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APPROXIMATION BY POLYNOMIALS SUBORDINATE TO A UNIVALENT FUNCTION[1]

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This paper is concerned with approximating a function $f(z)$ analytic and univalent in the unit disk $E=\{z : |z|<1\}$ by polynomials which are also univalent in E. We are interested in such approximations where exactly one polynomial of each degree n, $n \geqq 1$, is used and such that the polynomials are monotonically subordinate to each other.

Recall that $f(z)$ is called subordinate to $g(z)$ in E if both functions are analytic in E and if there exists a function $\varphi(z)$ analytic in E and satisfying $|\varphi(z)|<1$, $\varphi(0)=0$ such that $f(z)=g(\varphi(z))$. The existence of such a function $\varphi(z)$ is implied by the conditions that $g(z)$ is univalent in E, $f(0)=g(0)$ and $f(E)\subset g(E)$. If $f(z)$ is subordinate to $g(z)$ in E we will write $f(z)\subset g(z)$ in E.

Theorem 1 asserts that if $f(z)$ is analytic and univalent in E then there exists a sequence $\{p_n(z)\}$, $n=1, 2, \ldots$, such that $p_n(z)$ is a polynomial of degree n which is univalent in E,

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \cdots$$

and $p_n(z) \to f(z)$ as $n \to \infty$. This convergence is uniform in each compact subset of E. The idea of proving this is to appropriately relate $f(z)$ to the partial sums $s_n(z)$ of the power series for $f(z)$. Although $s_n(z)$ are not, in general, univalent in E (they are in $|z|<\frac{1}{4}$ [12]) they are univalent in $|z|<r$, $0<r<1$, for all large n once r is given. This eventually leads to a chain of univalent polynomials

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \cdots$$

such that $p_{n_j}(z)$ has degree n_j, $n_1<n_2<n_3<\cdots$, and $p_{n_j}(z) \to f(z)$ as $j \to \infty$. These polynomials are so related that it is still possible to find polynomials of the remaining degrees which fill in the chain and such that $p_n(z) \to f(z)$.

This method is equally adaptable when $f(z)$ maps E one-to-one onto a convex domain in order to produce a similar chain of convex, univalent polynomials. In fact, our argument may be applied to any one of a number of classes of univalent functions.

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The possibility of such an approximation to convex mappings was implicitly raised by G. Pólya and I. J. Schoenberg in [10]. They considered analytic functions

$$f(z) = \sum_{k=1}^{\infty} a_k z^k$$

which map E one-to-one onto a convex domain. For each integer n, $n \geqq 1$, the de la Vallée Poussin means of $f(z)$ are defined by

$$V_n(z) = \frac{n}{n+1} a_1 z + \frac{n(n-1)}{(n+1)(n+2)} a_2 z^2 + \cdots + \frac{n(n-1)\cdots 1}{(n+1)(n+2)\cdots(2n)} a_n z^n.$$

They showed that $V_n(z)$ is a convex, univalent function, $V_n(z) \subset f(z)$, and $V_n(z) \to f(z)$ in E, and conjectured that

$$V_1(z) \subset V_2(z) \subset V_3(z) \subset \cdots.$$

This conjecture remains open and its validity would yield a somewhat more explicit proof of the existence of such a chain except that we also require $p_n(z)$ to have the exact degree n.

Disregarding the demand that $p_n(z)$ be a polynomial, it is easy to find a chain

$$f_1(z) \subset f_2(z) \subset f_3(z) \subset \cdots$$

such that $f_n(z) \to f(z)$ and we do not have $f_n(z) = f(z)$ for all n. One simply sets $f_n(z) = f(r_n z)$ where $\{r_n\}$ is an increasing sequence of real numbers in the interval $0 < r < 1$ such that $r_n \to 1$. A less trivial situation in the case $f(z)$ is univalent and convex in E is obtained by considering the functions

$$f(z, \alpha) = \frac{1}{\alpha} \int_0^{\alpha} f(ze^{i\theta})\, d\theta.$$

If $\{\alpha_n\}$ is any sequence of real numbers in the interval $0 < \alpha < 2\pi$ which monotonically converges to zero then $f(z, \alpha_n)$ is univalent and convex, $f(z, \alpha_1) \subset f(z, \alpha_2) \subset f(z, \alpha_3) \subset \cdots$ and $f(z, \alpha_n) \to f(z)$ in E [6].

