in $\mathscr{Z}$ majorized by P. Setting $\psi=CV\cdot\phi$, we obtain an element ψ in S such that $P(1-L_\psi)\neq 0$. Hence H is a faithful AW^*-module over $\mathscr{Z}$.

We show that the map π is an isometry. Let A be a nonzero positive element in $\mathscr{A}$. It is enough to show that $\|\pi(A)\|=\|A\|$. Let $\varepsilon>0$ be an arbitrary number less than $\|A\|$. There is a ϕ in the unit sphere of N such that $\|\phi(A)\|>\|A\|-\varepsilon$ and thus there is a nonzero projection P in $\mathscr{Z}$ such that

$$|P\phi(A)| \geqq (\|A\|-\varepsilon)P.$$

There is a partial isometry V in $\mathscr{A}$ such that $V\cdot\phi$ is a positive functional and such that $VV^*\cdot\phi=\phi$ (Proposition 5). Then we have that

$$(\|A\|-\varepsilon)^2P \leqq |\phi(A)|^2 \leqq \phi(V)\phi(VA^2).$$

So there is a positive element C in $\mathscr{Z}$ such that $CV\cdot\phi=\psi$ is in S and such that

$\psi(1) \geqq P$. Since $PV \cdot \phi(1) \leqq P$, we see that $CP \geqq P$. Hence, we have that

$$(\|A\| - \varepsilon)^2 P \leqq P\phi(V)\phi(VA^2) \leqq \|A\| P\phi(V)\phi(VA) \leqq \|A\| P\psi(1)\psi(A) \leqq \|A\| \psi(A).$$

This proves that lub $\{\|\psi(A)\| \mid \psi \in S\} = \|A\|$ and that $\|\pi(A)\| = \|A\|$.

We show that $\pi(\mathscr{A})$ is equal to its double commutator $\mathscr{B}$ on H. Let B be an element in $\mathscr{B}$. There is a net $\{A_n\}$ in the sphere of $\mathscr{A}$ about the origin of radius $\|B\|$ such that $\lim \pi(A_n) = B$ weakly in $L(H)$ because $\pi(\mathscr{A})$ is an AW^*-algebra with center $\mathscr{Z}$ (Proposition 6) and thus Widom's lemma may be employed. Let $\phi \in N$ and let $V \cdot \phi$ be the polar decomposition of ϕ. There is a sequence $\{P_m\}$ of orthogonal projections in $\mathscr{Z}$ of sum P such that $P_m\phi(V)$ has inverse C_m in $\mathscr{Z}P_m$ and such that $(1-P)\phi(V) = 0$. By the hypothesis on N, we see that $\psi = \sum P_m(C_m V \cdot \phi)$ is in N and therefore in S and that $\phi(V)\psi = V \cdot \phi$. Setting $x = 1 - L_\psi$, we have that

$$\lim\, (\pi(A_n)x, \pi(V)x) = (Bx, \pi(V)x)$$

uniformly in $\mathscr{Z}$. This means that $\{\phi(A_n)\}$ is a Cauchy net in the uniform topology of $\mathscr{Z}$ and therefore $\{\phi(A_n)\}$ converges uniformly to an element $\Phi(\phi)$ in the sphere of radius $\|\phi\| \|B\|$ about the origin. Hence, we see that $\phi \to \Phi(\phi)$ defines an element Φ in $N^{\sim}$ and therefore we have an element A_0 in $\mathscr{A}$ such that $\Phi(\phi) = \phi(A_0)$ for every ϕ in N. Now for arbitrary ψ in S we have that $A \cdot \psi \cdot C$ is in N and therefore that

$$\begin{aligned}(\pi(A_0)\pi(C)x, \pi(A)^*x) &= \lim\, (A \cdot \psi \cdot C)(A_n) \\ &= \lim\, (\pi(A_n)\pi(C)x, \pi(A)^*x) = (B\pi(C)x, \pi(A)^*x)\end{aligned}$$

where $x = 1 - L_\psi$. Therefore, we have proved that $((\pi(A_0) - B)y, z) = 0$ for all y, z in $K = \{\pi(A)x \mid A \in \mathscr{A}\}$. Now given A in $\mathscr{A}$, there is a net $\{C_n\}$ in $\mathscr{A}$ such that $\{\pi(C_n)\}$ converges weakly to $(\pi(A_0) - B)\pi(A)$ since $(\pi(A_0) - B)\pi(A)$ is in $\mathscr{B}$. Therefore,

$$\|(\pi(A_0) - B)\pi(A)x\|^2 = \lim\, ((\pi(A_0) - B)\pi(A)x, \pi(C_n)x) = 0.$$

This means that $\pi(A_0) - B$ vanishes on K and therefore on H_ψ. Since ψ is arbitrary, we conclude that $\pi(A_0) = B$ and therefore that $\pi(\mathscr{A}) = \mathscr{B}$.

