BANACH SPACES OF LIPSCHITZ FUNCTIONS AND VECTOR-VALUED LIPSCHITZ FUNCTIONS([1]),([2])

BY

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Introduction. Given a metric space (S, d) and a Banach space E, let $\mathrm{Lip}_E (S, d)$ denote the set of bounded functions $f: S \to E$ such that

$$\|f\|_d = \sup \{\|f(s)-f(t)\|d^{-1}(s, t) \mid s, t \in S, s \neq t\}$$

is finite. For $f \in \mathrm{Lip}_E (S, d)$, let $\|f\|_\infty = \sup \{\|f(s)\| \mid s \in S\}$ and

$$\|f\| = \max (\|f\|_\infty, \|f\|_d).$$

It is routine to show that $\|\cdot\|$ is a norm for which $\mathrm{Lip}_E (S, d)$ is a Banach space. When E is the set of real or complex numbers, we drop the subscript and write $\mathrm{Lip} (S, d)$.

We define $\mathrm{lip}_E (S, d)$ to be the set of functions f in $\mathrm{Lip}_E (S, d)$ such that

$$\lim_{d(s,t) \to 0} \|f(s)-f(t)\|d^{-1}(s, t) = 0,$$

i.e., for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|f(s)-f(t)\|d^{-1}(s, t) < \varepsilon$ whenever $0 < d(s, t) < \delta$. Then $\mathrm{lip}_E (S, d)$ is a closed subspace of $\mathrm{Lip}_E (S, d)$. When (S, d) is locally compact, we will have occasion to consider the space $\mathrm{lip}_E^0 (S, d)$ of functions f in $\mathrm{lip}_E (S, d)$ which vanish at infinity; i.e., for every $\varepsilon > 0$, there exists a compact $K \subset S$ such that $\|f(s)\| < \varepsilon$ for $s \notin K$. As before, when E is the scalar field we write $\mathrm{lip} (S, d)$ and $\mathrm{lip}^0 (S, d)$.

When $0 < \alpha < 1$ and d is a metric, d^α is also a metric. It is easy to see that if $0 < \alpha < \beta \leq 1$, then $\mathrm{Lip} (S, d^\beta) \subset \mathrm{lip} (S, d^\alpha)$.

In [9, p. 401] it is stated that little is known about the Banach space properties of spaces of Lipschitz functions. Since then, several papers have appeared, yielding much information along these lines. In this paper we fill some of the remaining gaps and extend some known results.

In §1, we show how spaces of functions satisfying a very general "Lipschitz type" condition can be considered as Lipschitz spaces.

In §2, the weak Cauchy sequences in $\mathrm{lip} (S, d)$ are characterized and it is shown that for $0 < \alpha < 1$ and $E \neq \{0\}$, neither $\mathrm{Lip}_E (S, d^\alpha)$ nor $\mathrm{lip}_E (S, d^\alpha)$ is weakly sequentially complete unless S is finite. This result is then used to gain further information about the extreme points of the dual ball of $\mathrm{Lip} (S, d^\alpha)$.

In §3, the relatively compact subsets of lip (S, d) are characterized when (S, d) is compact and some other miscellaneous results concerning compactness in Lipschitz spaces are also derived.

In §4, it is proved that whenever E is a dual space, so is $\text{Lip}_E (S, d)$. Weak* sequential convergence in $\text{Lip}_E (S, d)$ is then characterized.

DEFINITION. A metric space is called *finitely compact* if its closed bounded subsets are compact.

Assume (S, d) is finitely compact and let $\text{lip}^0 (S, d)$ denote the space of functions in lip (S, d) that vanish at infinity. In [13] it was shown that, for real-valued functions, $\text{Lip}(S, d^\alpha)$ is isometrically isomorphic with the second dual of $\text{lip}^0 (S, d^\alpha)$, $0 < \alpha < 1$. For the complex case, an extra hypothesis on (S, d) was required. By using the fact that Lip (S, d) is a dual space, we are able to provide a different proof which shows the extra hypothesis to be unnecessary.

Finally, in §5, we prove that for a wide class of metric spaces (S, d) and Banach spaces E, $\text{Lip}_{E''} (S, d^\alpha)$ and $\text{lip}_E^0 (S, d^\alpha)''$ are canonically isometrically isomorphic, where A' and A'' denote the dual and second dual respectively of the Banach space A.

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1. Before the main body of work is begun, a few remarks seem in order about spaces more general than Lip (S, d). Fraser [10], for example, has considered spaces of functions satisfying a modulus of continuity condition. We show in this section that generalized Lipschitz spaces such as these are themselves Lipschitz spaces.

Let S be any set and ρ any nonnegative function on $S \times S$. Let $L(S, \rho)$ denote the set of bounded real or complex valued functions on S such that $|f(s) - f(t)| \le M\rho(s, t)$ for some $M \ge 0$ and all $(s, t) \in S \times S$. If $\|f\|_\rho$ denotes the smallest nonnegative constant M for which $|f(s) - f(t)| \le M\rho(s, t)$ for all $(s, t) \in S \times S$, and $\|f\|_\infty$ denotes the sup-norm of f, then $\|f\|$ is defined to be max $(\|f\|_\infty, \|f\|_\rho)$.

THEOREM 1.1. $L(S, \rho)$ *endowed with* $\|\cdot\|$ *is a Banach space.*

Proof. We let $W_\rho = \{(s, t) \in S \times S \mid \rho(s, t) > 0\}$, and assume $W_\rho \ne \varnothing$ since, if it were, $\rho = 0$ and $L(S, \rho)$ consists only of constants. Define $\tilde{f}(s) = f(s)$ for $s \in S$ and $\tilde{f}(s, t) = (f(s) - f(t))/\rho(s, t)$ for $\rho(s, t) > 0$. Then $\tilde{f} \in B(S \cup W_\rho)$, the bounded functions on $S \cup W_\rho$, and $\|\tilde{f}\|_\infty = \|f\|$. Since $f \to \tilde{f}$ is linear, $L(S, \rho)$ is a normed space. It is routine to show that $\{\tilde{f} \mid f \in L(S, \rho)\}$ is closed in $B(S \cup W_\rho)$ and hence that $L(S, \rho)$ is complete.   Q.E.D.

We remark that ρ may be assumed to be symmetric; i.e., $\rho(s, t) = \rho(t, s)$ for all $(s, t) \in S \times S$. This is so because ρ may be replaced with ρ_1 where

$$\rho_1(s, t) = \inf (\rho(s, t), \rho(t, s)).$$

Then $L(S, \rho) = L(S, \rho_1)$ and $\|\cdot\|_{\rho_1} = \|\cdot\|_\rho$.

THEOREM 1.2. *There is a pseudo-metric d on S such that $L(S, d) = L(S, \rho)$ and $\| \cdot \|_d = \| \cdot \|_\rho$. d is a metric if and only if $L(S, \rho)$ separates points.*

Proof. Define $d(s, t) = \sup \{ |f(s) - f(t)| \mid f \in L(s, \rho), \|f\|_\rho \leq 1 \}$. It is easy to see that d is a pseudo-metric and that $d \leq \rho$. If $f \in L(S, d)$, then $|f(s) - f(t)| \leq \|f\|_d d(s, t) \leq \|f\|_d \rho(s, t)$ for all $(s, t) \in S \times S$. Thus, $f \in L(S, \rho)$ and $\|f\|_\rho \leq \|f\|_d$. If $f \in L(S, \rho)$ then $|f(s) - f(t)| \leq \|f\|_\rho \rho(s, t)$. Let $g = f / \|f\|_\rho$. Then $g \in L(S, \rho)$ and $\|g\|_\rho \leq 1$. Hence, by definition of d, $|g(s) - g(t)| \leq d(s, t)$, so $|f(s) - f(t)| \leq \|f\|_\rho d(s, t)$. Therefore, $f \in L(S, d)$ and $\|f\|_d \leq \|f\|_\rho$. Hence $L(S, d) = L(S, \rho)$ and $\| \cdot \|_d = \| \cdot \|_\rho$. The definition of d makes the second assertion in the theorem obvious.   Q.E.D.

If (S, d) is a pseudo-metric space, then $s \sim t \Leftrightarrow d(s, t) = 0$ defines an equivalence relation on S. Let S' denote the set of equivalence classes and define $d'(s', t') = d(s, t)$ where s' is the class containing s. Then d' is well-defined on $S' \times S'$ and is a metric on S'. Furthermore, if $f \in L(S, d)$, then $f' \in \mathrm{Lip}\,(S', d')$, where $f'(s') = f(s)$ and $\|f'\|_{d'} = \|f\|_d$. The mapping $f \to f'$ is easily seen to be linear and to preserve multiplication. From these observations, we obtain the following

COROLLARY 1.3. *Given any nonnegative function ρ on $S \times S$, there is a metric space (X, d) and an isometric isomorphism of $L(S, \rho)$ onto $\mathrm{Lip}\,(X, d)$.*

In [19] and [20] Sherbert showed that $\mathrm{Lip}\,(S, d)$ endowed with the norm $\|| \cdot \|| = \| \cdot \|_\infty + \| \cdot \|_d$ is a Banach algebra and considered its ideal structure and Banach algebra properties. $L(S, \rho)$ under $\| \cdot \|_\infty + \| \cdot \|_\rho = \|| \cdot \||$ is likewise a Banach algebra. (Note that $\|| \cdot \||$ and $\| \cdot \|$ are equivalent norms.) The preceding theorems show that there is a metric space (X, d) such that $L(S, \rho)$ and $\mathrm{Lip}\,(X, d)$ are isometrically isomorphic as Banach algebras.

Since most interesting results occur when the underlying metric space is compact, the following theorem is useful.

THEOREM 1.4. *Let S be a compact topological space. Let ρ be nonnegative and symmetric. If $\rho(s, s) = 0$ for each $s \in S$ and if the function $t \to \rho(t, s)$ is continuous at s for each $s \in S$, then (S, d) is compact, where d is the pseudo-metric in 1.2.*

Proof. If $\{s_i\}$ is a net in S such that $s_i \to s$, then $\rho(s_i, s) \to \rho(s, s) = 0$. But $d \leq \rho \Rightarrow d(s_i, s) \to 0$. Hence the identity mapping $I: S \to (S, d)$ is continuous. Thus, (S, d) is compact.

COROLLARY 1.5. *If S is a compact topological space and if ρ is a function satisfying the conditions of 1.4, then there is a compact metric space (X, d) such that $\mathrm{Lip}\,(X, d)$ and $L(S, \rho)$ are isometrically isomorphic as Banach spaces and Banach algebras.*

Proof. This follows immediately from 1.3 and 1.4.

Thus the study of the general Banach space or Banach algebra structure of $L(S, \rho)$ is accomplished by restricting attention to Lipschitz spaces $\mathrm{Lip}\,(S, d)$.

2. In this section we characterize the weak Cauchy sequences in $\mathrm{lip}\,(S, d)$ and prove that for $0 < \alpha < 1$ neither $\mathrm{Lip}\,(S, d^\alpha)$ nor $\mathrm{lip}\,(S, d^\alpha)$ is weakly sequentially

complete unless S is a finite set. As an application of this result, we obtain some further information about the nature of the extreme points of the dual ball of Lip (S, d^α).

