TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 148, March 1970

COLLARING AN $(n-1)$-MANIFOLD IN AN n-MANIFOLD

BY
C. L. SEEBECK III[1]

1. **Introduction.** If M is a locally flat two-sided PL m-manifold in a PL $(m+1)$-manifold N then clearly [2] M can be approximated pointwise by locally flat embeddings from either side. Using a powerful result of Edwards and Kirby [7] we show conversely that M has a collar on one side if M can be approximated by locally flat embeddings from that side. As an application it follows that M is locally flat (even if M is one-sided in N) if $N\backslash M$ is 1-LC at each point of M, M can be approximated by locally flat embeddings, and $m \geqq 4$.

Let I denote the interval $[0, 1]$ and Id the identity mapping. Throughout we assume that M is a closed PL m-manifold, N is a PL n-manifold, $n=m+1$, and M is topologically embedded in N° with two sides. We choose a metric denoted by d on N and on $M\times[-1, 1]$ we choose the product metric ρ. In case A, B are subsets of N and h is a homeomorphism of N we say that h is an ε-push of (N, A) keeping B fixed if there is an isotopy h_t of N such that $h_0=\mathrm{Id}$, $h_1=h$, and for each $t\in I$ h_t is the identity on B and outside the ε-neighborhood of A and $d(h_t, \mathrm{Id})<\varepsilon$.

2. **Preliminary results.** Our proof depends heavily upon the following result of Edwards and Kirby [7].

LEMMA 1. *For each $\varepsilon>0$ there is a $\delta>0$ such that if $h\colon M\times[-\frac{1}{2},\frac{1}{2}]\to M\times[-1,1]$ is an embedding within δ of $\mathrm{Id}|M\times[-\frac{1}{2},\frac{1}{2}]$ then there is an isotopy $g_t\colon M\times[-\frac{1}{2},\frac{1}{2}]\to M\times[-1,1]$ such that $g_0=h$, $g_1|M\times 0=h|M\times 0$, and for each $t\in I$ $g_t|M\times\{-\frac{1}{2},\frac{1}{2}\}=h|M\times\{-\frac{1}{2},\frac{1}{2}\}$ and $\rho(g_t, \mathrm{Id})<\varepsilon$.*

Thus we may define an isotopy f_t on $M\times[-1,1]$ by $f_t=g_th^{-1}$ on $h(M\times[-\frac{1}{2},\frac{1}{2}])$ and $f_t=\mathrm{Id}$ elsewhere. Clearly $f_0=\mathrm{Id}$, $f_1h=\mathrm{Id}|M\times 0$, and $\rho(f_t,\mathrm{Id})<2\varepsilon$. Now let k be a homeomorphism of $M\times[-1,1]$ commuting with the projection onto M, taking $M\times\frac{3}{4}$ onto $M\times t_0$ for some small $t_0>0$, and equal the identity on $M\times[-1,0]$. Then, given $\varepsilon'>0$, if ε is small enough and $h(M\times 0)\subset M\times[-1,0]$ then $H=kf_1k^{-1}|M\times[0,t_0]$ is an embedding satisfying $H|M\times t_0=\mathrm{Id}|M\times t_0$, $H|M\times 0=h|M\times 0$, and diam $H(x\times[0,t_0])<\varepsilon'$ for each $x\in M$.

Where two disjoint, close embeddings h_0, h_1 of M into N bound a common complementary domain, let $[h_0, h_1]$ denote the one which is near M. Thus the

Received by the editors March 27, 1969.