We now identify $\mathscr{A}$ with $\mathscr{B}$ and we show that the $\sigma(\mathscr{A}, N)$-topology and the $\sigma(\mathscr{A}, \mathscr{A}_\sim)$-topology coincide on the unit sphere of $\mathscr{A}$. For each $\psi \in S$ let E_ψ be the projection of H on H_ψ [10, §6]. By the definition of H we have that the least upper bound of the family $\{E_\psi \mid \psi \in S\}$ is 1. Let x be in H and let ε be a strictly positive number. There is a set $\{P_i\}$ of mutually orthogonal projections in $\mathscr{Z}$ of sum 1 such that for each i there is a finite subset $n(i)$ of S with

$$P_i(|x|^2 - \sum \{|E_\psi x|^2 \mid \psi \in n(i)\}) < \varepsilon^2 P_i$$

since

$$\sum \{|E_\psi x|^2 \mid \psi \in S\} = |x|^2$$

[3, Lemma 4.2]. Let i be fixed and let $n(i) = \{\psi_1, \ldots, \psi_n\}$. Let $E_{\psi_k} = E_k$ and let

$x_k = 1 - L_{\psi_k}$. There is a set $\{Q_j\}$, not depending on k, of mutually orthogonal projections in $\mathscr{Z}$ of sum P_i such that for each $k = 1, 2, \ldots, n$ there is a set $\{A_{kj}\}_j$ in $\mathscr{A}$ with $\{\psi_k(A_{kj}^* A_{kj})\}_j$ bounded and

$$\left\| \sum_j Q_j A_{kj} x_k - E_k x \right\| < \varepsilon n^{-1}.$$

Let $y_j = \sum_k Q_j A_{kj} x_k$. Then we have that

$$\begin{aligned} |Q_j(y_j - x)| &\leqq \left| Q_j\left(y_j - \sum E_k x\right)\right| + \left| Q_j\left(1 - \sum E_k\right)x\right| \\ &\leqq \sum_k |Q_j(A_{kj}x_k - E_k x)| + \left| Q_j\left(1 - \sum E_k\right)x\right| \leqq 2\varepsilon, \end{aligned}$$

and

$$\begin{aligned} |y_j| &\leqq \left| Q_j\left(y_j - \sum E_k x\right)\right| + \left| Q_j \sum E_k x\right| \\ &\leqq \left(\varepsilon + \left(\left|\sum E_k x\right|^2\right)^{1/2}\right) Q_j \leqq (\varepsilon + \|x\|) Q_j. \end{aligned}$$

Then setting

$$\phi_j(A) = (Ay_j, y_j) = Q_j \sum_k \psi_k(A_{kj}^* A A_{kj}),$$

we obtain a positive functional in N of norm not exceeding $(\varepsilon + \|x\|)^2$. There is a unique y_i (respectively θ_i) in H (respectively in N) such that $Q_j y_i = y_j$ (respectively $Q_j \theta_i = \phi_j$) for each Q_j and $(1 - P_i)y_i = 0$ (respectively, $(1 - P_i)\theta_i = 0$). We have that $\theta_i(A) = (Ay_i, y_i)$ for each A in $\mathscr{A}$ and that $\|\theta_i\| \leqq (\varepsilon + \|x\|)^2$. Then we have that

$$\begin{aligned} P_i|(Ax, x)| &\leqq \|A\| \, |P_i(x - y_i)| \, |P_i x| \\ &\quad + \|A\| \, |P_i(x - y_i)| \, |P_i y_i| + |\theta_i(A)| \\ &\leqq 2\varepsilon \|A\| (\varepsilon + 2\|x\|) P_i + |\theta_i(A)|. \end{aligned}$$

Since $\theta_i(1) \leqq (\varepsilon + \|x\|)^2 P_i$, there is a unique θ in N such that $P_i \theta = \theta_i$ for each P_i. This means that

$$|(Ax, x)| \leqq 2\varepsilon \|A\| (\varepsilon + 2\|x\|) + |\theta(A)|$$

for every A in $\mathscr{A}$. Now it becomes obvious that the $\sigma(\mathscr{A}, N)$-topology is finer than the weak topology on the unit sphere of $\mathscr{A}$.

