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THE CONTINUITY OF FUNCTIONS ON CARTESIAN PRODUCTS

BY
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Introduction. A function f is sequentially continuous if the restriction of f to each convergent sequence (including its limit) is continuous, and f is k-continuous if its restriction to each compact subspace is continuous. If each sequentially continuous (resp. real valued sequentially continuous) function with domain X is continuous, X is a sequential space (resp. s_R-space); k-spaces and k_R-spaces are defined analogously.

In this paper we are concerned with determining conditions under which sequentially continuous or k-continuous functions on a product space $X=\prod_{\alpha\in A} X_\alpha$ will be continuous. Concerning sequentially continuous functions, our result includes conditions necessary and sufficient that a product of first countable spaces be a sequential space or an s_R-space. Concerning k-continuous functions, we show that if each X_α is either first countable or locally compact, then each k-continuous function on X with regular range is continuous, and that products of locally pseudocompact k_R-spaces are k_R-spaces. We also consider sequentially continuous and k-continuous group homomorphisms, and show, for instance, that the property "each k-continuous homomorphism with T_0 range is continuous" is preserved under arbitrary products.

All of these results are given in §5. §1 presents three fairly general conditions which can be combined to force the continuity of functions on product spaces, and these conditions are studied in §§2, 3 and 4. As incidental results we answer negatively a pair of questions: "Is the Hewitt-Nachbin realcompactification of a Fréchet space a k-space?" and "If X and Y are normal k-spaces, is the k-extension of $X\times Y$ completely regular?" raised by W. W. Comfort and E. A. Michael respectively (§2); answer a question posed by Keisler and Tarski in [10] by showing that a certain condition on cardinals is equivalent to measurability (§3); and prove an analogue of Tychonoff's Theorem by showing that k-compactness (introduced in [4]) is preserved under arbitrary products (§4).

1. **Conditions forcing continuity.** Recall that a subspace of $X=\prod_{\alpha\in A} X_\alpha$ is called a Σ-subspace if it has the form $\{x\in X\colon \delta(x, y)$ is countable$\}$ for some fixed y in X,

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where $\delta(x, y)=\{\alpha : x_\alpha \neq y_\alpha\}$. (Such subspaces have been studied in [3].) Call a Σ-subspace a Σ^0-subspace if in fact each $\delta(x, y)$ is finite, and call a function defined on a product space Σ-continuous (resp. Σ^0-continuous) if its restriction to each Σ-subspace (resp. Σ^0-subspace) is continuous. Also, call a function 2-continuous if it is continuous when restricted to each subspace of the form $\prod_{\alpha\in A} Y_\alpha$ where for each α, $1 \leqq \text{card}\,(Y_\alpha) \leqq 2$.

1.1 THEOREM. *Let Z be regular. If $f : \prod_{\alpha\in A} X_\alpha \to Z$ is Σ^0-continuous and 2-continuous, then f is continuous.*

Proof. Let U be an open subset of Z, let x be a point in $f^{-1}(U)$ and let V be an open neighborhood of x such that cl $V \subseteq U$. Let Y be the Σ^0-subspace which contains x; since $f^{-1}(V) \cap Y$ is open in Y, there exists a finite subset F of A and neighborhoods V_α of x_α such that for $W=\prod_{\alpha\in F} V_\alpha \times \prod_{\alpha\in A\setminus F} X_\alpha$, $W \cap Y \subseteq f^{-1}(V)$.

Now suppose there exists a point y in W such that $f(y) \notin U$, set $Y'=\prod_{\alpha\in A}\{x_\alpha, y_\alpha\}$ and note that, since f is 2-continuous, there exists a finite F' such that for $W'=\prod_{\alpha\in F'}\{y_\alpha\} \times \prod_{\alpha\in A\setminus F'}\{x_\alpha, y_\alpha\}$, $W' \subseteq f^{-1}(X\setminus \text{cl } V)$. Since $W' \cap W$ is clearly not empty, this is impossible, so $f(W) \subseteq U$ and hence f is continuous.