(1) This research was supported by the National Science Foundation.

remarks above can be interpreted as saying that $[h|M\times 0, \mathrm{Id}|M\times 0]$ is an ε-product if h is sufficiently close to $\mathrm{Id}|M\times[-\frac{1}{2},\frac{1}{2}]$. Using radial engulfing we shall show that h can be chosen close to $\mathrm{Id}|M\times[-\frac{1}{2},\frac{1}{2}]$ if $h|M\times 0$ is close enough to $\mathrm{Id}|M\times 0$. First we need the codimension three version of Bing's Engulfing Theorem A of [1]. Thus we adopt the following terminology of Bing. Let $\{A_\alpha\}$ be a collection of sets in $N^{\circ n}$, O an open subset of N, and U a neighborhood of $\bar{O}$. We say that finite r-complexes of U can be pulled into O along $\{A_\alpha\}$ if for each k-dimensional polyhedron $P\subset U$ and each subpolyhedron $Q\subset O$ such that $R=\overline{P\backslash Q}$ is compact and $R\subset N^\circ$, there is a homotopy $h\colon R\times I\to N^\circ$ such that $h_0=\mathrm{Id}$, $h_1(R)\subset 0$, for each $t\in I$ $h_t|Q\cap R=\mathrm{Id}|Q\cap R$, and for each $x\in R$ $h(x\times I)$ lies in an element of $\{A_\alpha\}$.

LEMMA 2. *Suppose $r\leqq n-3$ and $\{A_\alpha\}$ is a collection of sets in N^n such that finite r-complexes in U can be pulled into O along $\{A_\alpha\}$. Then for each compact k-dimensional polyhedron $P\subset U$, each q-dimensional polyhedron $Q\subset O$ such that $R=\overline{P\backslash Q}\subset N^\circ$, k, $q\leqq r$, and each $\varepsilon>0$ there is an isotopy h_t of N^n such that $h_0=\mathrm{Id}$, $h_1(O)\supset P$, for each $t\in I$ $h_t|Q=\mathrm{Id}|Q$, and for each $x\in N^n$ either $h(x\times I)$ is a point or else lies in the ε-neighborhood of the sum of some $k+1$ elements of $\{A_\alpha\}$ if $k\leqq n-4$ and some $k+3$ elements of $\{A_\alpha\}$ if $k=n-3$.*

A proof can be constructed using piping (see Lemma 48 of [11]) and following the proof of Lemma 2.7 of [5].

PROPOSITION 3. *For each $\varepsilon>0$ there is a $\delta>0$ such that for each pair of disjoint embeddings h_0, $h_1\colon M\to N$ within δ of $\mathrm{Id}|M$, there are strong ε-deformation retractions of $[h_0, h_1]$ onto $h_0(M)$ and $h_1(M)$.*

Proof. It is easy to show using the local contractibility of M that there is a neighborhood U of M and a $\delta>0$ such that for any δ-embedding $h\colon M\to N$, U $\varepsilon/2$-retracts onto $h(M)$ (see Proposition 2 of [3]). Since M is two-sided δ can be chosen so small that $[h_0, h_1]$ is defined for disjoint δ-embeddings of M and there is an $\varepsilon/2$ retraction r or U onto $[h_0, h_1]$. Now if h_0 and h_1 are sufficiently close to $\mathrm{Id}|M$ then there is a homotopy r_t of $[h_0, h_1]$ in U with $r_0=\mathrm{Id}|[h_0, h_1]$, for each $t\in I r_t|h_1(M)=\mathrm{Id}|h_1(M)$, $d(r_t, \mathrm{Id}|M)<\varepsilon/2$, and $r_1[h_0, h_1]=h_1(M)$. Thus rr_t is a strong ε-deformation retraction of $[h_0, h_1]$ onto $h_1(M)$.

PROPOSITION 4. *For each $\varepsilon>0$ there is a $\delta>0$ such that for each pair $h_0, h_1\colon M\to N$, $n\geqq 5$, of disjoint embeddings within δ of $\mathrm{Id}|M$ each neighborhood U of $[h_0, h_1]$, and each open set $O\supset h_0(M)$ there is an ε-push H of (N, M) fixed on $h_0(M)$ and outside U such that $H[h_0, h_1]\subset O$.*

Proof. The proof is standard using Proposition 3 to construct the sets $\{A_\alpha\}$ of Lemma 2. Then using the standard dual skeleton argument H is constructed by pushing O out over the $(n-3)$-skeleton across to the dual 2-skeleton and then out over the rest of $[h_0, h_1]$. The first and last pushes are made using Lemma 2 and the middle push (see Lemma 8.1 of [9]) preserves simplexes of some triangulation of N; thus H can be made an ε-push of (N, M) fixed outside U.