Interest in (continuous as well as discrete) chains of subordinate functions has occurred elsewhere and we would like to point out the recent papers by C. Pommerenke [7], [8], [9] as well as [1] by A. Bielecki and Z. Lewandowski.

The earliest consideration of this kind presumably is due to C. Carathéodory in his idea of kernels.

LEMMA 1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be analytic in* E *and set* $s_n(z) = \sum_{k=0}^{n} a_k z^k$.

Let $\{n_j\}$ *be a strictly increasing sequence of positive integers and let* $\{r_j\}$ *be a sequence of real numbers such that* $0 < r_j < 1$ *and* $r_j \to 1$ *as* $j \to \infty$. *Then* $s_{n_j}(r_j z) \to f(z)$ *uniformly in each disk* $|z| \leqq r$, $0 < r < 1$.

Proof. Suppose that $|z|\leqq r$ and $0<r<1$.

$$\begin{aligned}|f(z)-s_{n_j}(r_jz)| &= \left|\sum_{k=0}^{n_j} a_k(1-r_j^k)z^k+\sum_{k=n_j+1}^{\infty} a_kz^k\right| \\ &\leqq (1-r_j)\sum_{k=0}^{n_j} k|a_k|r^k+\sum_{k=n_j+1}^{\infty}|a_k|r^k \\ &\leqq (1-r_j)\sum_{k=0}^{\infty} k|a_k|r^k+\sum_{k=n_j+1}^{\infty}|a_k|r^k.\end{aligned}$$

Given a positive number ε there is an integer N such that $\sum_{k=n+1}^{\infty}|a_k|r^k<\varepsilon/2$ if $n\geqq N$. Since $n_j\geqq j$ this implies that for $j\geqq N$, $\sum_{k=n_j+1}^{\infty}|a_k|r^k<\varepsilon/2$. As $r_j\to 1$ there is an integer N' such that if $j\geqq N'$ then

$$(1-r_j)\sum_{k=0}^{\infty} k|a_k|r^k < \frac{\varepsilon}{2}.$$

Choosing $N''=\max(N, N')$ we see that $|f(z)-s_{n_j}(r_jz)|<\varepsilon$ for $j\geqq N''$ and this proves the lemma.

THEOREM 1. *Let $f(z)$ be analytic and univalent in E. There exists a sequence of polynomials $\{p_n(z)\}$, $n=1, 2, 3, \ldots$, where $p_n(z)$ has degree n and is univalent in E, such that*

$$p_1(z)\subset p_2(z)\subset p_3(z)\subset\cdots$$

in E and $p_n(z)\to f(z)$ in E. The convergence is uniform in compact subsets of E.

Proof. Let the power series for $f(z)$ be $f(z)=\sum_{k=0}^{\infty} a_kz^k$ and let $s_n(z)=\sum_{k=0}^{n} a_kz^k$.

Then $s_n(z)\to f(z)$ and $s_n'(z)\to f'(z)$ as $n\to\infty$ uniformly in each disk $|z|\leqq r$, where $0<r<1$.

Let us introduce the notation:

$$\begin{aligned}\Delta_g(u, v) &= (g(u)-g(v))/(u-v), \quad \text{if } u\neq v,\\ &= g'(u), \quad\quad \text{if } u=v,\end{aligned}$$

for each function $g(z)$ analytic in E. If $u\neq v$, $|u|\leqq r$, $|v|\leqq r$, $0<r<1$, we may write

$$g(u)-g(v) = \int_v^u g'(z)\,dz,$$

where the path of integration is the line segment from v to u. Since the points on this line segment satisfy $|z|\leqq r$, it is clear that

$$\max_{|u|\leqq r,\,|v|\leqq r}|\Delta_g(u, v)| = \max_{|z|\leqq r}|g'(z)|.$$

Applying this to $g_n(z)=f(z)-s_n(z)$ we conclude that $\Delta_{g_n}(u, v)$ is uniformly small for large n if $|u|\leqq r$, $|v|\leqq r$, since $g_n'(z)$ is uniformly small for large n in $|z|\leqq r$.