Call f Σ-semicontinuous (resp. Σ^0-semicontinuous) if whenever U is open and $x \in f^{-1}(U)$, there exists a finite $F \subseteq A$ such that $\Sigma_F(x) \subseteq f^{-1}(U)$ (resp. $\Sigma^0_F(x) \subseteq f^{-1}(U)$) where $\Sigma_F(X)=\{y : \delta(x, y)$ is countable and disjoint from $F\}$, and $\Sigma^0_F(X)=\{y : \delta(x, y)$ is finite and disjoint from $F\}$. Recall that a map is closed if it carries closed sets to closed sets, and let π_B denote the projection: $\prod_{\alpha\in A} X_\alpha \to \prod_{\alpha\in B} X_\alpha$.

1.2 THEOREM. *Let Z be regular. If $f: \prod_{\alpha\in A} X_\alpha \to Z$ is 2-continuous and Σ^0-semicontinuous, and if $\pi_F \circ f^{-1}$ is closed for each finite $F \subseteq A$, then f is continuous.*

Proof. Given $x \in f^{-1}(U)$ with U open, choose V open with $x \in V$ and cl $V \subseteq U$ and let $F \subseteq A$ be finite such that $\Sigma^0_F(x) \subseteq f^{-1}(V)$. Let $X'=\pi_F(X)$, $X''=\pi_{A\setminus F}(X)$ and $x'=\pi_F(x)$ and note that, as in the proof of 1.1, $x' \times X'' \subseteq f^{-1}(U)$. Thus x' is not in the closed set $S=\pi_F \circ f^{-1}(Z\setminus U)$, so $x \in (X'\setminus S) \times X'' \subseteq f^{-1}(U)$. Therefore f is continuous.

Note that the proof actually shows: If $f: \prod_\alpha X_\alpha \to Z$ is Σ^0-semicontinuous and for each Σ^0-subspace Y of $\prod_\alpha X_\alpha$, $\pi_F|_Y \circ f|_Y^{-1}$ is closed for each finite subset F of A, then f is Σ^0-continuous. By the obvious adaptation of our proof, the corresponding result for Σ-semicontinuous functions and Σ-subspaces holds.

For conditions under which $\pi_F \circ f^{-1}$ will be closed, see [6], [9], [16] or [17].

Where $G=\prod_\alpha G_\alpha$ with each G_α a topological group, we consider subproducts of G to be embedded in G in the natural way (i.e., $\prod_{\alpha\in B} G_\alpha=\prod_{\alpha\in B} G_\alpha \times \prod_{\alpha\in A\setminus B}\{e_\alpha\}$ where e_α is the identity of G_α). A subgroup H of G is invariant under projections if $\pi_B(H) \subseteq H$ for each $B \subseteq A$. Note that the Σ^0-subgroup (= the direct sum of the G_α), the Σ-subgroup, and of course G, are invariant under projections.

1.3 THEOREM. *Let $G=\prod_\alpha G_\alpha$ where each G_α is a topological group, let H be a subgroup of G which is invariant under projections, and let ψ be a homomorphism* *on H with T_0 range. If ψ is separately continuous (continuous on each factor) and Σ^0-semicontinuous (resp. Σ-semicontinuous) then ψ is Σ^0-continuous (resp. Σ-continuous).*

Proof. Let us first note that, like any separately continuous homomorphism (since for $\psi(x) \in V$ and $V^n \subseteq U$, $\psi(\prod_{i=1}^n \psi^{-1}(V) \cap G_i) \subseteq U$), ψ is continuous on finite products. For $x \in f^{-1}(U)$ and V such that $V^2 \subseteq U$ let $F \subseteq A$ be finite such that $\Sigma_F^0(x) \subseteq \psi^{-1}(V)$ and let $V' = \psi^{-1}(V) \cap \prod_{\alpha \in F} X_\alpha$. By the comment above, V' is open; since $\psi(V' \cdot \Sigma_F^0(x)) \subseteq \psi(V') \cdot \psi(\Sigma_F^0(x)) \subseteq V^2 \subseteq U$, this shows that ψ is Σ^0-continuous. The obvious adaptation proves the remaining statement.

2. **Σ-subspaces.** A space X is called a Fréchet space if whenever x is in the closure of $S \subseteq X$, S contains a sequence which converges to x. Each Fréchet space is sequential and each subspace of a Fréchet space is a Fréchet space—indeed, a space is hereditarily sequential if and only if it is a Fréchet space. Our first result generalizes the relevant portions of [12, Theorem II] and [5, Theorem 3].