LEMMA 5. *For each* $\varepsilon>0$ *there is a* $\delta>0$ *such that if* $h\colon M\times 0\to M\times[-1,1]$ *is a locally flat embedding within* δ- *of* $\mathrm{Id}|M\times 0$ *such that* $h(M\times 0)\cap M\times 0=\varnothing$ *then there is a homeomorphism* $H\colon M\times I\to[M\times 0, h]$ *such that* $H_0=\mathrm{Id}|M$, $H_1=h$, *and* diam $H(x\times I)<\varepsilon$ *for each* $x\in M$.

Outline of Proof. The proof is implicit in Wright's proof in [10]. From the remarks following Lemma 1 it is sufficient to show that h can be extended to $M\times[-\frac{1}{2},\frac{1}{2}]$ so that $\rho(h, \mathrm{Id}|M\times[-\frac{1}{2},\frac{1}{2}])<\delta$ where δ is given by Lemma 1 with ε replaced by some positive number depending on ε. Thus we need only to produce a $\delta>0$ such that for each locally flat δ-approximation h of $\mathrm{Id}|M\times 0$, each extension of h to $M\times[-\frac{1}{2},\frac{1}{2}]$ so that for each $t\in[-\frac{1}{2},\frac{1}{2}]\rho(h(x,t),(x,0))<\delta$, and each number $t_0\in(0,1)$ there is an ε-push H of $(M\times[-1,1], M\times 0)$ fixed on $h(M\times[-1,t_0])$ such that $h(M\times t_0)$ is separated from $Hh(M\times 1)$ by $M\times t'$ for some $t'\in(-\varepsilon,\varepsilon)$. However for δ chosen by Proposition 4, an embedding $h\colon M\times[-\frac{1}{2},\frac{1}{2}]\to M\times[-1,1]$ such that $\rho(h(x,t),(x,0))<\delta$ for all $t\in[-\frac{1}{2},\frac{1}{2}]$, and $0<t_0<t_1<\frac{1}{2}$ we apply Proposition 4 with $h_0(x)=(x,\lambda)$ for each $x\in M$ ($\lambda=\delta$ if $h(M\times 0)$ is separated from $M\times\delta$ by $h(M\times\frac{1}{2})$ and $\lambda=-\delta$ in the other case), $h_1(x)=h(x,t_1)$ for all $x\in M$, $U=$ the component of $M\times[-1,1]\backslash h(M\times t_0)$ containing $[h_0,h_1]$, and $O=$ the component of $M\times[-1,1]\backslash M\times\eta$ which does not contain $M\times 0$ where $0<|\eta|<|\lambda|$ and $[M\times\eta, h_0]\subset[h_0,h_1]$. Thus there is an ε-push G of $(M\times[-1,1], M\times 0)$ fixed outside U (and hence on $h(M\times t_0)$) such that $G[h_0,h_1]\subset O$ and therefore $Gh(M\times(t_0,t_1))$ contains $M\times\eta$. The embedding H is obtained now by choosing a fine partition $0=t_0<t_1<\cdots<t_k=\frac{1}{2}$ and an embedding $H'\colon M\times[-\frac{1}{2},\frac{1}{2}]\to M\times[-\delta,\delta]$ extending $h|M\times 0$ such that for each i, $H'(M\times(t_i,t_{i+1}))\supset M\times\eta_i$ for some $\eta_i\in(-\delta,\delta)$ and diam $H'(M\times[t_i,t_{i+1}])$ is small. Finally move η_i to $\pm i/2k$ by a homeomorphism F of $M\times[-1,1]$ leaving $h(M\times 0)$ fixed. We define $H=FH'$. This completes the outline of the proof.