THEOREM 2.1. *Let (f_n) be a sequence in* lip (S, d) *where (S, d) is any metric space. The following assertions are equivalent*:

(a) (f_n) *is a weak Cauchy sequence.*

(b) $\{\|f_n\|_d \mid n = 1, 2, \ldots\}$ *is bounded and (f_n) is a weak Cauchy sequence in $BC(S)$, the space of bounded continuous functions on S with $\|\cdot\|_\infty$.*

(c) (f_n) *is bounded and every sequence (s_m) in S has a subsequence (s_{m_i}) such that* $\lim_{n \to \infty} \lim_{i \to \infty} f_n(s_{m_i})$ *exists.*

Proof. (a) $\Rightarrow$ (b) because $\|\cdot\|_\infty \leq \|\cdot\|$ and weak Cauchy sequences are bounded. (b) $\Rightarrow$ (c) is immediate from [9, IV, 12.42, p. 346]. We use this same theorem to prove that (c) $\Rightarrow$ (a). Thus, assume (c) holds. We consider the linear isometric embedding of lip (S, d) in $BC(S \cup W)$ defined by $f \to \tilde{f}$ where

$$W = \{(s, t) \in S \times S \mid s \neq t\},$$

$\tilde{f}(s) = f(s)$ for $s \in S$, and $\tilde{f}(s, t) = (f(s) - f(t))/d(s, t)$ for $(s, t) \in W$. Now, to show (f_n) is weakly Cauchy, we must show $(\tilde{f}_n)$ is weakly Cauchy in $BC(S \cup W)$. This will be done when we show that every sequence $\{(s_m, t_m)\}$ in W has a subsequence $\{(s_{m_i}, t_{m_i})\}$ such that $\lim_{n \to \infty} \lim_{i \to \infty} \tilde{f}_n(s_{m_i}, t_{m_i})$ exists. If

$$\sup \{d(s_m, t_m) \mid m = 1, 2, \ldots\} = +\infty,$$

then there is a subsequence $\{(s_{m_i}, t_{m_i})\}$ such that $\lim_{i \to \infty} d(s_{m_i}, t_{m_i}) = +\infty$. Then,

$$\lim_{i \to \infty} |\tilde{f}_n(s_{m_i}, t_{m_i})| \leq 2\|f_n\|_\infty \lim_{i \to \infty} 1/d(s_{m_i}, t_{m_i}) = 0, \qquad \forall n.$$

Hence we may assume that $\{d(s_m, t_m)\}$ is bounded. Then there is a subsequence $\{(s_{m_i}, t_{m_i})\}$ such that $\lim_{i \to \infty} d(s_{m_i}, t_{m_i}) = \delta$ exists. If $\delta = 0$, then, since $f_n \in$ lip (S, d), $\lim_{i \to \infty} \tilde{f}_n(s_{m_i}, t_{m_i}) = 0$ for each n. We may therefore assume $\delta > 0$. By (c) there is a subsequence $(s_{m_{i_k}}) = (s_k')$ of (s_{m_i}) such that $\lim_{n \to \infty} \lim_{k \to \infty} f_n(s_k') = L_1$ exists. Consider the corresponding subsequence $(t_{m_{i_k}}) = (t_k')$ of (t_{m_i}). Again, by (c), there is a subsequence $(t_{k_l}') = (t_l'')$ of (t_k') such that $\lim_{n \to \infty} \lim_{l \to \infty} f_n(t_l'') = L_2$ exists. Now, let $(s_{k_l}') = (s_l'')$ be the corresponding subsequence of (s_k') and consider the sequence $\{(s_l'', t_l'')\}$. It is a subsequence of $\{(s_m, t_m)\}$ and $\lim_{n \to \infty} \lim_{l \to \infty} \tilde{f}_n(s_l'', t_l'') = (L_1 - L_2)/\delta$ exists.   Q.E.D.

COROLLARY 2.2. *If (S, d) is compact and (f_n) is a sequence in* lip (S, d), *the following are equivalent.*

(a) (f_n) *is a weak Cauchy sequence.*

(b) (f_n) *is bounded and $\lim_{n \to \infty} f_n(s)$ exists for each $s \in S$.*

Proof. By [9, IV, 13.40, p. 345], (b) above is equivalent with 2.1(b) when (S, d) is compact.   Q.E.D.

We note that 2.2 was essentially established by Jenkins [13, Lemma 4.9]. However, his proof is quite different since it depends upon the fact that the point evaluations span a dense subspace of lip $(S, d)'$ when (S, d) is compact.

If K is a subset of S and $s \in S$, let $d(s, K) = \inf \{d(s, t) \mid t \in K\}$. It is easy to see that the function $f(s) = d(s, K)$ has the property that $\|f\|_d = 1$. Letting $a \wedge b = \inf (a, b)$, we see that the function $g(s) = 1 \wedge d(s, K)$ belongs to Lip (S, d) and that $\|g\| \leq 1$.

COROLLARY 2.3. *Let $0 < \alpha < 1$ and $K \subseteq S$. Let (β_n) be a sequence converging to β with $\alpha \leq \beta < \beta_n \leq 1$. Define $f_n(s) = d^{\beta_n}(s, K) \wedge 1$. Then (f_n) is a weak Cauchy sequence in lip (S, d^α).*

Proof. Since $\beta_n > \alpha$, the above remarks and the discussion in the introduction show that $f_n \in$ lip (S, d^α) for each n. If $0 < \nu < \mu \leq 1$ and $f \in$ Lip (S, d^μ), then the norm $\|f\|_\nu$ of f in Lip (S, d^ν) is shown in [13, Lemma 2.4] to be not greater than $2^{(\mu - \nu)/\nu}\|f\|_\mu$, where $\|f\|_\mu$ is the norm of f in lip (S, d^μ). The norm $\|f_n\|$ of each f_n taken in lip (S, d^α) is therefore dominated by $2^{(\beta_n - \alpha)/\alpha}$, since the norm of f_n in Lip (S, d^{β_n}) is not larger than 1. Thus, $\|f_n\| \leq 2^{(\beta_n - \alpha)/\alpha} \leq 2^{(1 - \alpha)/\alpha}$. Hence (f_n) is bounded. Let (s_m) be a sequence in S. If infinitely many of the s_m are such that $d(s_m, K) \geq 1$, then we can extract a subsequence (s_{m_i}) such that $f_n(s_{m_i}) = 1$ for all n and all i. Hence we may assume that $d(s_m, K) \leq 1$ for all m. Then there is a subsequence (s_{m_i}) such that $\lim_{i \to \infty} d(s_{m_i}, K) = L \leq 1$. Then

$$\lim_{n \to \infty} \lim_{i \to \infty} f_n(s_{m_i}) = \lim_{n \to \infty} L^{\beta_n} = L^\beta. \qquad \text{Q.E.D.}$$

LEMMA 2.4. *Let K be a proper, closed, nonempty subset of S and define $f(s) = d(s, K) \wedge 1$. If $f \in$ lip (S, d), then $d(K, S \sim K) > 0$.*

Proof. If $f \in$ lip (S, d), then there is a number δ, with $0 < \delta < 1$, such that $|f(s) - f(t)|/d(s, t) \leq \frac{1}{2}$ for $0 < d(s, t) < \delta$. Suppose there is a point $s_0 \in S$ such that $0 < d(s_0, K) < \delta$. Then the set $K_0 = \{t \in K \mid d(s_0, t) < \delta\} \neq \varnothing$ and $d(s_0, K) = d(s_0, K_0)$. Thus, for each $t \in K_0$, we have

$$\frac{1}{2} \geq \frac{|f(s_0) - f(t)|}{d(s_0, t)} = \frac{f(s_0)}{d(s_0, t)} = \frac{d(s_0, K)}{d(s_0, t)}.$$

Hence $\frac{1}{2} \geq d(s_0, K)/\inf \{d(s_0, t) \mid t \in K_0\} = 1$, a contradiction. Thus, if $f \in$ lip (S, d), there can be no point s_0 with $0 < d(s_0, K) < \delta$. Since K is closed, this implies that $d(K, S \sim K) \geq \delta > 0$.   Q.E.D.

DEFINITION. A metric space (S, d) is called *uniformly discrete* if there is a number $\delta > 0$ such that $d(s, t) \geq \delta$ for all $s, t \in S$ with $s \neq t$.

LEMMA 2.5. *The following are equivalent*:
(a) (S, d) *is uniformly discrete.*
(b) lip $(S, d) = B(S)$.
(c) lip $(S, d) =$ Lip (S, d).

(d) *For each proper, closed, nonempty subset K of S, the function $s \to d(s, K) \wedge 1$ belongs to* lip (S, d).

Proof. (a) $\Rightarrow$ (b). Let $\delta > 0$ be such that $d(s, t) \geq \delta$ for $s \neq t$. If $f \in B(S)$, then $|f(s) - f(t)|/d(s, t) \leq 2\|f\|_\infty/\delta$ for $s \neq t$. Thus $f \in$ Lip (S, d). For functions

$$f \in \text{Lip } (S, d),$$

the condition

$$\lim_{d(s,t) \to 0} \frac{|f(s) - f(t)|}{d(s, t)} = 0$$

is vacuously satisfied. (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are trivial. Assume (d) holds. Since points are closed, it follows from 2.4 that (S, d) is discrete. Thus all subsets of S are closed. Hence, from 2.4 we deduce that for any proper nonempty subset K of S, $d(K, S \sim K) > 0$. If (a) were false, then we could choose two points s_1, t_1 such that $0 < d(s_1, t_1) < 1$. Since $d(\{s_1, t_1\}, S \sim \{s_1, t_1\}) > 0$, we could choose points s_2, t_2 distinct from s_1 and t_1 such that $0 < d(s_2, t_2) < \frac{1}{2}$. Continuing this process, we obtain disjoint sets $\{s_1, s_2, \ldots\}$ and $\{t_1, t_2, \ldots\}$ the distance between which is zero, a contradiction.   Q.E.D.

We are now ready to prove the main theorem of this section. We recall that a Banach space E is called weakly sequentially complete (w.s.c.) if every weak Cauchy sequence has a weak limit in E.