2.1 THEOREM. *Each Σ-subspace of a product of first countable spaces is a Fréchet space.*

Proof. Let Y be a Σ-subspace of $\prod_{\alpha \in A} X_\alpha$ where each X_α is first countable, let $S \subseteq Y$ and let x be any point in the closure of S. For each α let $\{U_\alpha^n : n = 1, 2, \ldots\}$ be a nested base for the neighborhoods of x_α. Choose y' in S and, inductively, choose $y^n \in S \cap \prod_{\beta \in B(n)} U_\beta^n \times \prod_{\beta \in A \setminus B(n)} X_\beta$ where $B(n) = \{\alpha_j^i : 1 \leqq i, j < n\}$ and the α_j^i are determined by index $\delta(x, y^i)$ as $\{\alpha_j^i : i = 1, 2, \ldots\}$. Such points y^n can always be chosen since neighborhoods of x must meet S.

We complete the proof by showing that $\{y^n\}$ converges to x. Let U be any neighborhood of x and, without loss of generality, suppose $U = (\prod_{\alpha \in F} U_\alpha^m \times \prod_{\alpha \in A \setminus F} X_\alpha) \cap Y$ for some integer m and some finite subset F of A. Since F is finite and $\{(Bn)\}$ are increasing, there exists an integer n_0 such that $F \cap U_n B(n) \subseteq B(n_0)$. Since for $\alpha \notin U_n B(n)$, $y_\alpha^n = x_\alpha$, y^n is in U whenever $n > \max\{n_0, m\}$, so $\{y^n\}$ converges to x.

2.2 COROLLARY. *Each sequentially continuous function on $\prod_\alpha X_\alpha$ is Σ-semicontinuous.*

Proof. Retopologize each X_α to be discrete and apply 2.1.

Incidentally, Theorem 2.1 provides examples answering questions raised by W. W. Comfort and E. A. Michael in the context of [2] and [13] respectively. Comfort asked if the Hewitt-Nachbin realcompactification νY of a Fréchet space Y is a k-space; the answer is no since if X is a product of uncountably many copies of the reals, and Y is any Σ-subspace of X, then $\nu Y = X$ [3, Theorem 2] but X is not a k-space [11, p. 240]. Michael's question was: "Is the k-extension of $X \times Y$ completely regular when X and Y are k-spaces and are $\langle$completely regular, normal, paracompact, $\aleph_0$-spaces$\rangle$? We answer the first two: Let Y' be a product of uncountably many copies of the unit interval, and let Y be a Σ-subspace of Y'. Since Y is not locally compact, there exists [14, Theorem 3.1] a paracompact

Hausdorff (hence normal) k-space X such that $X\times Y$ is not a k-space. Also, Y is normal by [3, Theorem 1]. Since $\nu Y=Y'$ is compact, Y is pseudocompact, so $X\times Y$ is a k_R-space [18, Theorem 4]. It follows that the k-extension of $X\times Y$ (the smallest topology on $X\times Y$ which makes $X\times Y$ a k-space and coincides with the product topology on compact sets) cannot be completely regular.

Note that (by Theorem 2.1) each Σ-subspace of $\prod_{\alpha\in A} X_\alpha$ will be a Fréchet space if each X_α is itself a Σ-subproduct of a product of first countable spaces. One cannot, however, allow the factors to be arbitrary Fréchet spaces: Let X_0 denote the quotient formed by identifying the limits of a countable collection of disjoint convergent sequences. Then X_0 is a Fréchet space, but (as in the proof of [20, Theorem 6.1]) $X_0\times Y$ is a Fréchet space, for Y any T_1 space, if and only if Y is discrete. It follows that if X is an infinite product of T_1-spaces, one of whose factors contains a copy of X_0, then no Σ-subspace of X is a Fréchet space.