3. The main results.

THEOREM 6. *Suppose* M *is a closed* PL *m-manifold*, $m\geqq 4$, N *is a* PL *n-manifold, and* $n=m+1$. *If* M *is topologically embedded in the interior of* N *as a two-sided subset then* M *has a collar on one side if* M *can be pointwise approximated by locally flat embeddings on that side.*

Proof. Clearly it is sufficient to show that for any $\varepsilon>0$ there is a $\delta>0$ such that for any pair of disjoint locally flat embeddings $h_0, h_1\colon M\to N$ within δ of $\mathrm{Id}|M$, there is a homeomorphism $H\colon M\times I\to[h_0,h_1]$ such that $H_0=h_0$, $H_1=h_1$, and diam $H(x\times I)<\varepsilon$ for each $x\in M$. H is constructed roughly as follows: Using a sequence of small engulfings move collars on $h_0(M)$ and $h_1(M)$ so that the whole collars are very close. Then apply local contractibility [7] to make the collars agree on a little stretch in the middle. Thus $[h_0, h_1]$ is homeomorphic to $M \times I$ by the standard push-pull technique. We now give a rigorous argument.

First select $\delta_1 > 0$ and $\eta_1 > 0$ so that $d(x, y) < \eta_1$, $x, y \in M \Rightarrow d(h(x), h(y)) < \varepsilon/3$ for any δ_1-embedding of M into N. Next pick η_2 using Lemma 5 with ε replaced by η_1. Now there are $\delta_2 > 0$ and $\delta_3 > 0$ such that for any δ_2-approximation h of Id/M and any pair (x, y) of points in $h(M)$ with $d(x, y) < 3\delta_3$, $d(h^{-1}(x), h^{-1}(y)) < \eta_2$. Finally we choose $\delta_4 > 0$ using Proposition 4 with ε replaced by $\min\{\varepsilon/3, \delta_3\}$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$.

Now let $h_0, h_1 \colon M \to N$ be disjoint locally flat embeddings within δ of $\mathrm{Id}|M$. We can now assume that δ is so small that $h_0(M)$ is two-sided; then identifying M with $M \times 0$ extend h_0 to an embedding (still denoted h_0) of $M \times [-1, 1]$ into N such that

$$x, y \in M \times [-1, 1],\ \rho(x, y) < \eta_1 \Rightarrow d(h_0(x), h_0(y)) < \varepsilon/3.$$

Let $O \subset h_0(M \times [-1, 1])$ be a neighborhood of $h_0(M)$ so small that $x, y \in 0$, $d(x, y) < 3\delta_3 \Rightarrow \rho(h_0^{-1}(x), h_0^{-1}(y)) < \eta_2$. Now apply Proposition 4 to obtain a $\min\{\varepsilon/3, \delta_3\}$-push F of (N, M), fixed on $h_0(M)$, such that $F[h_0, h_1] \subset 0$. Then $d(h_0(x), Fh_1(x)) \leq d(h_0(x), h_1(x)) + d(h_1(x), Fh_1(x)) < 3\delta_3$. Thus $\rho(x, h_0^{-1}Fh_1(x)) < \eta_2$. Therefore we can apply Lemma 5 and obtain an embedding $G \colon M \times I \to M \times [-1, 1]$ such that $G_0 = \mathrm{Id}|M$, $G_1 = h_0^{-1}Fh_1$, and $\operatorname{diam} G(x \times I) < \eta_1$. Thus $\operatorname{diam} h_0G(x \times I) < \varepsilon/3$. Now define H to be $F^{-1}h_0G$. Then $H_0 = F^{-1}h_0G_0 = F^{-1}h_0 = h_0$, $H_1 = F^{-1}h_0G_1 = F^{-1}h_0h_0^{-1}Fh_1 = h_1$, and for each $x \in M$, $H(x \times I) = F^{-1}(h_0G(x \times I))$ has diameter $< \varepsilon$. This completes the proof of Theorem 6.

For the proof of the next theorem we need one more preliminary result in order to apply Theorem 2.

PROPOSITION 7. *Suppose that the closed m-manifold M embedded in the interior of* N^n, $n = m+1$, *separates N into two components U and V and U is* 1-ULC. *For each* $\varepsilon > 0$ *there is a neighborhood* 0 *of* $U \cup M$ *such that for each closed set* $C \subset U$ *there is a closed set* B, $C \subset B \subset U$, *and a homotopy* h_t *of O in N such that*:

1. $h_0 = \mathrm{Id}$,
2. $h_1(O) \subset U$,
3. $h_t|B = \mathrm{Id}|B$ *for each* $t \in I$,
4. $h_t(O \backslash B) \cap C = \varnothing$ *for each* $t \in I$, *and*
5. $d(h_t, \mathrm{Id}) < \varepsilon$ *for each* $t \in I$.