THEOREM 2.6. *For any metric space (S, d) and $0 < \alpha < 1$, neither* lip (S, d^α) *nor* Lip (S, d^α) *is w.s.c. unless S is a finite set.*

Proof. Suppose lip (S, d^α) is w.s.c. Let K be a nonempty proper closed subset of S, and let (β_n) be a sequence converging to α with $\alpha < \beta_n \leq 1$. Define $f_n(s) = 1 \wedge d^{\beta_n}(s, K)$. By 2.3, (f_n) is a weak Cauchy sequence in lip (S, d^α). Thus there is a function f in lip (S, d^α) such that (f_n) converges weakly to f. In particular, $\lim_{n \to \infty} f_n(s) = f(s)$ for each $s \in S$. Thus, $f(s) = d^\alpha(s, K) \wedge 1$. But $f \in$ lip (S, d^α), so, by 2.5, we have that (S, d^α) is uniformly discrete (say $d^\alpha(s, t) \geq \delta > 0$ if $s \neq t$) and lip $(S, d^\alpha) = B(S)$. For $f \in B(S)$, we have $|f(s) - f(t)|/d^\alpha(s, t) \leq (2/\delta)\|f\|_\infty$ if $s \neq t$. Hence $\|f\|_{d^\alpha} \leq (2/\delta)\|f\|_\infty$. Thus, the Lipschitz norm max $(\|\cdot\|_\infty, \|\cdot\|_{d^\alpha})$ and the sup-norm $\|\cdot\|_\infty$ are equivalent. Therefore $B(S)$ is w.s.c. But $B(S)$ is never w.s.c. unless S is a finite set (see [9, IV,13.5, p. 339]). Thus, lip (S, d^α) is never w.s.c. unless S is a finite set. If Lip (S, d^α) is w.s.c., then so is every closed subspace. In particular, lip (S, d^α) is w.s.c., so S is a finite set.   Q.E.D.

REMARK. If E is a Banach space and if $x \in E$ with $\|x\| = 1$, then the mapping $f \to f \cdot x$, where $(f \cdot x)(s) = f(s)x$, is an isometric embedding of Lip (S, d^α) [resp. lip (S, d^α)] in Lip$_E$ (S, d^α) [resp. lip$_E$ (S, d^α)]. Thus, if $E \neq \{0\}$, neither Lip$_E$ (S, d^α) nor lip$_E$ (S, d^α) is w.s.c. unless S is finite.

In [6], deLeeuw characterized the extreme points of the dual ball of lip (S, d^α), $0 < \alpha < 1$, for a special compact space (S, d). Jenkins [13] generalized this result to arbitrary compact metric spaces, and gave an incomplete description of the

extreme points of the dual ball of Lip (S, d^α). We state these results below in 2.7 for reference.

As in the proof of 2.1, we consider Lip (S, d^α) as a subspace of $BC(S \cup W)$ by the linear, isometric embedding $f \to \tilde{f}$, where $W = \{(s, t) \in S \times S \mid s \neq t\}$, $\tilde{f}(s) = f(s)$ if $s \in S$, and $\tilde{f}(s, t) = (f(s) - f(t))/d^\alpha(s, t)$ for $(s, t) \in W$. Now, each function $\tilde{f}$ extends to an element $\hat{f}$ of $C(S \cup \beta W)$, where βW is the Stone-Čech compactification of W. Let ε_s denote the evaluation functional $f \to f(s)$. Then we have:

Theorem 2.7 (deLeeuw-Jenkins). *Let* $Q_1 = \{\lambda \varepsilon_s \mid s \in S, |\lambda| = 1\}$ *and*

$$Q_2 = \{\lambda d^{-\alpha}(s, t)(\varepsilon_s - \varepsilon_t) \mid 0 < d^\alpha(s, t) < 2, |\lambda| = 1\}.$$

Then for S compact and $0 < \alpha < 1$, the set of extreme points of the unit ball of lip $(S, d^\alpha)'$ *is* $Q_1 \cup Q_2$. *If* $w \in \beta W \sim W$, *let* ε_w *denote the element of* Lip $(S, d^\alpha)'$ *defined by* $\varepsilon_w(f) = \hat{f}(w)$. *Then the set of extreme points of the unit ball of* Lip $(S, d^\alpha)'$ *is* $Q_1 \cup Q_2 \cup Q_0$, *where Q_0 is some subset of* $\{\lambda \varepsilon_w \mid w \in \beta W \sim W, |\lambda| = 1\}$.

The problem left open in [13, p. 44] is that of describing Q_0. In particular, the question was raised as to whether Q_0 is empty. Theorem 2.6 provides an answer to the latter problem.

Theorem 2.8. *Let (S, d) be a compact metric space and $0 < \alpha < 1$. If S is an infinite set, then* $Q_0 \neq \varnothing$.

Proof. Suppose $Q_0 = \varnothing$. Let (f_n) be a weak Cauchy sequence in Lip (S, d^α). Then (f_n) is bounded, say, $\|f_n\| \leq M$ for each n, and $\lim_{n \to \infty} f_n(s) = f(s)$ exists for each $s \in S$. Thus, for each $(s, t) \in W$, we have $M \geq \lim_{n \to \infty} |\tilde{f}_n(s, t)| = |\tilde{f}(s, t)|$. Therefore, $f \in$ Lip (S, d^α) and (f_n) converges to f at each point of $Q_1 \cup Q_2$. Now, if $Q_0 = \varnothing$, then (f_n) is bounded and converges at each extreme point of the unit ball of Lip $(S, d^\alpha)'$ to $f \in$ Lip (S, d^α). By Rainwater's theorem [15, p. 33], this implies that (f_n) converges weakly to f. But (f_n) was an arbitrary weak Cauchy sequence. Hence if $Q_0 = \varnothing$, Lip (S, d^α) is w.s.c., so S is finite.   Q.E.D.

3. One of the fundamental problems in a discussion of the properties of a given Banach space is that of describing its relatively compact subsets. The basic tool in such a description in Lipschitz spaces is the following:

Lemma 3.1. *Let (S, d) be any metric space and $\mathscr{F}$ any bounded subset of* Lip (S, d). *Denote by $\tilde{\mathscr{F}}$ the set $\{\tilde{f} \in BC(S \cup W) \mid f \in \mathscr{F}\}$. Then $\tilde{\mathscr{F}}$ is equicontinuous at each point of $S \cup W$.*

Proof. If $\|f\| \leq M$ for each $f \in \mathscr{F}$, then we have $|f(s) - f(t)| \leq Md(s, t)$, for all $f \in \mathscr{F}$ and $s, t \in S$. Then $\tilde{\mathscr{F}}$ is clearly equicontinuous on S. Let ε_s denote the functional $f \to f(s)$ in Lip $(S, d)'$. Now, since $|f(s) - f(t)| \leq d(s, t)$ for f in the unit ball of Lip (S, d), we have $\|\varepsilon_s - \varepsilon_t\| \leq d(s, t)$. Hence the mapping $s \to \varepsilon_s$ is continuous from (S, d) into Lip $(S, d)'$. Define $\sigma: W \to$ Lip $(S, d)'$ by

$$\sigma(s, t) = d^{-1}(s, t)(\varepsilon_s - \varepsilon_t).$$

Since $s \to \varepsilon_s$ is continuous and $(s, t) \to d(s, t)$ is continuous, it follows that σ is continuous. Hence, given $\varepsilon > 0$ and $(s_0, t_0) \in W$, there is a neighborhood V of (s_0, t_0) such that if $(s, t) \in V$, then $\|\sigma(s, t) - \sigma(s_0, t_0)\| < \varepsilon/M$. Therefore, for each $f \in \mathscr{F}$ and $(s, t) \in V$, we have $|\tilde{f}(s, t) - \tilde{f}(s_0, t_0)| \leq \|\tilde{f}\|_\infty \varepsilon/M \leq \varepsilon$.   Q.E.D.

THEOREM 3.2. *Let (S, d) be compact and $\mathscr{F}$ a subset of* lip (S, d). *The following are equivalent*:

(a) $\mathscr{F}$ *is relatively compact.*

(b) $\mathscr{F}$ *is bounded and* $\lim_{d(s,t) \to 0} |f(s) - f(t)|/d(s, t) = 0$ *uniformly on $\mathscr{F}$.*

Proof. As usual, consider lip (S, d) embedded in $C(S \cup W)$ by $f \to \tilde{f}$. Since (S, d) is compact it is clear that $\{(s, t) \mid d(s, t) \geq \delta > 0\}$ is compact in W for each $\delta > 0$ and that W is locally compact. Now it becomes apparent that the functions $\tilde{f}$ vanish at infinity on W. Letting $W^* = W \cup \{\omega\}$ denote the one-point compactification of W, we see that by defining $\tilde{f}(\omega) = 0$, lip (S, d) is embedded in $C(S \cup W^*)$. Since $S \cup W^*$ is compact, the Arzela-Ascoli theorem shows that the set $\mathscr{F}$ is relatively compact if and only if it is bounded and equicontinuous. By 3.1, $\mathscr{F}$ is equicontinuous on $S \cup W$ and by hypothesis, $\mathscr{F}$ is bounded. Thus, $\mathscr{F}$ is relatively compact if and only if $\mathscr{F}$ is equicontinuous at the point at infinity, ω. But this is precisely condition (b).   Q.E.D.

COROLLARY 3.3. *If (S, d) is precompact and $0 < \alpha < \beta \leq 1$, then the unit ball of* Lip (S, d^β) *is compact in* lip (S, d^α).

Proof. We claim it is enough to prove the result for (S, d) compact. For, if (S, d) is precompact its completion is compact, and since the functions in Lip (S, d^γ), for any $\gamma \in (0, 1]$, are uniformly continuous on (S, d), they extend uniquely to the completion of (S, d) in a norm preserving way. Thus, the compactness of the unit ball U of Lip (S, d^β) in lip (S, d^α) is preserved.

Hence, assume (S, d) compact. We have

$$\frac{|f(s) - f(t)|}{d^\alpha(s, t)} = \frac{|f(s) - f(t)|}{d^\beta(s, t)} d^{\beta-\alpha}(s, t) \leq d^{\beta-\alpha}(s, t) \qquad \text{for all } f \in U \text{ and } s \neq t.$$

Hence $\lim_{d(s,t) \to 0} |f(s) - f(t)|/d^\alpha(s, t) = 0$ uniformly on U. Thus, U is relatively compact. To show that U is closed in lip (S, d^α), it is enough to show that it is closed in a coarser topology, namely $\|\cdot\|_\infty$. If (f_n) converges to f in sup-norm, then $\|f\|_\infty \leq 1$ and

$$1 \geq \lim_{n \to 0} \frac{|f_n(s) - f_n(t)|}{d^\beta(s, t)} = \frac{|f(s) - f(t)|}{d^\beta(s, t)} \quad \text{for } s \neq t.$$

Hence, $f \in U$.   Q.E.D.

REMARK. Since the unit ball of Lip (S, d^β) is compact for the topology of Lip (S, d^α), it is *a fortiori* compact for the coarser topology of the sup-norm. However, for (S, d) compact, this can be deduced directly from the Arzela-Ascoli theorem because U is equicontinuous and bounded in $C(S)$.

We now give a result which provides the converses to the preceding remark and 3.3.

THEOREM 3.4. *Suppose the unit ball of* Lip (S, d^α) *is relatively compact in the sup-norm topology for some* α *with* $0 < \alpha \leq 1$. *Then* (S, d) *is precompact.*

Proof. We have shown in the proof of 3.3 that the unit ball U of Lip (S, d^α) is closed for the sup-norm topology. Thus U is compact in the space $BC(S)$ of bounded continuous functions on S. Let $\varepsilon > 0$. From [9, IV.6.5, p. 266] we can find a cover $S_1, \ldots, S_n$ of S and points $s_j \in S_j$ such that $|f(s) - f(s_j)| < \varepsilon$ for every $f \in U$, $s \in S_j$, and $j = 1, \ldots, n$. Now, given $s \in S$, there is a j such that $s \in S_j$. If $f(t) = d^\alpha(t, s_j) \wedge 1$, then $f \in U$ and $\varepsilon > |f(s) - f(s_j)| = |f(s)| = d^\alpha(s, s_j)$ (taking $\varepsilon < 1$). Thus (S, d^α), and hence (S, d), is precompact.   Q.E.D.