Conversely, let ϕ be a functional in N and let U be a partial isometry of $\mathscr{A}$ such that $U \cdot \phi$ is positive and $UU^* \cdot \phi = \phi$. There is a sequence $\{P_n\}$ of mutually orthogonal projections in $\mathscr{Z}$ such that $P_n \phi(U) = C_n$ is invertible with inverse D_n in $\mathscr{Z} P_n$ and such that $(1 - \sum P_n)\phi(U) = 0$. Then $\sum D_n U \cdot \phi = \psi$ is in N. Since $\psi(1) = \sum P_n$, the functional ψ is in S. We then have that $\phi(A) = (Ax, y)$ where $x = \sum C_n(1 - L_\psi)$ and $y = U(1 - L_\psi)$. Thus a net $\{A_n\}$ in the unit sphere of $\mathscr{A}$ converges to A in the $\sigma(\mathscr{A}, N)$-topology whenever $\{A_n\}$ converges to A in the $\sigma(\mathscr{A}, \mathscr{A}_\sim)$-topology. Q.E.D.

REMARK. In the notation of Theorem 7 we have that the closure of N in the uniform topology of $\mathscr{A}^\sim$ is equal to the closure of $\mathscr{A}_\sim$ in $\mathscr{A}^\sim$ and that $\mathscr{A}$ is the dual of the closure of the module N.

BIBLIOGRAPHY

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957. MR **20** #1234.

2. ———, *Les C*-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964. MR **30** #1404.

3. M. Goldman, *Structure of AW*-algebras*. I, Duke Math. J. **23** (1956), 23–34. MR **17**, 512; MR **17**, 1437.

4. H. Halpern, *A spectral decomposition for self-adjoint elements in the maximum GCR ideal of a von Neumann algebra with applications to noncommutative integration theory*, Trans. Amer. Math. Soc. **133** (1968), 281–306. MR **37** #5704.

5. ———, *Module homomorphisms of a von Neumann algebra into its center* (Trans. Amer. Math. Soc. **140** (1969), 183–193.

6. R. Kadison, *Isometries of operator algebras*, Ann. of Math. (2) **54** (1951), 325–338. MR **13**, 256.

7. I. Kaplansky, *A theorem of rings of operators*, Pacific J. Math. **1** (1951), 227–232. MR **14**, 291.

8. ———, *Projections in Banach algebras*, Ann. of Math. (2) **53** (1951), 235–249. MR **13**, 48.

9. ———, *Algebras of type* I, Ann. of Math. (2) **56** (1952), 460–472. MR **14**, 291.

10. ———, *Modules over operator algebras*, Amer. J. Math. **75** (1953), 839–858. MR **15**, 327.

11. M. Nakai, *Some expectations in AW*-algebras*, Proc. Japan Acad. **34** (1958), 411–416. MR **20** #1924.

12. S. Sakai, *A characterization of W*-algebras*, Pacific J. Math. **6** (1956), 763–773. MR **18**, 811.

13. ———, *On linear functionals of W*-algebras*, Proc. Japan Acad. **34** (1958), 571–574. MR **21** #5915.

14. M. Takesaki, *On the conjugate space of operator algebra*, Tôhoku Math. J. (2) **10** (1958), 194–203. MR **20** #7227.

15. ———, *On the Hahn-Banach type theorem and the Jordan decomposition of module linear mapping over some operator algebras*, Kōdai Math. Sem. Rep. **12** (1960), 1–10. MR **22** # 5914.

16. J. Tomiyama, *On the projection of norm one in W*-algebras*, Proc. Japan Acad. **33** (1957), 608–612. MR **20** #2635.

17. G. Vincent-Smith, *The Hahn-Banach theorem for modules*, Proc. London Math. Soc. (3) **17** (1967), 72–90. MR **35** #766.

18. H. Widom, *Embedding in algebras of type I*, Duke Math. J. **23** (1956), 309–324. MR **17**, 1228.

19. T. Yen, *Trace on finite AW*-algebras*, Duke Math. J. **22** (1955), 207–222. MR **16**, 1033.

ILLINOIS INSTITUTE OF TECHNOLOGY,
CHICAGO, ILLINOIS