Since $f(z)$ is univalent in E, $f'(z)$ does not vanish, and, therefore,

$$m_f(r) = \min_{|u|\leqq r,\,|v|\leqq r}|\Delta_f(u, v)| > 0,$$

where $0<r<1$. Because $\Delta_{s_n}(u, v)=\Delta_f(u, v)-\Delta_{g_n}(u, v)$ there exists an integer N such that if $n \geqq N$ then $m_{s_n}(r) \geqq m_f(r)/2$. In particular, this implies that $s_n(z)$ is univalent in $|z| \leqq r$ for all $n \geqq N$.

Let $D(r)$ denote the image of $|z|<r$ under $f(z)$ and let $D_n(r)$ denote the image of $|z|<r$ under $s_n(z)$, where $0<r \leqq 1$. Also, let $d(A, B)$ denote the distance between the sets A and B and let δA denote the boundary of the set A. If we set $d(r)=d(D(r), \delta D(1))$, where $0<r<1$, then $d(r)<\infty$ since $D(r)$ is bounded and $D(1)$ is not the whole plane. Also, $d(r)>0$ as $f(z)$ is an open mapping. There is an integer N' such that

$$|f(z)-s_n(z)| < d(r)/2$$

for $|z| \leqq r$, $0<r<1$, if $n \geqq N'$. Consequently, if $n \geqq N'$ then $d_n(r)=d(D_n(r), \delta D(1)) > d(r)/2$.

Let r_1 be any fixed number such that $0<r_1<1$ and choose an integer $n_1 \geqq \max(N, N')$, where $N=N(r_1)$, $N'=N'(r_1)$ was obtained by the two previous arguments. This implies that $m_{s_{n_1}}(r_1) \geqq m_f(r_1)/2$ and $d_{n_1}(r_1)>d(r_1)/2$.

We continue our argument with the assumption that $f(z)$ is not a polynomial. Then the selection of n_1 can be made so that $s_{n_1}(z)$ has exactly the degree n_1. If $E_n(r)=f^{-1}(D_n(r))$ then there is a number ρ_1, $0<\rho_1<1$, such that $E_{n_1}(r_1)$ is contained in $|z| \leqq \rho_1$, as, otherwise, $d_{n_1}(r_1)=0$. Choose the number r_2 such that $\rho_1<r_2<1$ and $r_2>(1+r_1)/2$. Then $d(D_{n_1}(r_1), \delta D(r_2))>0$ and, as always, $d(r_2)>0$.

Arguing as before we conclude that there is an integer $N=N(r_2)$ such that if $n \geqq N$ then $m_{s_n}(r_2) \geqq m_f(r_2)/2$. Also there is an integer $N'=N'(r_2)$ such that if $n \geqq N'$ then $|f(z)-s_n(z)|<d^*/2$ for $|z| \leqq r_2$, where

$$d^* = \min(d(r_2), d(D_{n_1}(r_1), \delta D(r_2))).$$

This implies that $d_n(r_2)>d(r_2)/2>0$ and

$$d(D_{n_1}(r_1), \delta D_n(r_2)) > \tfrac{1}{2} d(D_{n_1}(r_1), \delta D(r_2)) > 0$$

for $n \geqq N'$. The last condition and the facts that $D_{n_1}(r_1)$ is connected and $D_{n_1}(r_1)$ and $D_n(r_2)$ have a common point, namely a_0, implies that $D_{n_1}(r_1) \subset D_n(r_2)$.