We now sum up the previous results in the following

THEOREM 3.5. *Let* (S, d) *be any metric space. Then the following are equivalent:*
(a) (S, d) *is precompact.*
(b) *The unit ball of* Lip (S, d^α), $0 < \alpha \leq 1$, *is compact for the sup-norm topology.*
(c) *The unit ball of* Lip (S, d^β) *is compact in* lip (S, d^α) *for* $0 < \alpha < \beta \leq 1$.

The weakly relatively compact subsets of lip (S, d^α) do not have as elegant a characterization as the relatively compact sets. However, 3.1 does provide one, which we give below. For the definition of quasi-equicontinuity and quasi-uniform convergence, see [9].

THEOREM 3.6. *Let* (S, d) *be compact and let* $\mathscr{F}$ *be a subset of* lip (S, d). *Denote by* $\tilde{\mathscr{F}}$ *the set of functions* $\tilde{f}$ *in* $C(S \cup W^*)$ *for which* $f \in \mathscr{F}$(*see the proof of* 3.2). *Then the following are equivalent:*

(a) $\mathscr{F}$ *is weakly relatively compact.*

(b) $\mathscr{F}$ *is bounded and* $\tilde{\mathscr{F}}$ *is quasi-equicontinuous at infinity on* W.

(c) $\mathscr{F}$ *is bounded and given any net* $\{(s_i, t_i)\}_{i \in I}$ *in* W *such that* $\lim_{i \in I} d(s_i, t_i) = 0$, *any* $\varepsilon > 0$, *and any* $i_0 \in I$, *there exist* $i_1, \ldots, i_k \in I$ *such that* $i_j \geq i_0$ *for each* j *and such that*

$$\min \left\{ \frac{|f(s_{i_j}) - f(t_{i_j})|}{d(s_{i_j}, t_{i_j})} : j = 1, \ldots, n \right\} < \varepsilon \quad \text{for } f \in \mathscr{F}.$$

Proof. From 3.1 and [9, IV,6.14(3), p. 269] it is obvious that (a) and (b) are equivalent. Since the sets $\{(s, t) \mid d(s, t) \geq \varepsilon\}$ form an exhaustive family of compact subsets of W, it is clear that a net $\{(s_i, t_i)\}$ converges to the point at infinity in W^* if and only if $d(s_i, t_i) \to 0$. Now (c) is simply the definition of the quasi-equicontinuity of $\tilde{\mathscr{F}}$ at infinity on W.   Q.E.D.

4. There have been several papers establishing the fact that certain spaces of Lipschitz functions are dual spaces; i.e., they are isometrically isomorphic with the dual of some Banach space. The first result along these lines appears to have been that of Arens and Eells [1] which appeared in 1956. They established the fact that

the set of real valued functions f on any metric space (S, d) that vanish at a fixed point and satisfy $\|f\|_d < \infty$ is a dual space when endowed with the norm $\|\cdot\|_d$. In 1958, Kantorovič and Rubenšteǐn [14] proved that for compact (S, d), Lip (S, d) is a dual space.

In 1961, de Leeuw [6], proved that the space of periodic functions in Lip (S, d^α) of period 1 is isometrically isomorphic with the second dual of the space of functions in lip (S, d^α) of period 1, where (S, d) in the real line with the usual metric and $0 < \alpha < 1$. Then, in 1967, T. M. Jenkins [13] generalized de Leeuw's result as follows: Let (S, d) be finitely compact (see introduction). Then (S, d) is locally compact and we can consider the space $\text{lip}^0 (S, d^\alpha)$. In the case of real-valued functions, Lip (S, d^α) is isometrically isomorphic with $\text{lip}^0 (S, d^\alpha)''$. In the case of complex-valued functions, the proof would not carry through without an additional hypothesis on the metric space.

In this section we give a different proof of Jenkins' result which shows this extra hypothesis to be unnecessary. We also show that for any metric space (S, d), $\text{Lip}_E (S, d)$ is a dual space whenever E is.

We will use the following notation: If E is a Banach space and E' its dual, $\langle x, x' \rangle$ is defined to be $x'(x)$ for each $x \in E$, $x' \in E'$.

THEOREM 4.1. *Let (S, d) be any metric space and E any Banach space. For each $x \in E$ and $s \in S$, let $x \otimes \varepsilon_s$ be the element of the dual of $\text{Lip}_{E'} (S, d)$ defined by $\langle f, x \otimes \varepsilon_s \rangle = \langle x, f(s) \rangle$. If $\mathscr{V}$ denotes the norm closed linear span of*

$$\{ x \otimes \varepsilon_s \mid x \in E, s \in S \}$$

in $\text{Lip}_{E'} (S, d)'$, then $\mathscr{V}'$ is isometrically isomorphic with $\text{Lip}_{E'} (S, d)$.

Proof. For each $F \in \text{Lip}_{E'} (S, d)''$, define $\Lambda F: S \to E'$ by $\langle x, \Lambda F(s) \rangle = F(x \otimes \varepsilon_s)$. For $s \neq t$, and $x \in E$ we have

$$\left| \left\langle x, \frac{\Lambda F(s) - \Lambda F(t)}{d(s, t)} \right\rangle \right| = \left| \frac{F(x \otimes \varepsilon_s) - F(x \otimes \varepsilon_t)}{d(s, t)} \right| \leqq \|F\| \left\| x \frac{\otimes \varepsilon_s - x \otimes \varepsilon_t}{d(s, t)} \right\|$$

$$= \|F\| \sup \left\{ \left| \left\langle x, \frac{f(s) - f(t)}{d(s, t)} \right\rangle \right| : f \in \text{Lip}_{E'} (S, d), \|f\| \leqq 1 \right\}$$

$$\leqq \|F\| \|x\|.$$

Thus $\|\Lambda F(s) - \Lambda F(t)\| \leqq \|F\| d(s, t)$ for all $s, t \in S$. Also,

$$|\langle x, \Lambda F(s) \rangle| = |F(x \otimes \varepsilon_s)| \leqq \|F\| \|x \otimes \varepsilon_s\|$$

$$= \|F\| \sup \{ |\langle x, f(s) \rangle| : f \in \text{Lip}_{E'} (S, d), \|f\| \leqq 1 \}$$

$$\leqq \|F\| \|x\|, \quad \text{for all } x \in E \text{ and } s \in S.$$

Thus, $\|\Lambda F(s)\| \leqq \|F\|$ for all $s \in S$. Hence we have that $\Lambda F \in \text{Lip}_{E'} (S, d)$, $\|\Lambda F\|_\infty \leqq \|F\|$, and $\|\Lambda F\|_d \leqq \|F\|$. It is routine to show that Λ is linear. Suppose $F \in \text{Lip}_{E'} (S, d)''$ is the image of $f \in \text{Lip}_{E'} (S, d)$ under the canonical embedding. This means that for $\varphi \in \text{Lip}_{E'} (S, d)'$, $\langle f, \varphi \rangle = \langle \varphi, F \rangle$. In particular, $\langle x, f(s) \rangle$

$= \langle f, x \otimes \varepsilon_s \rangle = F(x \otimes \varepsilon_s)$ for $s \in S$, $x \in E$. Thus, $\langle x, \Lambda F(s) \rangle = F(x \otimes \varepsilon_s) = \langle x, f(s) \rangle$ for all $x \in E$ and $s \in S$. Hence $\Lambda F = f$. Now, if $\text{Lip}_{E'}(S, d)$ is identified with its image in $\text{Lip}_{E'}(S, d)''$ under the canonical embedding, the above shows that Λ induces a projection of $\text{Lip}_{E'}(S, d)''$ on $\text{Lip}_{E'}(S, d)$ which is norm decreasing. Furthermore, $\Lambda F = 0$ if and only if $F(x \otimes \varepsilon_s) = 0$ for all $x \in E$ and $s \in S$. Hence, the null space of Λ is the annihilator of $\mathscr{V}$, which, in particular, implies that it is weak* closed. It now follows from a theorem due to Dixmier [8, Théorème 18, p. 1069] that $\text{Lip}_{E'}(S, d)$ is a dual space. $\mathscr{V}$ is clearly weak* dense in $\text{Lip}_{E'}(S, d)'$ since if $\langle x, f(s) \rangle = 0$ for all $s \in S$ and $x \in E$ then $f = 0$. It is now clear from Theorems 9, 11, and 16 of [8] that $\mathscr{V}'$ is isometrically isomorphic with $\text{Lip}_{E'}(S, d)$.

COROLLARY 4.2. *Let (S, d) be any metric space and let V denote the closed linear span of the point evaluations in $\text{Lip}(S, d)'$. Then $\text{Lip}(S, d)$ is isometrically isomorphic with V'.*

It is interesting to note that, as a result of §1, all the spaces $L(S, \rho)$ are dual spaces. The technique used in proving 4.1 can also be applied to the space $B(S)$ of all bounded functions on a set S. The result obtained is that $B(S)$ is the dual of the closed linear span of the point evaluations in $B(S)'$. The same is true if the supremum norm is replaced by the equivalent norm $\max(\|\cdot\|_\infty, w(\cdot))$ where

$$w(f) = \sup\{|f(s) - f(t)| : s, t \in S\},$$

the oscillation of f on S. This is seen by taking ρ to be identically one in $L(S, \rho)$.