Proof. The proof is in two steps.

Step 1. There is a neighborhood P of $N \backslash U$ such that for any closed set $C \subset U$ there is a closed neighborhood D of $N \backslash U$, $C \cap D = \varnothing$, and an $\varepsilon/2$-map r of P into $N \backslash C$ such that $r|D = \mathrm{Id}|D$ and $r(P \cap U) \subset U \backslash C$.

Step 2. There is a neighborhood O' of $U \cup M$ such that for any closed set $B \subset U$ there is a closed set E, $B \subset E \subset U$, and a strong $\varepsilon/2$-retraction r_t of O' onto E such that $r_t(N \backslash E) \subset P$ for each $t \in I$.

Now to complete the proof let $O=O'$ from Step 2 where B=closure of $U\backslash D$. Then define $h_t(x)=x$ for $x \in B$ and $t \in I$ and $h_t(x)=r\ r_t(x)$ for $x \in O \cap D$. Since $r_t(x) \in P$ for $x \in O \cap D$, $h_t(x)$ is defined for all $t \in I$ and $x \in O \cap D$. Since $r|B \cap D = r_t|B \cap D = \mathrm{Id}|B \cap D$, h_t is continuous for all $t \in I$. It is clear that h_t satisfies the conclusion of the proposition. Next we prove Step 1. Since M is an ANR and separates N there is a polyhedral neighborhood P of $N\backslash U$ and an $\varepsilon/8$-retraction f of P onto $N\backslash U$. Now let $\eta=\min\{\varepsilon/8, d(M, C)/2\}$. Since U is 1-ULC, U is ULC^{n-1}. Thus there is a sequence $0<\varepsilon_1<\varepsilon_2<\cdots<\varepsilon_n=\eta$ such that each map $f: s^i \to U$ with $\operatorname{diam} f(s^i)<2\varepsilon_i$ can be extended to a map of the $(i+1)$-ball B^i into U such that $\operatorname{diam} f(B^{i+1})<\varepsilon_{i+1}$. Now take a triangulation T' of P with mesh less than $2\varepsilon_1/3$ and so small that for each $\sigma \in T'$ $\operatorname{diam} f(\sigma)<2\varepsilon_1/3$. Since $f|M=\mathrm{Id}|M$ there is a neighborhood D' of $N\backslash U$ such that $d(f|D', \mathrm{Id}|D')<2\varepsilon_1/3$. Choose a refinement T of T' so fine that the simplicial neighborhood D of $N\backslash U$ in T' is contained in D'. Define $r|D=\mathrm{Id}|D$ and extend r skeletonwise to P as follows. For each vertex $\sigma^\circ \in P\backslash D$ pick $r(\sigma^\circ) \in U$ satisfying $d(r(\sigma^\circ), f(\sigma^\circ))<2\varepsilon_1/3$. Then for each 1-simplex $\sigma' \in P\backslash D$, $\operatorname{diam} r(\sigma')<2\varepsilon_1$. Suppose r has been extended to $r: D \cup |T^k| \to N$ so that for each $\sigma^{k+1} \in T$, $\operatorname{diam} r(\dot\sigma^{k+1})<2\varepsilon_{k+1}$. Thus for each $\sigma^{k+1} \in T$ we can extend $r|\dot\sigma^{k+1}$ to σ^{k+1} so that $\operatorname{diam} r(\sigma^{k+1})<2\varepsilon_{k+2}$. By induction we have r defined on all of P so that $\operatorname{diam} r(\sigma)<\eta$ for all simplexes $\sigma \in T$. However, $d(r(\sigma), N\backslash U)<2\varepsilon_1/3<\eta$ and $\operatorname{diam} r(\sigma)<\eta$ imply that $r(\sigma)\subset N\backslash C$ for each $\sigma \in T$. Moreover for each point $p \in P$ there is a vertex v of T such that

$$d(p, r(P)) \leqq d(p, v)+d(v, r(v))+d(r(v), r(p)) < \eta+(\varepsilon/8+\eta)+\eta < \varepsilon/2.$$

This completes the proof of Step 1. Step 2 can be proved similarly.