If we set $N''=\max(N, N')$ then for $n \geqq N''$ $m_{s_n}(r_2) \geqq (m_f(r_2)/2)$, $d(D_{n_1}(r_1), \delta D_n(r_2)) > 0$ and $d_n(r_2)>0$. The first two conditions and $s_n(0)=s_{n_1}(0)=a_0$ show that $s_n(r_2 z)$ is univalent in $|z| \leqq 1$ and $s_{n_1}(r_1 z) \subset s_n(r_2 z)$ in E. Since $f(z)$ is not a polynomial there is such an integer $n=n_2$ so that $n_2 \geqq N''$, $n_2>n$, and $s_{n_2}(z)$ has degree n_2. Moreover, since $d_{n_2}(r_2)>0$ this whole argument may be repeated.

Namely, we first consider the set $E_{n_2}(r_2)$ to show that there is a number r_3 such that $(1+r_2)/2<r_3<1$ and $d(D_{n_2}(r_2), \delta D(r_3))>0$.

We then find an integer N associated with the condition $m_{s_n}(r_3) \geqq m_f(r_3)/2$ and an integer N' associated with the inequality $|f(z)-s_n|<d^*/2$ for $|z| \leqq r_3$, where

$$d^* = \min(d(r_3), d(D_{n_2}(r_2), \delta D(r_3))).$$

This leads to an integer n_3 so that $n_3 > n_2$, $s_{n_3}(z)$ has degree n_3, $m_{s_{n_3}}(r_3) \geqq m_f(r_3)/2$, $d(D_{n_2}(r_2), \delta D_{n_3}(r_3)) > 0$, and $d_{n_3}(r_3) > 0$. In particular, $s_{n_3}(r_3 z)$ is univalent in $|z| \leqq 1$ and $s_{n_2}(r_2 z) \subset s_{n_3}(r_3 z)$ in E.

This argument, therefore, can be continued indefinitely and it yields the following conclusion. There is a sequence of real numbers $\{r_j\}$, a strictly increasing sequence of integers $\{n_j\}$ and a sequence of polynomials $\{p_{n_j}(z)\}$ such that $0 < r_j < 1$, $r_j \to 1$ and $p_{n_j}(z) = s_{n_j}(r_j z)$ are univalent in $|z| \leqq 1$. Moreover, $p_{n_j}(z)$ has degree n_j and

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \cdots$$

in E. In addition,

$$m_{p_{n_j}}(1) \geqq m_f(r_j)/2$$

and

$$d(D_{n_j}(r_j), \delta D(r_{j+1})) > 0.$$

The fact that $r_j \to 1$ follows from the demand that

$$(1+r_j)/2 < r_{j+1} < 1.$$

We also note that because of Lemma 1 $p_{n_j}(z) \to f(z)$ uniformly in every disk $|z| \leqq r$, $0 < r < 1$.

The remaining part of our argument consists of interjecting appropriate polynomials between successive pairs in the chain

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \cdots.$$

Suppose that $p(z)$ and $q(z)$ are such a pair and have degrees n and k, respectively, where $n < k$. Then $m_p(1) > 0$ and therefore $p(z)$ is univalent in $|z| < R$ for some $R > 1$. If $1 < \rho < R$ and $P(z) = p(\rho z)$ then $m_P(1) > 0$. Let

$$Q(z) = P(z) + bz^{n+1}$$

so that

$$|\Delta_Q(u, v)| \geqq m_P(1) - (n+1)|b|$$

in $|z| \leqq 1$. This implies that for all sufficiently small values of b

$$m_Q(1) \geqq \tfrac{1}{2} m_P(1)$$

so that, in particular, $Q(z)$ is univalent in E. If $A = p(E)$ and $B = q(E)$ then we have the condition $d(A, \delta B) > 0$. Because of the kind of arguments made earlier in this proof it follows that for $\rho > 1$ and sufficiently near 1 we can conclude that $p(E) \subset P(E) \subset q(E)$ from the relations $d(p(E), \delta P(E)) > 0$, $d(P(E), \delta q(E)) > 0$, and $p(0) = P(0) = q(0)$.