THEOREM 4.3. *Let $\{f_i\}$ be a bounded net in $\text{Lip}_{E'}(S, d)$. Considering $\text{Lip}_{E'}(S, d)$ as a dual space, the following are equivalent:*

(a) *$\{f_i\}$ converges to zero weak*.*

(b) *$\{f_i(s)\}$ converges to zero weak* in E' for each $s \in S$. In particular, a sequence $\{f_n\}$ in $\text{Lip}_{E'}(S, d)$ converges weak* to zero if and only if $\{f_n(s)\}$ converges weak* to zero in E' for each $s \in S$, and $\{f_n\}$ is bounded.*

Proof. If (a) holds, then, in particular, $\langle f_i, x \otimes \varepsilon_s \rangle \to 0$ for each $x \in E$, $s \in S$. Thus, $\langle x, f_i(s) \rangle \to 0$ for each $x \in E$, $s \in S$. Hence (b) follows. Assume that (b) holds. Let $\varphi \in \mathscr{V}$ (see 4.1) and $\varepsilon > 0$. $\{f_i\}$ is bounded, say $\|f_i\| \le M$ for all i. There exist $s_1, \ldots, s_k \in S$ and $x_1, \ldots, x_k \in E$ such that $\|\varphi - \psi\| < \varepsilon/2M$ where

$$\psi = \sum_{j=1}^{k} x_j \otimes \varepsilon_{s_j}.$$

Since $\lim_i \langle x_j, f_i(s_j) \rangle = 0$ for each j, $\lim_i \psi(f_i) = 0$. Thus, there is an i_0 such that for $i \ge i_0$, $|\psi(f_i)| < \varepsilon/2$. Hence, for $i \ge i_0$,

$$|\varphi(f_i)| \le |\varphi(f_i) - \psi(f_i)| + |\psi(f_i)| \le \|\varphi - \psi\| M + \varepsilon/2 \le \varepsilon. \qquad \text{Q.E.D.}$$

COROLLARY 4.4. *A bounded net in $\text{Lip}(S, d)$ converges weak* to zero if and only if it converges pointwise to zero. If (S, d) is precompact, a bounded net $\{f_i\}$ converge weak* to zero if and only if $\|f_i\|_\infty \to 0$.*

Proof. The first assertion is an obvious consequence of 4.3. Let (S, d) be precompact, $\|f_i\| \leqq M$, and suppose $\{f_i\}$ converges weak* to zero. Let $\varepsilon > 0$. There exist $s_1, \ldots, s_n \in S$ such that min $\{d(s, s_j) \mid j = 1, \ldots, n\} < \varepsilon/2M$ for each $s \in S$. $\{f_i\}$ converges pointwise to zero, so there is an i_0 such that $i \geqq i_0$ implies $|f_i(s_j)| < \varepsilon/2$ for each $j = 1, \ldots, n$. Let $i \geqq i_0$ and $s \in S$. There is a $j = 1, \ldots, n$ so that $d(s, s_j) < \varepsilon/2M$. Then

$$|f_i(s)| \leqq |f_i(s) - f_i(s_j)| + |f_i(s_j)| \leqq Md(s, s_j) + \varepsilon/2 \leqq \varepsilon.$$

Hence if $i \geqq i_0$, $\|f_i\|_\infty \leqq \varepsilon$.   Q.E.D.

Suppose that (S, d) is separable. Since $|f(s) - f(t)| \leqq d(s, t)$ for each f in the unit ball of Lip (S, d), it is clear that the mapping $s \to \varepsilon_s$ of S into Lip $(S, d)'$ is continuous. Thus, V, the closed linear span of $\{\varepsilon_s \mid s \in S\}$, is separable. It is well known, that if (and only if) E is a separable Banach space, there is a metric on E' which induces the weak* topology on each bounded subset of E' (see [9, VI. 5.1, p. 426]). What is interesting is that, in case (S, d) is precompact, there is a norm, namely $\| \cdot \|_\infty$, on Lip (S, d) which induces the weak* topology on bounded sets[1].

From 4.4 and the Krein-Milman theorem, we know that the unit ball of Lip (S, d) is the closed convex hull of its extreme points in the topology of pointwise convergence. The natural problem that arises here is to characterize these extreme points and, in case (S, d) is compact, to decide whether the unit ball of Lip (S, d) is the closed convex hull of its extreme points in the norm topology. When (S, d) is the unit interval and d the usual metric, Roy [16] has characterized the extreme points and shown that their closed convex hull is indeed the entire unit ball. The general problem appears to be much more difficult.

Before we begin the proof of our next result, we need some preliminaries.

THEOREM 4.5 (DE LEEUW-JENKINS). *Let (S, d) be a finitely compact metric space. Then the linear span of the point evaluations is dense in the dual of $\mathrm{lip}^0 (S, d)$.*

The idea of the proof is to consider $\mathrm{lip}^0 (S, d)$ embedded in the usual way in $C^0(S \cup W)$, the continuous functions on $S \cup W$ vanishing at infinity. Then each element φ of $\mathrm{lip}^0 (S, d)'$, by the Hahn-Banach theorem, is given by a measure on $S \cup W$. One shows from this that φ can be approximated by measures on S with compact support. To complete the proof, it is shown that the measures of compact support on S, considered as elements of the dual of $\mathrm{lip}^0 (S, d)$, can be approximated by measures of finite support; i.e., functionals in the linear span of $\{\varepsilon_s \mid s \in S\}$.

In order to establish the fact that Lip (S, d^α), $0 < \alpha < 1$, is isometrically isomorphic with the bidual of $\mathrm{lip}^0 (S, d^\alpha)$, Jenkins makes use of the fact that if $f \in$ Lip (T, d) is real-valued, $T \subset S$, then there is a norm preserving extension of f to an element of Lip (S, d) (see [20, Proposition 1.4]). In order for this proof to carry over to the complex case, it is necessary to assume that the metric space S

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[1] The author discovered after submitting this paper that this property is common to all separable dual spaces.

has the property that complex functions can likewise be extended in a norm preserving way. This leads to the added assumption in [13] that (S, d) has the Lipschitz four-point property, which guarantees precisely this needed hypothesis.

In Theorem 4.7, we will prove that Lip (S, d^α) is isometrically isomorphic with $\text{lip}^0 (S, d^\alpha)''$ for $0 < \alpha < 1$ and (S, d) finitely compact, without this extra condition. We need one further crucial observation. Although not stated explicitly in [13], one can, by a careful examination of the proofs of Lemmas 4.7 and 4.8 in that paper, deduce the following:

LEMMA 4.6. *Let* (S, d) *be finitely compact and* $0 < \alpha < 1$. *Given a real-valued function* $f \in \text{Lip} (S, d^\alpha)$, $\varepsilon > 0$, *and* $s_1, \ldots, s_n \in S$, *there exists a function* $g \in \text{lip}^0 (S, d^\alpha)$ *(real-valued) such that* $\|g\| \leq (1+\varepsilon)^3 \|f\|$ *and* $g(s_i) = f(s_i)$, $i = 1, \ldots, n$.

We define the mapping of $\text{lip}^0 (S, d^\alpha)''$ into Lip (S, d^α) in the same way as in 4.1 and denote it by Λ as well. Thus, for $F \in \text{lip}^0 (S, d^\alpha)''$ and $s \in S$, $\Lambda F(s) = F(\varepsilon_s)$ where ε_s is the evaluation functional at s. That $\Lambda F \in \text{Lip} (S, d^\alpha)$ can be seen from the proof of 4.1.

THEOREM 4.7. *Let* (S, d) *be finitely compact and* $0 < \alpha < 1$. *Then* Λ *is a surjective isometric isomorphism.*

Proof. Let V_0 [resp. U_0] denote the linear span of the point evaluations in Lip $(S, d^\alpha)'$ [resp. $\text{lip}^0 (S, d^\alpha)'$] endowed with its respective induced dual norm. For any finite collection $s_1, \ldots, s_n \in S$, and any $j = 1, \ldots, n$, there is $f_j \in \text{lip}^0 (S, d^\alpha)$ such that $f_j(s_i) = \delta_{ij}$. Hence the set of point evaluations $\{\varepsilon_s \mid s \in S\}$ is a linearly independent set in $\text{lip}^0 (S, d^\alpha)'$ and, *a fortiori*, in Lip $(S, d^\alpha)'$. (We use the same symbol, ε_s, for point evaluations in both dual spaces, since the context will make clear which is meant.) Define $J: V_0 \to U_0$ to be the (unique) bijective linear function such that $J(\varepsilon_s) = \varepsilon_s$. Since $\text{lip}^0 (S, d^\alpha)$ is a subspace of Lip (S, d^α), it is obvious that J is norm decreasing. The adjoint, J^*, of J maps U_0' into V_0'. By 4.5, U_0' is identical with $\text{lip}^0 (S, d^\alpha)''$. By 4.2, V_0' is isometrically isomorphic with Lip (S, d^α). If we employ these two canonical identifications to consider J^* as a mapping from $\text{lip}^0 (S, d^\alpha)''$ into Lip (S, d^α), we see that $J^* = \Lambda$. Now, let $\varphi = \sum_{i=1}^n \lambda_i \varepsilon_{s_i} \in U_0$ where the λ_i's are real or complex numbers and $s_1, \ldots, s_n$ are distinct. Given $f \in \text{Lip} (S, d^\alpha)$ real valued and $\varepsilon > 0$, choose $g \in \text{lip}^0 (S, d^\alpha)$ as in 4.6. Then

$$|J^{-1}\varphi(f)| = \left| \sum_{i=1}^n \lambda_i f(s_i) \right| = \left| \sum \lambda_i g(s_i) \right| = |\varphi(g)| \leq \|\varphi\| \|g\| \leq \|\varphi\|(1+\varepsilon)^3\|f\|.$$

Since $\varepsilon > 0$ was arbitrary, we have $|J^{-1}\varphi(f)| \leq \|\varphi\| \|f\|$ for all $\varphi \in U_0$ and $f \in \text{Lip} (S, d^\alpha)$ real valued. Thus, in case we restrict ourselves to real scalars, we have J and J^{-1} both norm decreasing. Therefore, J is an isometry and hence, so is $\Lambda = J^*$.

Now, suppose $f \in \text{Lip} (S, d^\alpha)$ is arbitrary. Then $f = f_1 + if_2$ where f_j is real valued $(j = 1, 2)$ and, since

$$\|f\| = \|\tilde{f}\|_\infty = \|\tilde{f_1} + i\tilde{f_2}\|_\infty \geq \max(\|\tilde{f_1}\|_\infty, \|\tilde{f_2}\|_\infty) = \max(\|f_1\|, \|f_2\|),$$

$$f_j \in \text{Lip} (S, d^\alpha), \qquad j = 1, 2.$$

Furthermore, we have

$$|J^{-1}\varphi(f)| = |J^{-1}\varphi(f_1) + iJ^{-1}\varphi(f_2)| \leqq |J^{-1}\varphi(f_1)| + |J^{-1}\varphi(f_2)|$$
$$\leqq 2\|\varphi\|\,\|f\| \quad \text{for all } f \in \text{Lip}\,(S, d^\alpha) \text{ and } \varphi \in U_0.$$

Thus, J is an isomorphism and therefore so is Λ.

To show that Λ is an isometry, we consider $\text{lip}^0\,(S, d^\alpha)$ canonically embedded in the space of continuous functions on $S \cup W$, vanishing at infinity and let $\mathscr{E}_1$ and $\mathscr{E}_2$ denote respectively the subsets

$$\{\lambda\varepsilon_s \mid s \in S, |\lambda| = 1\} \quad \text{and} \quad \{\lambda(\varepsilon_s - \varepsilon_t)/d^\alpha(s, t) \mid s \neq t, |\lambda| = 1\}$$

of $\text{lip}^0\,(S, d^\alpha)'$. By the above embedding and by [9, V. 8.6, p. 441], it follows that $\mathscr{E} = \mathscr{E}_1 \cup \mathscr{E}_2$ is a subset of the unit ball $\mathscr{U}$ of $\text{lip}^0\,(S, d^\alpha)'$ containing the extreme points. (We note that the proof of 2.7 gives the extreme points exactly; but this is unnecessary here.) But for each $F \in \text{lip}^0\,(S, d^\alpha)''$, $\|F\| = \sup\,\{|F(\varphi)| \mid \varphi \in \mathscr{U}\}$ and $\|\Lambda F\| = \sup\,\{|F(\varphi)| \mid \varphi \in \mathscr{E}\}$. Now, by 4.5, we have that $\text{lip}^0\,(S, d^\alpha)'$ is separable. Hence by [2], $\mathscr{U}$ is the closed convex hull of $\mathscr{E}$. Thus, $\|F\| = \|\Lambda F\|$ and Λ is an isometry.   Q.E.D.