Of course we do not really need Σ-subspaces to be Fréchet spaces. To apply Theorem 1.1 it suffices to know that certain functions on Σ-subspaces are continuous. Where Y is any space, let $\mathscr{C}(Y)$ denote the collection of compact subsets of Y which, as subsets, have countable neighborhood bases. (For $K\subseteq Y$ a neighborhood base for K is a collection of open sets U_α such that for $V\supseteq K$ and V open, $V\supseteq U_\alpha\supseteq K$ for some α.) Let $\mathscr{C}^*(Y)=\{K\in\mathscr{C}(Y)$: as a space, K is first countable$\}$. A space Y is said to be of pointwise countable type (we abbreviate this as "of type $\mathscr{C}$") if there is a subcollection $\mathscr{C}_0(Y)$ of $\mathscr{C}(Y)$ such that for each y in Y, each C in $\mathscr{C}_0$ with y in C, and each neighborhood U of y, there exists a C' in $\mathscr{C}_0$ with $y\in C'\subseteq U\cap C$. We say that Y is of type $\mathscr{C}^*$ if $\mathscr{C}_0(Y)$ can be chosen as a subcollection of $\mathscr{C}^*(Y)$.

Spaces of type $\mathscr{C}$ were introduced and studied by Arhangel'skiĭ in [1] where it is shown that locally compact T_2 spaces and spaces complete in the sense of Čech (absolute G_δ's) and of course first countable spaces, are of this type. Arhangel'skiĭ also proved the portion of the following proposition which applies to spaces of type $\mathscr{C}$.

2.3 Proposition. (i) *Each space of type $\mathscr{C}$ is a k-space, and each space of type $\mathscr{C}^*$ is a sequential space.*

(ii) *Countable products of spaces of type $\mathscr{C}$ are of type $\mathscr{C}$ and countable products of spaces of type $\mathscr{C}^*$ are of type $\mathscr{C}^*$.*

Proof. (i) Let Y be of type $\mathscr{C}$, let $U\subseteq Y$ be k-open (i.e. $U\cap K$ is open in K for each compact K) and let $y\in U$. Choose C' in $\mathscr{C}_0(Y)$ with y in C'; since $U\cap C'$ is open in C' there exists an open U' such that $U'\cap C'=U\cap C'$ and hence, by the definition of spaces of type $\mathscr{C}$, there exists a C in $\mathscr{C}$ with $y\in C\subseteq U'\cap C'\subseteq U$. Now suppose U is not a neighborhood of y and let $\{V^n\}$ be a base for the neighborhoods of C. Then for each n, $V^n\nsubseteq U$ so there exists a point $y^n\in V^n\backslash U$. Set $K=C\cup\{y^n : n=1,2,\ldots\}$. Clearly K is compact, but since $U\cap K=C$ is not open in K, this yields a contradiction.

Now let Y be of type $\mathscr{C}^*$, and proceed as above when U is sequentially open. Construct K as before and note that K is sequential so that $U \cap K$ should be (but is not) open.

(ii) This follows by the obvious construction.

2.4 THEOREM. *Let $X=\prod_{\alpha\in A} X_\alpha$. If each X_α is of type $\mathscr{C}$, then each k-continuous function on X is Σ-continuous. If each X_α is of type $\mathscr{C}^*$, then each Σ-subspace of X is a sequential space (so each sequentially continuous function on X is Σ-continuous).*

Proof. Let f be a k-continuous function on X, let Y be a Σ-subspace of X, let U be an open subset of the range of f, and let $y \in Y \cap f^{-1}(U)$. For each α choose $C_\alpha \in \mathscr{C}_0(X_\alpha)$ with $y_\alpha \in C_\alpha$, and note that, since $f^{-1}(U) \cap \prod_\alpha C_\alpha$ must be relatively open, we may suppose that $\prod_\alpha C_\alpha$ is contained in $f^{-1}(U)$. Now suppose $Y \cap f^{-1}(U)$ is not a neighborhood of y in Y. Then, as in the proof of 2.1, we construct a sequence $\{y^n\} \subseteq Y \backslash f^{-1}(U)$ which "converges" to $\prod_\alpha C_\alpha$. Let $K_\alpha = C_\alpha \cup \{y_\alpha^n\}$; then by construction each K_α must be compact, so $\{y^n\} \subseteq \prod_\alpha K_\alpha$ must have an accumulation point. Again by the construction, such an accumulation point must lie in $\prod_\alpha C_\alpha$, but this is impossible since $f^{-1}(U) \cap \prod_\alpha K_\alpha$ is open.