As $Q(z) \to P(z)$ as $b \to 0$ uniformly in $|z| \leqq 1$ it is also possible to deduce that

$$d(p(E), \delta Q(E)) > 0 \quad \text{and} \quad d(Q(E), \delta q(E)) > 0$$

for sufficiently small b. Choosing $b \neq 0$ and sufficiently small so that these several conditions hold we see that $Q(z)$ is univalent in E, has degree $n+1$ and $p(z) \subset Q(z)$ in E. Moreover, $m_Q(1) > 0$ and $d(Q(E), \delta q(E)) > 0$ so that the argument that just was applied to $p(z)$ and $q(z)$ can again be applied to $Q(z)$ and $q(z)$. Continuing in this manner we obtain the chain of univalent polynomials

$$p(z) \subset p_{n+1}(z) \subset p_{n+2}(z) \subset \cdots \subset p_{k-1}(z) \subset q(z),$$

where $p_j(z)$ has degree j. In fact, there is an infinite such chain between $p(z)$ and $q(z)$, but for our purposes we end with $p_{k-1}(z)$.

Now we return to the earlier chain

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \cdots.$$

If $n_1 \neq 1$ set $p_1(z) = a_0 + b_1 z$ where $b_1 \neq 0$ and is sufficiently small so the closed disk $|w - a_0| \leqq |b_1|$ is contained in $p_{n_1}(E)$. Between successive pairs of polynomials in this new chain interject appropriate polynomials, as just described. This produces the chain

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \cdots,$$

where $p_n(z)$ is univalent in E and is a polynomial of degree n. Moreover, a subsequence of $\{p_n(z)\}$, namely $\{p_{n_j}(z)\}$, converges to $f(z)$ uniformly in compact subsets of E. The proof will be complete if we can show that if $\{\varepsilon_j\}$ is any sequence of positive numbers such that $\varepsilon_j \to 0$ then the interjected polynomials can be so chosen to fill out the chain

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \cdots$$

so that

$$|p_n(z) - p_{n_j}(z)| < \varepsilon_j$$

for z in E and for $n_j < n < n_{j+1}$, $j = 1, 2, \ldots$.

In order to do this consider the previous argument where $p(z)$ and $q(z)$ are successive pairs in the sequence $\{p_{n_j}(z)\}$ and have degrees n and k, respectively, with $n < k$. That argument yielded real numbers R_μ and δ_μ, $\mu = n+1, \ldots, k-1$, such that $R_\mu > 1$, $\delta_\mu > 0$ and successively defining $p_\mu(z)$ by

$$p_{\mu+1}(z) = p_\mu(\rho_\mu z) + b_{\mu+1} z^{\mu+1}$$

the chain between $p(z)$ and $q(z)$ results, where ρ_μ and b_μ are restricted only by the conditions $1 < \rho_\mu < R_\mu$ and $0 < |b_\mu| < \delta_\mu$. The recursive definitions lead to the formula

$$\begin{aligned} p_\mu(z) = {} & p_n(\rho_n \rho_{n+1} \cdots \rho_{\mu-1} z) + b_{n+1}(\rho_{n+1}\rho_{n+2} \cdots \rho_{\mu-1})^{n+1} z^{n+1} \\ & + b_{n+2}(\rho_{n+2}\rho_{n+3} \cdots \rho_{\mu-1})^{n+2} z^{n+2} + \cdots + b_{\mu-1}\rho_{\mu-1}^{\mu-1} z^{\mu-1} + b_\mu z^\mu. \end{aligned} \tag{1}$$