REMARK. If $\text{lip}^0\,(S, d^\alpha)$ is considered as a Banach algebra, its bidual can be made into a Banach algebra by Arens multiplication (see [4]). It is routine to show that Arens multiplication coincides with the usual multiplication on $\text{Lip}\,(S, d^\alpha)$ via the isomorphism Λ, and hence is commutative.

5. We have observed in the previous section that for finitely compact metric spaces (S, d) (see introduction) and $0 < \alpha < 1$, $\text{Lip}\,(S, d^\alpha)$ is canonically isometrically isomorphic with the bidual of $\text{lip}^0\,(S, d^\alpha)$. We also observed that if E is a dual space, so is $\text{Lip}_E\,(S, d^\alpha)$. Thus, it is natural to conjecture that if E is any Banach space, (S, d) any finitely compact metric space, and $0 < \alpha < 1$, then the bidual of $\text{lip}^0_E\,(S, d^\alpha)$ is isometrically isomorphic with $\text{Lip}_{E''}\,(S, d^\alpha)$.

This section is devoted to an attempt to prove the above conjecture. Although we are unable to prove the result in its full generality, we are able to establish it for wide classes of Banach spaces E and metric spaces (S, d).

From here on we will always assume that (S, d) is finitely compact and $0 < \alpha < 1$ unless otherwise stated.

Given $F \in \text{lip}^0_E\,(S, d^\alpha)''$, define $\Lambda F: S \to E''$ by $\langle x', \Lambda F(s)\rangle = F(x' \otimes \varepsilon_s)$, where $x' \otimes \varepsilon_s$ is the element of $\text{lip}^0_E\,(S, d^\alpha)'$ defined by $(x' \otimes \varepsilon_s)(f) = \langle f(s), x'\rangle$. Then it is easily seen from the proof of 4.1 that $\Lambda F \in \text{Lip}_{E''}\,(S, d^\alpha)$ and that Λ defines a norm-decreasing linear mapping from $\text{lip}^0_E\,(S, d^\alpha)''$ into $\text{Lip}_{E''}\,(S, d^\alpha)$. We will try to prove that Λ is a surjective isometry. If we proceed along the lines of the proof of 4.7, our first step is to prove that the linear span of $\{x' \otimes \varepsilon_s \mid x' \in E', s \in S\}$ is dense in $\text{lip}^0_E\,(S, d)'$. For reasons that will be clear later, we denote this linear span by $E' \otimes V_0$.

We embed $\text{lip}_E^0(S, d)$ in the space of continuous functions on $S \cup W$ to E vanishing at infinity in the usual way; i.e.

$$\tilde{f}(s) = f(s), \quad s \in S, \qquad \tilde{f}(s, t) = d^{-\alpha}(s, t)(f(s) - f(t)), \quad s \neq t, \quad \text{and} \quad \tilde{f}(\omega) = 0$$

where ω is the point at infinity. Hence $\text{lip}_E^0(S, d^\alpha)$ can be considered as a subspace of $C_E(Z)$, where Z is a compact Hausdorff space. For any compact Hausdorff space Z and Banach space E, the dual $C_E(Z)'$ of $C_E(Z)$ can be represented as the space of E'-valued regular Borel measures on Z with finite variation (see [7, Corollary 2, p. 387]). For notation and definitions concerning vector-valued measures, see [7]. We will use $C_E(Z)'$ to denote the set of regular E'-valued Borel measures on Z with finite variation.

Given $m \in C_E(S^*)'$, where S^* denotes the one-point compactification of S, the functional $\varphi_m: f \to \int f \, dm, f \in \text{lip}_E^0(S, d^\alpha)$, is an element of $\text{lip}_E^0(S, d^\alpha)'$. Denote by $\mathscr{S}$ the set of all those φ_m such that m has compact support in S. Clearly the functionals in $E' \otimes V_0$ are those which are represented by measures m with finite support in S. Our first goal is to show that $\mathscr{S}$ is dense in $\text{lip}_E^0(S, d^\alpha)'$. The proof given below is a modification of that given by Jenkins for the scalar-valued case.

LEMMA 5.1. $\mathscr{S}$ is dense in $\text{lip}_E^0(S, d^\alpha)'$.

Proof. Given $\varphi \in \text{lip}_E^0(S, d^\alpha)'$, there exists $m \in C_E(Z)'$, where $Z = (S \cup W)^*$ is the one point compactification of $S \cup W$, such that $\varphi(f) = \int \tilde{f} \, dm$ for each $f \in \text{lip}_E^0(S, d^\alpha)$. This is immediate from the Hahn-Banach theorem. Fix a point $s_0 \in S$ and let $S_n = \{s \in S \mid d(s, s_0) \leq n\}$. Let

$$W_n = \{(s, t) \in W \mid s, t \in S_n \text{ and } d(s, t) \geq 1/(n+1)\}.$$

Then $\{S_n \cup W_n\}$ is an increasing sequence of compact sets whose union is $S \cup W$. For each $n = 1, 2, \ldots$ and each $f \in \text{lip}_E^0(S, d^\alpha)$ we define $\varphi_n(f) = \int_{S_n \cup W_n} \tilde{f} \, dm$. Then

$$|(\varphi - \varphi_n)(f)| = \left| \int_Z \tilde{f} \, dm - \int_{S_n \cup W_n} \tilde{f} \, dm \right|$$

$$= \left| \int_{Z \sim (S_n \cup W_n)} \tilde{f} \, dm \right| \leq \int_{Z \sim (S_n \cup W_n)} |\tilde{f}| \, d|m|,$$

$$\text{where } |g|(z) = \|g(z)\| \quad \text{for} \quad g \in C_E(Z)$$

and $|m|$ is the variation of m. Since $|\tilde{f}|$ vanishes at infinity, the last integral is not larger than $\|\tilde{f}\|_\infty |m|\{(S \cup W) \sim (S_n \cup W_n)\}$. Since $|m|$ is countably additive and $\bigcup_{n=1}^\infty S_n \cup W_n = S \cup W$, we get $\|\varphi_n - \varphi\| \to 0$. We now show that $\varphi_n \in \mathscr{S}$.

$$\varphi_n(f) = \int_{S_n} f(s) \, dm(s) + \int_{W_n} \tilde{f}(s, t) \, dm(s, t)$$

and

$$\int_{W_n} \tilde{f} \, dm = \int_{W_n} f(s) d^{-\alpha}(s, t) \, dm(s, t) - \int_{W_n} f(t) d^{-\alpha}(s, t) \, dm(s, t).$$

Since $d^\alpha(s, t)$ is bounded away from zero on W_n, there are measures m_1 and m_2 in $C_E(S_n)'$, i.e. in $C_E(S^*)'$ with support contained in S_n, such that

$$\int_{W_n} f(s)\, d^{-\alpha}(s, t)\, dm(s, t) = \int_{S_n} f\, dm_1$$

and

$$\int_{W_n} f(t) d^{-\alpha}(s, t)\, dm(s, t) = \int_{S_n} f\, dm_2.$$

Thus, we get $\varphi_n(f) = \int_{S_n} f d(m + m_1 - m_2)$.   Q.E.D.

We next want to show that $E' \otimes V_0$ is dense in $\mathscr{S}$. The proof given by Jenkins for the scalar case does not apply here. In his proof, essential use was made of the fact that the unit ball of lip (S, d^α) is relatively compact in $C(S)$ when S is compact. However, this is not the case for vector-valued functions unless the range space E is finite dimensional. To see this, note that the mapping $f \to f(s)$ of $C_E(S)$ onto E sends the unit ball of $\mathrm{lip}_E\,(S, d^\alpha)$ onto that of E and is continuous for the sup-norm topology. Thus, the unit ball of E would have to be relatively compact, making E finite dimensional.

However, this difficulty can be circumvented.

LEMMA 5.2. $E' \otimes V_0$ is dense in $\mathscr{S}$.

Proof. Let $\varphi_m \in \mathscr{S}$ and $\varepsilon > 0$. Let K be the support of m. K is compact, so there exist $t_1, \ldots, t_n \in K$ such that $\bigcup_{j=1}^n B_j \supset K$ where $B_j = \{s \in S \mid d^\alpha(s, t_j) < \varepsilon/2|m|(K)\}$. Let $A_1 = B_1$ and $A_j = B_j \sim \bigcup_{i<j} B_i$, $(2 \leqq j \leqq n)$. Then (A_j) is a pairwise disjoint family of Borel sets covering K. Since we may discard any A_j that is empty, we assume that each $A_j \neq \varnothing$. Choose $s_j \in A_j$ for each j and define $\varphi = \sum_{j=1}^n x'_j \otimes \varepsilon_{s_j}$, where $x'_j = m(A_j) \in E'$. Let $f \in \mathrm{lip}_E^0\,(S, d^\alpha)$ with $\|f\| \leqq 1$. We will show that $|\varphi(f) - \varphi_m(f)| < \varepsilon$.

Define $g = \sum_{j=1}^n f(s_j)\Psi_{A_j}$ where Ψ_{A_j} is the characteristic function of A_j. Given $t \in K$, there is a unique i so that $t \in A_i$. Then

$$\|g(t) - f(t)\| = \|f(s_i) - f(t)\| \leqq d^\alpha(s_i, t) \leqq d^\alpha(s_i, t_i) + d^\alpha(t_i, t) < \varepsilon/|m|(K),$$

since $A_i \subset B_i$. Thus $\|g - f\|_K = \sup\{\|g(s) - f(s)\| \mid s \in K\} < \varepsilon/|m|(K)$. Since

$$\int g\, dm = \sum_{j=1}^n \langle f(s_j), m(A_j)\rangle = \sum_{j=1}^n \langle f(s_j), x'_j\rangle = \varphi(f),$$

we have

$$|\varphi(f) - \varphi_m(f)| = \left|\int g\, dm - \int f\, dm\right| = \left|\int g - f\, dm\right| \leqq \int |g - f|\,d|m|$$

$$= \int_K |g - f|\,d|m| \leqq \|g - f\|_K |m|(K) < \varepsilon.$$

Since f was arbitrary in the unit ball of $\mathrm{lip}_E^0\,(S, d^\alpha)$, we have $\|\varphi - \varphi_m\| < \varepsilon$.   Q.E.D.

THEOREM 5.3. *The vector space $E' \otimes V_0$ spanned by $\{x' \otimes \varepsilon_s \mid x' \in E', s \in S\}$ is dense in $\mathrm{lip}_E^0 (S, d^\alpha)'$.*

The proof of 5.3 is immediate from Lemmas 5.1 and 5.2.

COROLLARY 5.4. *The canonical mapping Λ is one-to-one.*

Proof. By definition of Λ, we have that $\Lambda F = 0$ if and only if $F(x' \otimes \varepsilon_s) = 0$ for all $x' \in E'$ and $s \in S$. Thus $\Lambda F = 0$ if and only if F vanishes on the dense subspace $E' \otimes V_0$ of $\mathrm{lip}_E^0 (S, d^\alpha)$.   Q.E.D.