Lemma 8. *Suppose that M^m is a closed two-sided submanifold of the interior of N^n, $n=m+1 \geqq 5$. If $N\backslash M$ is* 1-ULC *then for each $\varepsilon>0$ there is a* PL *ε-push H of (N, M) such that $H(M) \cap M=\varnothing$.*

Proof. Suppose that W is a connected open neighborhood of M which is separated into U and V by M. Then apply Proposition 7 with N replaced by W and ε by $\varepsilon/3n$ to obtain polyhedral neighborhoods O_1 of $W\backslash U$ and O_2 of $W\backslash V$. Apply Proposition 7 for each closed subset of $W\backslash M$ containing $W\backslash O_1 \cap O_2$ and let $\{A_\alpha\}$ be the tracks of all points under all such homotopies of $O_1 \cap O_2$ into $O_1 \cap U$ and $O_2 \cap V$. Now take a triangulation T of $O_1 \cap O_2$ with mesh less that $\varepsilon/3$ and apply Lemma 2 with P replaced by $|T^{n-3}|$, N by O_1, O by $O_1 \cap U$ and U by $O_1 \cap O_2$. Thus there is an $\varepsilon/3$-push H_1 of (W, M) fixed on $U\backslash O_1$ such that $H_1(U) \supset T^{n-3}$. Similarly there is an $\varepsilon/3$-push H_2 of (W, M) fixed on $V\backslash O_2$ such that $H_2(V)\supset \tilde{T}^2$ the dual 2-skeleton of T^{n-3}. Using Lemma 8.1 of [9] there is an $\varepsilon/3$-push H_3 of (W, M) such that $H_3H_1(U) \cup H_2(V)=W$. Thus $H_2^{-1}H_3H_1(U) \cup V = W\supset M$. Let $H=H_2^{-1}H_3H_1$. Then H is an ε-push of (N, M) such that $H(M)\subset V$.

Clearly $H^{-1}(M)\subset U$ therefore if M can be approximated sufficiently close by locally flat embeddings then it can be approximated from both sides and so M is bicollared. In fact, we can construct a double cover $\tilde{N}$ of N in the case that M is one-sided in N and $\tilde{M}$ (the part of $\tilde{N}$ covering M) is two-sided in $\tilde{N}$. Moreover, if M can be approximated then so can $\tilde{M}$. Therefore $\tilde{M}$ is bicollared and so M has a closed normal 1-disk bundle neighborhood. Thus we have the following.

THEOREM 9. *Suppose that M is a closed (possibly one-sided) m-manifold in the interior of the n-manifold N, $n=m+1\geqq 5$. If M can be pointwise approximated by locally flat embeddings and $N\backslash M$ is* 1-ULC, *then M has a normal* 1-*disk bundle neighborhood.*

We remark in conclusion that it follows from the annulus conjecture [8], Connell's approximation theorem [6], and [7] that any locally flat $(n-1)$-sphere in S^n, $n\geqq 5$, is ε-tame; thus for the case of spheres, Theorems 6 and 9 can be strengthened to ε-taming results. In fact, since the locally flat side approximations become levels in the collar, there is an ε-taming result if M can be approximated by PL embeddings from both sides. Therefore it follows from the results here and in [4] that a closed PL m-manifold M in the interior of an n-manifold N is ε-tame if its complement is 1-ULC, it can be approximated by PL embeddings, $n\geqq 5$, and $m\neq n-2$. Moreover, given one of these embeddings $h: M\to N$ and an $\varepsilon>0$ there is a $\delta>0$ such that if g is also one that is within δ of h then there is an ε-push p of $(N, h(M))$ such that $h=pg$.

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MICHIGAN STATE UNIVERSITY,
EAST LANSING, MICHIGAN