Let ε be any given positive number. Since $p_n(z)$ is uniformly continuous in E there exist numbers ρ_μ such that $1 < \rho_\mu < R_\mu$ and the product $\rho_{n+1}\rho_{n+2} \cdots \rho_{k-1}$ is so close to 1 that

$$|p_n(z) - p_n(\rho_n \rho_{n+1} \cdots \rho_{\mu-1} z)| < \varepsilon/2$$

for z in E, $\mu=n+1,\ldots,k-1$. Let such a choice of ρ_μ be made. From (1) it follows that if $z\in E$ then

$$\begin{aligned}|p_\mu(z)-p_n(\rho_n\rho_{n+1}\cdots\rho_{\mu-1}z)| &\leqq |b_{n+1}|(\rho_{n+1}\cdots\rho_{\mu-1})^{n+1}+|b_{n+2}|(\rho_{n+2}\cdots\rho_{\mu-1})^{n+2}\\ &\qquad+\cdots+|b_{\mu-1}|\rho_{\mu-1}^{\mu-1}+|b_\mu|\\ &\leqq(\rho_{n+1}\rho_{n+2}\cdots\rho_{k-1})^{n+1}\{|b_{n+1}|+|b_{n+2}|+\cdots+|b_\mu|\}\\ &\leqq(\rho_{n+1}\rho_{n+2}\cdots\rho_{k-1})^{n+1}\{|b_{n+1}|+|b_{n+2}|+\cdots+|b_{k-1}|\}.\end{aligned}$$

Now choose the numbers b_μ, $\mu=n+1,\ldots,k-1$, so that $0<|b_\mu|<\delta_\mu$ and the last expression in the previous line does not exceed $\varepsilon/2$. This produces an appropriate chain

$$p(z)\subset p_{n+1}(z)\subset\cdots\subset p_{k-1}(z)\subset q(z)$$

satisfying the additional condition

$$|p_\mu(z)-p_n(z)|<\varepsilon$$

for z in E, $\mu=n+1,\ldots,k-1$. This argument applied to $p_{n_j}(z)$ and $p_{n_{j+1}}(z)$, with $\varepsilon=\varepsilon_j$, yields the desired conclusion that $p_n(z)\to f(z)$ as $n\to\infty$ uniformly in compact subsets of E. This completes the proof in the case $f(z)$ is not a polynomial.

The case when $f(z)$ is a univalent polynomial can be treated in an even simpler way. Suppose that $f(z)=\sum_{k=0}^N a_kz^k$, $a_N\neq 0$, is univalent in E. Let $\{r_j\}$ be a sequence of real numbers such that $0<r_1<r_2<r_3<\cdots$ and $r_j\to 1$ and set $f_j(z)=f(r_jz)$. Choose $b_1\neq 0$ so small that $p_1(z)=a_0+b_1z$ is subordinate to $f_1(z)$ say by demanding the disk $|w-a_0|\leqq|b_1|$ is covered by $f_1(E)$. The relation between $p_1(z)$ and $p_N(z)=f_1(z)$ implies the existence of polynomials $p_n(z)$, $n=2,\ldots,N-1$ univalent in E such that $p_n(z)$ has degree n and

$$p_1(z)\subset p_2(z)\subset p_3(z)\subset\cdots\subset p_{N-1}(z)\subset p_N(z),$$

as was already shown in an earlier argument in this proof.

Next choose $b_{N+1}\neq 0$ so small that

$$p_{N+1}(z)=f_2(z)+b_{N+1}z^{N+1}$$

is univalent in E and such that $d(p_N(E),\ \delta p_{N+1}(E))>0$ and $d(p_{N+1}(E),\ \delta f_3(E))>0$.

This is possible since $f_2(z)$ is univalent in a disk $|z|<R$ with $R>1$, $d(f_1(E),\ \delta f_2(E))>0$ and $d(f_2(E),\ \delta f(E))>0$.

At the next step we select $b_{N+2}\neq 0$ so small that $p_{N+2}(z)=f_3(z)+b_{N+2}z^{N+1}$ is univalent in E and such that $d(p_{N+1}(E),\ \delta p_{N+2}(E))>0$ and $d(p_{N+2}(E),\ \delta f_4(E))>0$.