The vector space $E' \otimes V_0$ can be considered as a subspace of either $\mathrm{Lip}_{E''} (S, d^\alpha)'$ or $\mathrm{lip}_E^0 (S, d^\alpha)'$. We write $E' \otimes_L V_0$ [resp. $E' \otimes_l V_0$] to denote $E' \otimes V_0$ endowed with the norm induced by $\mathrm{Lip}_{E''} (S, d^\alpha)'$ [resp. $\mathrm{lip}_E^0 (S, d^\alpha)'$]. In a manner analagous to the proof of 4.7, we consider the "identity" mapping $J: E' \otimes_L V_0 \to E' \otimes_l V_0$. Since $\mathrm{lip}_E^0 (S, d^\alpha)$ is a subspace of $\mathrm{Lip}_{E''} (S, d^\alpha)$, J is norm-decreasing and the adjoint J^* of J maps $(E' \otimes_l V_0)'$ into $(E' \otimes_L V_0)'$. Now, by 5.3 $(E' \otimes_l V_0)'$ $= \mathrm{lip}_E^0 (S, d^\alpha)''$ and by 4.1, $(E' \otimes_L V_0)'$ is isometrically isomorphic with $\mathrm{Lip}_{E''} (S, d^\alpha)$. When these two identifications are made, we see that $J^* = \Lambda$. Hence we get the following

THEOREM 5.5. *Λ is an isometric isomorphism if and only if the norms induced on $E' \otimes V_0$ by $\mathrm{Lip}_{E''} (S, d^\alpha)'$ and $\mathrm{lip}_E^0 (S, d^\alpha)'$ are the same.*

In order to apply the techniques of 4.7 to the problem, some ability to extend Lipschitz functions in a norm preserving way is required. Since the range of our functions is an arbitrary Banach space, a different approach must be used. As we will see, the theory of tensor products of Banach spaces provides a convenient set of tools with which to attack the problem. We will soon formally recognize $E' \otimes V_0$ as a tensor product.

If E and F are normed spaces, we denote by $E \otimes F$ their (algebraic) tensor product (see [17, Chapter III, §6] or [18]). From the norms on E and F there can be constructed various norms on $E \otimes F$. A crossnorm on $E \otimes F$ is a norm p such that $p(x \otimes y) = \|x\| \|y\|$ for all $x \in E$, $y \in F$. We write $E \otimes_p F$ to denote $E \otimes F$ endowed with the norm p, and $E \check{\otimes}_p F$ to denote the completion of the normed space $E \otimes_p F$. In particular, we will be concerned with two special crossnorms: λ, the least crossnorm (a misnomer perpetuated in the literature), and γ, the greatest crossnorm. For their definitions, see [11] or [18]. Schatten's monograph [18] provides a readable discussion of the elementary ideas and definitions in the theory.

We now proceed to the basic result which allows the profitable application of tensor products to our problem. If E and F are normed spaces, $\mathscr{L}(E, F)$ denotes the space of continuous linear mappings from E into F with the usual norm.

THEOREM 5.6. *Let (S, d) be any metric space and E any Banach space. Then $\mathrm{Lip}_{E'} (S, d)$ and $\mathscr{L}(E, \mathrm{Lip} (S, d))$ are isometrically isomorphic.*

Proof. Given $T \in \mathscr{L}(E, \mathrm{Lip}\,(S, d))$, define $f = f_T \colon S \to E'$ by (*) $\langle x, f(s) \rangle = Tx(s)$. Let $\|T\|_\infty$ and $\|T\|_d$ denote $\sup_{\|x\| \leqq 1} \|Tx\|_\infty$ and $\sup_{\|x\| \leqq 1} \|Tx\|_d$ respectively. Then

$$\|T\| = \sup_{\|x\| \leqq 1} (\|Tx\|_\infty \vee \|Tx\|_d) = \|T\|_\infty \vee \|T\|_d.$$

Now,

$$(1) \quad \begin{aligned} \|T\|_\infty &= \sup_{\|x\| \leqq 1} \|Tx\|_\infty = \sup_{\|x\| \leqq 1} \sup_{s \in S} |Tx(s)| \\ &= \sup_{s \in S} \sup_{\|x\| \leqq 1} |\langle x, f(s) \rangle| = \sup_{s \in S} \|f(s)\| = \|f\|_\infty, \end{aligned}$$

and

$$(2) \quad \begin{aligned} \|T\|_d &= \sup_{\|x\| \leqq 1} \|Tx\|_d = \sup_{\|x\| \leqq 1} \sup_{s \neq t} \frac{|Tx(s) - Tx(t)|}{d(s, t)} \\ &= \sup_{\|x\| \leqq 1} \sup_{s \neq t} \frac{|\langle x, f(s) - f(t) \rangle|}{d(s, t)} \\ &= \sup_{s \neq t} \sup_{\|x\| \leqq 1} \left| \left\langle x, \frac{f(s) - f(t)}{d(s, t)} \right\rangle \right| = \sup_{s \neq t} \frac{\|f(s) - f(t)\|}{d(s, t)} \\ &= \|f\|_d. \end{aligned}$$

Thus, the mapping $T \to f_T$ is an isometry of $\mathscr{L}(E, \mathrm{Lip}\,(S, d))$ into $\mathrm{Lip}_{E'}(S, d)$. It is clear that, given $f \in \mathrm{Lip}_{E'}(S, d)$, one can define T by (*). Then equations (1) and (2) show that $T \in \mathscr{L}(E, \mathrm{Lip}\,(S, d))$ and $f_T = f$. Hence the mapping $T \to f_T$ is a surjective isometry. Q.E.D.

REMARK. When (S, d) is compact, the mapping $T \to f_T$ takes the compact operators with range in $\mathrm{lip}\,(S, d)$ onto $\mathrm{lip}_{E'}(S, d)$. We will prove this (see 5.15) and discuss its implications later.

By the proof of 4.7, we know that the norms induced on the linear span V_0 of the point evaluations by both $\mathrm{lip}^0(S, d^\alpha)'$ and $\mathrm{Lip}\,(S, d^\alpha)'$ are equal. Therefore the identity mapping ($\varepsilon_s \to \varepsilon_s$) extends to an isometric isomorphism between $\mathrm{lip}^0(S, d^\alpha)'$ and the closed linear span V of V_0 in $\mathrm{Lip}\,(S, d^\alpha)'$. The tensor product $E' \otimes V$ [resp. $E' \otimes \mathrm{lip}^0(S, d^\alpha)'$] can be identified (algebraically) with a subspace of $\mathrm{Lip}_{E''}(S, d^\alpha)'$ [resp. $\mathrm{lip}_E^0(S, d^\alpha)'$] in a canonical way by letting $(x' \otimes \varphi)(f) = \varphi(x' \circ f)$ where $x' \circ f$ is the element of $\mathrm{Lip}\,(S, d^\alpha)$ [resp. $\mathrm{lip}^0(S, d^\alpha)$] such that $(x' \circ f)(s) = \langle x', f(s) \rangle$ [resp. $\langle f(s), x' \rangle$].

THEOREM 5.7 (SCHATTEN [18, THEOREM 3.2, p. 47]). *Let E and F be Banach spaces. Then $(E \otimes_\gamma F)'$ is isometrically isomorphic with $\mathscr{L}(E, F')$. The isometry makes $\varphi \in (E \otimes_\gamma F)'$ correspond with $T \in \mathscr{L}(E, F')$ such that*

$$\varphi\left(\sum x_i \otimes y_i \right) = \sum \langle y_i, Tx_i \rangle.$$

THEOREM 5.8. *Let (S, d) be any metric space and E any Banach space. The norm induced on $E \otimes V$ as a subspace of $\mathrm{Lip}_{E'}(S, d)$ is the greatest crossnorm γ.*

Proof. Let $E \otimes_L V$ denote, as usual, $E \otimes V$ with the induced dual norm of $\mathrm{Lip}_{E'}(S, d)$. First we show that this is, in fact, a crossnorm. Let $\varphi \in V$ and $x \in E$. We must show that $\|x \otimes \varphi\|$, defined by $\sup \{ |(x \otimes \varphi)(f)| \mid \|f\| \leqq 1, f \in \mathrm{Lip}_{E'}(S, d) \}$,

equals $\|x\| \|\varphi\|$. Let $g \in \mathrm{Lip}\,(S, d)$, $\|g\| \leq 1$, and $x' \in E'$, $\|x'\| \leq 1$. Then the function $g \cdot x': s \to g(s)x'$ belongs to $\mathrm{Lip}_{E'}(S, d)$ and $\|g \cdot x'\| = \|g\| \|x'\| \leq 1$. Thus, $\|x \otimes \varphi\|$ $\geq |(x \otimes \varphi)(g \cdot x')| = |\varphi(g)x'(x)|$ for all $g \in \mathrm{Lip}\,(S, d)$, $x' \in E'$, $\|g\| \leq 1$, $\|x'\| \leq 1$. Therefore $\|x \otimes \varphi\| \geq \|x\| \|\varphi\|$. If $f \in \mathrm{Lip}_{E'}(S, d)$, $\|f\| \leq 1$, define $x \circ f: s \to \langle x, f(s) \rangle$. Then $x \circ f \in \mathrm{Lip}\,(S, d)$ and $\varphi(x \circ f) = (x \otimes \varphi)(f)$. Now,

$$|(x \otimes \varphi)(f)| = |\varphi(x \circ f)| \leq \|\varphi\| \|x \circ f\| \leq \|\varphi\| \|x\|.$$

Hence $\|x \otimes \varphi\| = \|x\| \|\varphi\|$ for all $x \in E$ and $\varphi \in V$.

Consider now, the identity mapping $I: E \otimes_\gamma V \to E \otimes_L V$. This is norm-decreasing since γ is the greatest crossnorm. Now, the adjoint I^* of I maps $(E \otimes_L V)'$ into $(E \otimes_\gamma V)'$. Let us write

$$\mathrm{Lip}_{E'}(S, d) \to (E \otimes_L V)' \to (E \otimes_\gamma V)'$$
$$\to \mathcal{L}(E, V') \to \mathcal{L}(E, \mathrm{Lip}\,(S, d)) \to \mathrm{Lip}_{E'}(S, d)$$

where, except for I^*, each arrow represents a canonical isometric isomorphism. Then the composition is seen to be the identity map on $\mathrm{Lip}_{E'}(S, d)$. Therefore I^* is an isometric isomorphism, and γ is the induced dual norm.   Q.E.D.

The following is an immediate consequence of 5.5 and 5.8.