Now suppose each X_α is of type $\mathscr{C}^*$, let Y be a Σ-subspace of X, let $U \subseteq Y$ be sequentially open and let y in U. Choosing C in $\mathscr{C}^*$ with y_α in C_α we may, by Theorem 2.1, suppose that $C=\prod_\alpha C_\alpha \cap Y$ is contained in U. If U is not a neighborhood of y, then as before we construct $\{y^n\} \subseteq Y \backslash U$ which "converges" to $\prod_\alpha C_\alpha$, hence to C. Let $K = \mathrm{cl}\,(C_\alpha \cup \{y_\alpha^n\})$ and let $B = \{\alpha : K_\alpha \neq C_\alpha\}$; B is countable so $\prod_{\alpha\in B} K_\alpha$ is sequential by 2.3. Now $(\prod_{\alpha\in A\backslash B} C_\alpha) \cap Y$ is sequential (by Theorem 2.1) and contains no infinite closed discrete subspace (since $\prod_{\alpha \in A\backslash B} C_\alpha$ is compact) so by [18, Theorem 2], $\prod_\alpha K_\alpha \cap Y = (\prod_{\alpha\in B} K_\alpha) \times (\prod_{\alpha\in A\backslash B} C_\alpha \cap Y)$ is sequential. But, as before, $U \cap (\prod_\alpha K_\alpha \cap Y)$ is not sequentially open and we have the desired contradiction.

Let us mention, although we do not wish to pursue it, that Theorems 2.3 and 2.4 remain true if, in the definition of spaces of type $\mathscr{C}^*$, one replaces the requirement that members of $\mathscr{C}$ be compact with the condition: Each countable subset has compact closure, or any other condition which insures countable compactness and is preserved under products.

3. **Sequential cardinals.** Theorems 2.1, 2.3 (and in fact 1.3) give conditions under which sequentially continuous functions will be Σ-continuous. In this section we investigate conditions under which a sequentially continuous function will be 2-continuous. As one would expect, such conditions involve only the cardinality of the index set A.

Let A be a set; 2^A denotes the set of subsets of A. For $\{S_n\} \subseteq 2^A$, we use $\lim S_n = S$ to mean that $\{S_n\}$ converge to S in the set-theoretic sense, i.e.,

$$S = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} S_{n+m} = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} S_{n+m}.$$

(Although it is not relevant to our considerations, it is interesting to note that this is actually convergence with respect to a standard topology, the Vietoris finite topology, on 2^A.) A function with domain 2^A will be called sequentially continuous if it preserves this type of sequential convergence.

Recall that the cardinal of A is measurable if there exists a nonzero (completely additive) measure $\mu: 2^A \to 2$ (as usual 2 denotes the set $\{0, 1\}$ with the discrete topology) which maps finite sets to zero. Temporarily, call a cardinal strongly sequential if there exists a nonzero sequentially continuous function $\sigma: 2^A \to 2$ mapping finite sets to zero. Although they are not given a name, strongly sequential cardinals are considered briefly in [10, p. 270] and our next result answers one of the questions posed there.

3.1 THEOREM. *A cardinal is strongly sequential if and only if it is measurable.*

Proof. Since each measure is sequentially continuous, each measurable cardinal is strongly sequential. Suppose A has strongly sequential cardinality and let $\sigma: 2^A \to 2$ be as above. We first note that there exists a subset A_0 of A such that $\sigma(A_0)=1$ and whenever $A_0=B \cup C$ with $B \cap C=\varnothing$, $\sigma(B)$ or $\sigma(C)$ is zero. (If not, we can construct disjoint sets $\{A_n\}$ with $A=\bigcup_n A_n$ and $\sigma(A_n)=1$ for each n. But then $\bigcup_n \bigcap_m A_{n+m}=\varnothing=\bigcap_n \bigcup_m A_{n+m}$ so $\sigma(\varnothing)=\lim_n \sigma(A_n)=1$ contrary to the assumption that σ maps finite sets to zero.) Now define $\mu: 2^{A_0} \to 2$ by the rule: $\mu(B)=\max\{\sigma(B'): B' \subseteq B\}$; a straightforward computation shows that μ is a measure, so A_0 and hence A has measurable cardinal.