This process can be continued to yield the chain

$$p_1(z)\subset p_2(z)\subset p_3(z)\subset\cdots,$$

where each $p_n(z)$ is a polynomial of degree n univalent in E. Moreover, given any sequence of positive real numbers $\{\varepsilon_j\}$ such that $\varepsilon_j\to 0$ we can choose the numbers b_{N+j} such that $|b_{N+j}|<\varepsilon_j$. This ensures that $p_n(z)\to f(z)$ uniformly in E.

THEOREM 2. *Suppose that $f(z)$ is analytic and univalent in E and maps E onto a convex domain. There exists a sequence $\{p_n(z)\}$, $n=1,2,\dots$, such that $p_n(z)$ is a polynomial of degree n that maps E one-to-one onto a convex domain,*

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \cdots$$

in E, and $p_n(z) \to f(z)$ uniformly in each compact subset of E.

Proof. A function $g(z)$ analytic in E and satisfying $g'(0) \neq 0$ maps E one-to-one onto a convex domain if and only if

$$\operatorname{Re}\left\{\frac{zg''(z)}{g'(z)}+1\right\} > 0$$

for z in E. Thus, if $0<r<1$ then such a function satisfies

$$m_g^*(r) = \min_{|z|\leqq r} \operatorname{Re}\left\{\frac{zg''(z)}{g'(z)}+1\right\} > 0.$$

The proof of this theorem is essentially the same as that given for Theorem 1 where the role of $m_g(r)$ in that theorem is now replaced by $m_g^*(r)$. Again letting $s_n(z)$ be the nth partial sum of the power series for $f(z)$, we note that $s_n'(z) \to f'(z)$ and $s_n''(z) \to f''(z)$ uniformly in $|z| \leqq r$, $0<r<1$. Consequently, there exists an integer $N=N(r)$ such that if $n \geqq N$ then

$$m_{s_n}^*(r) \geqq m_f^*(r)/2.$$

In particular, such functions $s_n(z)$ are univalent and convex in $|z| \leqq r$. Once this can be done it is clear that the other parts of the argument in Theorem 1 can be combined with this to yield (in the case $f(z)$ is not a polynomial) the polynomials $p_{n_j}(z)=s_{n_j}(r_jz)$ such that $0<r_j<1$, $r_j \to 1$, $p_{n_j}(z)$ is univalent and convex in E and has degree n_j, and $n_1<n_2<n_3<\cdots$. Also,

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \cdots$$

in E,

$$m_{p_{n_j}}^*(1) \geqq m_f^*(r_j)/2$$

and

$$d(D_{n_j}(r_j), \delta D(r_{j+1})) > 0.$$

In order to interject convex univalent polynomials between two polynomials $p(z)$ and $q(z)$ which are consecutive in the sequence $\{p_{n_j}(z)\}$ first notice that $m_p^*(1)>0$ and therefore $p(z)$ is univalent and convex in $|z|<R$ for some $R>1$. Setting $P(z) = p(\rho z)$ for $1<\rho<R$ we see that $m_P^*(1)>0$, and, consequently, if

$$Q(z) = P(z)+bz^{n+1}$$

then

$$m_Q^*(1) \geqq \tfrac{1}{2}m_P^*(1) > 0$$

for all sufficiently small values of b. This implies that $Q(z)$ is univalent and convex in E. All the other arguments of Theorem 1 are applicable. In particular, at this point the relations between $p(z)$ and $q(z)$ now hold between $Q(z)$ and $q(z)$ so that the argument may be repeated. This produces the appropriate chain

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \cdots,$$

and, as before, a more careful construction of the interjected polynomials shows that such polynomials can be chosen which also satisfy $p_n(z) \to f(z)$ uniformly in compact subsets of E.

This outlines the proof when $f(z)$ is not a polynomial, and an argument like that given at the end of Theorem 1 takes care of the case when $f(z)$ is a polynomial.