COROLLARY 5.9. *Let (S, d) be a finitely compact metric space, $0 < \alpha < 1$, and E any Banach space. Then Λ is a surjective isometry if and only if the dual norm of $\mathrm{lip}_E^0(S, d^\alpha)$ induces the greatest crossnorm γ on $E' \otimes V$.*

LEMMA 5.10. *Let (S, d) be any metric space and E any Banach space. Then the mapping $x \otimes f \to f \cdot x$ of $E \otimes_\lambda \mathrm{Lip}\,(S, d)$ into $\mathrm{Lip}_E(S, d)$ is an isometry.*

Proof. Let $f = \sum_{i=1}^n f_i x_i$.

$$\|f\|_\infty = \sup_{s \in S} \left\| \sum_{i=1}^n f_i(s)x_i \right\| = \sup_{s \in S} \sup_{\|x'\| \leq 1} \left| \sum_{i=1}^n f_i(s)x'(x_i) \right|$$
$$= \sup_{\|x'\| \leq 1} \left\| \sum_{i=1}^n x'(x_i)f_i \right\|_\infty.$$

Likewise,

$$\|f\|_d = \sup_{s \neq t} \left\| \sum_{i=1}^n \frac{f_i(s) - f_i(t)}{d(s, t)} x_i \right\| = \sup_{s \neq t} \sup_{\|x'\| \leq 1} \left| \sum_{i=1}^n x'(x_i) \frac{f_i(s) - f_i(t)}{d(s, t)} \right|$$
$$= \sup_{\|x'\| \leq 1} \left\| \sum x'(x_i)f_i \right\|_d.$$

Now,

$$\lambda \left(\sum_{i=1}^n x_i \otimes f_i \right) = \sup_{\|x'\| \leq 1} \left\| \sum x'(x_i)f_i \right\|$$
$$= \sup_{\|x'\| \leq 1} \max \left(\left\| \sum x'(x_i)f_i \right\|_\infty, \left\| \sum x'(x_i)f_i \right\|_d \right)$$
$$= \max (\|f\|_\infty, \|f\|_d) = \|f\|.$$ Q.E.D.

By [18, Lemma 2.12, p. 35], the previous lemma, in particular, implies that if (S, d) is finitely compact, then $\text{lip}_E^0 (S, d^\alpha)$ induces on $E \otimes \text{lip} (S, d^\alpha)$ the least crossnorm λ. Now, $E' \otimes \text{lip}^0 (S, d^\alpha)'$, as a subspace of $(E \otimes_\lambda \text{lip}^0 (S, d^\alpha))'$, can be endowed with the dual λ' of the least crossnorm λ or with the larger dual norm of $\text{lip}_E^0 (S, d^\alpha)$. Clearly they are equal if and only if $E \otimes \text{lip}^0 (S, d^\alpha)$ is dense in $\text{lip}_E^0 (S, d^\alpha)$. Thus, we obtain the following

THEOREM 5.11. *The norms λ' and γ agree on $E' \otimes \text{lip}^0 (S, d^\alpha)'$ if and only if* (1) $E \otimes \text{lip}^0 (S, d^\alpha)$ *is dense in* $\text{lip}_E^0 (S, d^\alpha)$ *and* (2) Λ *is a surjective isometry.*

Proof. $\lambda' = \gamma$ implies that the dual norm of $\text{lip}_E^0 (S, d^\alpha)$ on $E' \otimes \text{lip}^0 (S, d^\alpha)'$, which, as a crossnorm, is always between λ' and γ, equals λ'. Thus, we get (1). Also, it equals γ, so we get (2) from 5.9. Conversely if (1) holds, the dual norm of $\text{lip}_E^0 (S, d^\alpha)$ equals λ' on $E' \otimes \text{lip}^0 (S, d^\alpha)'$ and if (2) holds, it equals γ by 5.9. Q.E.D.

The next theorem is due to Grothendieck [12] and is found, as stated here, in [11, Corollary 5.2, p. 110]. Although we use the term "Phillips space", we need for our purposes to know only that every separable dual space is a Phillips space [11, p. 108]. Thus, $\text{lip}^0 (S, d^\alpha)'$ is a Phillips space. For a definition and discussion of the approximation property, see [17, Chapter 10, §9].

THEOREM 5.12 (GROTHENDIECK). *Let E and F be Banach spaces such that either E' or F' has the approximation property and such that either E' or F' is a Phillips space. Then $E' \tilde{\otimes}_\gamma F' = (E \otimes_\lambda F)'$.*

We are now ready to state the main result.

THEOREM 5.13. *Let E be a Banach space, (S, d) a finitely compact metric space, and $0 < \alpha < 1$. If either E' or $\text{lip}^0 (S, d^\alpha)'$ has the approximation property, then Λ is a surjective isometry.*

Proof. Since $\text{lip}^0 (S, d^\alpha)'$ is a Phillips space, and either E' or $\text{lip}^0 (S, d^\alpha)'$ has the approximation property, $\lambda' = \gamma$ on $E' \otimes \text{lip}^0 (S, d^\alpha)'$ by 5.12. The result now follows from 5.11. Q.E.D.

Although it is an open question whether every Banach space has the approximation property, most of the standard ones do; e.g. abstract L and M spaces, L^p-spaces, and spaces having a Schauder basis (see [17, Chapter 10, §9]). However, it does not appear to be known in general whether the Lipschitz spaces have the approximation property. If, for a finitely compact metric space and $0 < \alpha < 1$, one could prove that $\text{Lip} (S, d^\alpha)$ has the approximation property, it would follow from 4.7 that both $\text{lip}^0 (S, d^\alpha)$ and its dual do (see [17, Remark, p. 113]). This would establish our initial conjecture in its full generality.

Suppose (S, d) is the unit interval with the usual metric. Ciesielski in [5] proved that the space of functions in $\text{lip} (S, d^\alpha)$ vanishing at 0 with the norm $\| \cdot \|_d$, has a Schauder basis and, in fact, is isomorphic with c_0. Thus, the same is true of $\text{lip} (S, d^\alpha)$

itself. From this, we deduce that Lip (S, d^α) is isomorphic with l^∞ and hence has the approximation property.

Recently, it has been shown in [3] that if S is an infinite compact subset of R^m and E is a Banach space, then $\mathrm{lip}_E (S, d^\alpha)$, $0 < \alpha < 1$, is isomorphic with the space $c_0(E)$ of null sequences of E. Thus, we know that if S is an infinite compact subset of R^m, then Lip (S, d^α) has the approximation property. Hence Λ is an isometric isomorphism. We remark that the proofs in [5] and [3] depend very heavily upon the geometry of the underlying metric spaces and seem to offer little hope of generalization.

To attack the problem of showing that Lipschitz spaces in general have the approximation property, the following ideas may be useful.

THEOREM 5.14. *Let (S, d) be any metric space and let $T \in \mathscr{L}(E, \mathrm{Lip}\,(S, d))$. If f is the element of $\mathrm{Lip}_{E'}(S, d)$ corresponding to T under the canonical identification in 5.6, then the following are equivalent.*

(a) *T is compact.*

(b) *$f(S)$ and $\tilde{f}(W) = \{d^{-1}(s, t)(f(s) - f(t)) \mid s \neq t\}$ are relatively compact in E'.*

Proof. Consider Lip (S, d) as embedded in $B(S \cup W)$. Then T can be considered as an operator $\tilde{T}$ from E into $B(S \cup W)$ and T is compact if and only if $\tilde{T}$ is. But, by [21, Theorem 2, p. 597], $\tilde{T}$ is compact if and only if $\tilde{f}(S \cup W)$ is relatively compact in E'. But this condition is equivalent to (b).   Q.E.D.

REMARK. If (S, d) has finite diameter, the condition in 5.14 (b) that $f(S)$ be relatively compact is superfluous. For, let $r = \sup \{d(s, t) \mid s, t \in S\}$ be the diameter of S and let $\tilde{f}(W)$ be relatively compact. This is equivalent with $\tilde{f}(W)$ being precompact since E' is complete. Let Z be the circled hull of $\tilde{f}(W)$. Fix any $t_0 \in S$ and consider the precompact set $f(t_0) + rZ$. For each $s \in S$,

$$f(s) = f(t_0) + d(s, t_0)\left(\frac{f(s) - f(t_0)}{d(s, t_0)}\right) \in f(t_0) + rZ$$

because $0 < d(s, t_0) \leq r$ and Z is circled. Hence $f(S) \subset f(t_0) + rZ$, so $f(S)$ is precompact.   Q.E.D.

THEOREM 5.15. *Let (S, d) be compact. Then the canonical mapping in 5.6 identifies $\mathrm{lip}_{E'}(S, d)$ with the space $\mathscr{K}(E, \mathrm{lip}\,(S, d))$ of compact operators of E into $\mathrm{lip}\,(S, d)$.*

Proof. Let $T \in \mathscr{K}(E, \mathrm{lip}\,(S, d))$. Then $T(U)$ is relatively compact in $\mathrm{lip}\,(S, d)$, where U is the unit ball of E. Thus, by 3.2,

$$\lim_{d(s,t) \to 0} \left| \frac{Tx(s) - Tx(t)}{d(s, t)} \right| = 0 \quad \text{uniformly for } x \in U.$$

Therefore, we have

$$0 = \lim_{d(s,t) \to 0} \sup_{x \in U} \left| \left\langle x, \frac{f(s) - f(t)}{d(s, t)} \right\rangle \right| = \lim_{d(s,t) \to 0} \frac{\|f(s) - f(t)\|}{d(s, t)},$$

where $f \in \text{Lip}_{E'}(S, d)$ corresponds with T. The converse is immediately obtained by reversing the steps of the proof.   Q.E.D.

Recall that for a scalar-valued function h and an element $x \in E$, the function $h \cdot x$ is defined by $h \cdot x: s \rightarrow h(s)x$. If $x \otimes h$ is identified with $h \cdot x$, then $E' \otimes \text{Lip}(S, d)$ [resp. $E' \otimes \text{lip}(S, d)$] can be identified with the subspace of finite-dimensional mappings in $\mathscr{L}(E, \text{Lip}(S, d))$ [resp. $\mathscr{L}(E, \text{lip}(S, d)]$ and in $\text{Lip}_{E'}(S, d)$ [resp. $\text{lip}_{E'}(S, d)]$. Now, using [17, Theorem 9.5, p. 113], we obtain the following two corollaries from 5.14 and 5.15 respectively.

COROLLARY 5.16. *Let (S, d) be any metric space. Then the following are equivalent*:

(a) Lip (S, d) *has the approximation property*.

(b) *For each Banach space E, $\varepsilon > 0$, and $f \in \text{Lip}_{E'}(S, d)$ satisfying condition* (b) *of 5.14, there exist $x'_1, \ldots, x'_n \in E'$ and $h_1, \ldots, h_n \in \text{Lip}(S, d)$ such that $\|f - g\| < \varepsilon$, where $g = \sum_{i=1}^{n} h_i x'_i$.*

COROLLARY 5.17. *If (S, d) is compact, then the following are equivalent*:

(a) lip (S, d) *has the approximation property*.

(b) *For any Banach space E, $\varepsilon > 0$, and $f \in \text{lip}_{E'}(S, d)$, there exist $x'_1, \ldots, x'_n \in E'$ and $h_1, \ldots, h_n \in \text{lip}(S, d)$ so that $\|f - g\| < \varepsilon$ where $g = \sum_{i=1}^{n} h_i x'_i$.*

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