REMARKS. 1. The method used to prove Theorems 1 and 2 also applies if $f(z)$ maps E one-to-one onto a domain starlike with respect to the point $f(0)$. This yields univalent polynomials $\{p_n(z)\}$ which are starlike with respect to $f(0)$. One simply needs to note that a function $g(z)$ analytic in E and satisfying $g'(0) \neq 0$ is univalent and starlike (with respect to $g(0)$) in E providing

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)-g(0)}\right\} > 0.$$

The role of $m_g(r)$ in Theorem 1 is now replaced by

$$\min_{|z|\leq r} \operatorname{Re}\left\{\frac{zg'(z)}{g(z)-g(0)}\right\}.$$

One can readily state suitable general conditions on a given function $f(z)$ analytic and univalent in E in order that the ideas of Theorem 1 yield a proof of the existence of such polynomials which additionally have the given property of $f(z)$. One such general statement is the following.

Let $w = F(u_1, u_2, \ldots, u_n)$ be defined for certain complex n-tuples $(u_1, u_2, \ldots, u_n)$. Then $f(z)$ has a certain "given property" if

$$w = G(z) = F(z, f(z), f'(z), \ldots, f^{(n-1)}(z))$$

is a nonconstant analytic function in E and if the values of $G(z)$ are assumed to lie in a given open set. This includes as special cases starlike and convex (univalent) functions; namely, the value region for $G(z)$ in each case is $\operatorname{Re} w > 0$ and the functions $w = F(u_1, \ldots, u_n)$ are

$$w = u_1u_3/(u_2 - f(0)) \quad \text{and} \quad w = u_1u_4/u_3 + 1,$$

respectively. This idea also can be expressed if $G(z)$ is replaced by a function of several variables. In particular, this would include the case of all univalent functions (Theorem 1). There the value region is $w \neq 0$ for the function $\Delta_f(u, v)$.

Finally, we mention one more example of this type of approximation, namely for the case $f(z)$ is close-to-convex. Such functions are defined by the conditions that they be analytic in E and that there exist an analytic, univalent and convex function $g(z)$ such that $\operatorname{Re}\{f'(z)/g'(z)\} > 0$ in E. Such functions $f(z)$ are univalent in E [4]. If $0 < r < 1$ then $|g'(z)| \geqq m_g(r)$ and, consequently,

$$\left|\frac{f'(z)}{g'(z)} - \frac{s_n'(z)}{g'(z)}\right| \leqq \frac{1}{m_g(r)}\,|f'(z) - s_n'(z)|$$

for $|z| \leqq r$. This shows that

$$s_n'(z)/g'(z) \to f'(z)/g'(z)$$

uniformly in $|z| \leqq r$. Setting

$$\tilde{m}_f(r) = \min_{|z| \leqq r} \operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\}$$

we see that $\tilde{m}_f(r)$ can play the role of $m_f(r)$ in the argument of Theorem 1. This produces an appropriate chain of polynomials each of which is close-to-convex (relative to $g(z)$) in E.

2. The proof of Theorem 1 is very constructive and when combined with precise inequalities for univalent functions gives specific information on how well $p_n(z)$ approximates $f(z)$. What is sought is an estimate on

$$\max_{|z| \leqq r} |f(z) - p_n(z)|$$

in terms of n and r. We may assume that $f'(0) = 1$, as otherwise the factor $|f'(0)|$ will generally appear. It is known that $s_n(z)$ are univalent in

$$|z| < 1 - 6(\log n)/n$$

for $n \geqq 17$ [5], and, in fact, this can be slightly improved using the more recent result, $m_f(r) \geqq (1-r^2)/r^2$ (see [2, p. 120] or [3]). The quantity $d(r)$ can also be estimated from below in a fairly simple fashion to get

$$d(r) \geqq \tfrac{1}{4}((1-r)/(1+r))^2.$$

One only needs consider the interplay of these numbers $1 - 6(\log n)/n$ and $\frac{1}{4}((1-r)/(1+r))^2$, noticing that the interjected polynomials bear no significance as the sequence $\{\varepsilon_j\}$ in the proof is arbitrary. One finally applies these relations between r_j and n_j into the proof of Lemma 1 along with the estimate $|a_k| < ek$. A similar procedure is possible for other classes of univalent functions. In particular, use could be made of the results in [11, pp. 404–408].

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