Since the term is available, we call a cardinal squential if there exists a nonzero real valued sequentially continuous function $\sigma: 2^A \to R$ which maps finite sets to zero. It is shown in [12] that each cardinal less than the first weakly inaccessible cardinal is not sequential. Let II denote the space $\{0, 1\}$ with topology $\{\varnothing, \{0, 1\}, \{1\}\}$, and let R denote the real line.

3.2 THEOREM. *Let $X=\prod_{\alpha\in A} X_\alpha$ where each X_α is not indiscrete and, for a space Z, let $P(Z)$ denote the assertion: Each sequentially continuous function $f: X \to Z$ is 2-continuous. Then:*

(i) *P(II) if and only if A is countable;*

(ii) *$P(R)$ if and only if* card *A is not sequential;*

(iii) *$P(2)$ if and only if* card *A is not measurable.*

Proof. (i) If A is uncountable, choose proper closed $Y_\alpha \subseteq X_\alpha$ and define $f(x)=0$ iff $\{\alpha : x_\alpha \notin Y_\alpha\}$ is countable. Then $f^{-1}(0)$ is proper, dense, and sequentially closed, so f is sequentially continuous but not continuous. The converse is trivial.

(ii) Suppose card A is sequential and suppose $\sigma: 2^A \to R$ is nontrivial and sequentially continuous. Define $f: \prod_\alpha X_\alpha \to R$ by the rule $f(x)=\sigma(\{\alpha : x_\alpha \notin Y_\alpha\})$ where Y_α are as in (i). It is easily verified that f is sequentially continuous but not continuous.

Now suppose card A is not sequential, let $Y_\alpha \subseteq X_\alpha$ with $1 \leqq \text{card}\,(Y_\alpha) \leqq 2$ for each α, choose $x \in \prod_\alpha Y_\alpha$ and let $f\colon \prod_\alpha Y_\alpha \to R$ be sequentially continuous. By Theorem 2.1, the restriction of f to the Σ-subspace Y containing x is continuous, so by for instance [5, Theorem 1] $f|_Y$ extends to a continuous function $f^*\colon \prod_\alpha Y_\alpha \to R$. Choose any point y in $\prod_{\alpha\in A} Y_\alpha$ and define $\sigma\colon 2^A \to R$ by the rule $\sigma(B)=f(xBy) -f^*(xBy)$ where xBy is the point with coordinates y_α for α in B and x_α otherwise. Since f and f^* coincide on Y, σ maps finite (indeed countable) sets to zero, and σ is easily seen to be sequentially continuous. Since card A is not sequential, it follows that σ is identically zero, hence that $0=\sigma(A)=f(xAy)-f^*(xAy)=f(y)-f^*(y)$. Since y was arbitrary, $f=f^*$ and is therefore continuous.

It should be mentioned that the proofs of this section are based on refinements of techniques introduced by Mazur in [12] and Varopoulos in [19]. Also, the statement in (ii) holds with R replaced by any regular Hausdorff space Z for which either each point of Z, or the diagonal of $Z \times Z$, is a sequential G_δ, by essentially the same proof. This yields a corresponding improvement of Theorems 5.1 and 5.2. For results concerning ranges which are not regular, see [12] and [5].

4. **Products of k-compact spaces.** Where Y is any space, let kY (resp. k_RY) denote the set underlying Y endowed with the smallest topology making each k-continuous (resp., real valued k-continuous) function on Y continuous. Define sY and s_RY analogously, using sequentially continuous functions instead of k-continuous functions. Note that Y is a k-space (resp. sequential space) if and only if $Y=kY$ (resp. $Y=sY$); we call a completely regular space Y a k_R-space (resp. s_R-space) if $k_RY=Y$ (resp. $s_RY=Y$).

In contrast to §§2 and 3, whose results will yield applications of Theorem 1.1, the results of this section will yield applications of Theorem 1.2. What we will need are conditions under which $k(X)$ or $s(X)$ will be countably compact, and conditions under which $k_R(X)$ and $s_R(X)$ will be pseudocompact, where X is a product space $\prod_{\alpha\in A} X_\alpha$. We begin with a result for more general spaces. Call a subset U of X k_R-open, s_R-open, etc., if it is open in k_RX, s_RX, etc.

4.1 THEOREM. *Let X be any T_1 space.*

(i) *kX is countably compact if and only if whenever $S \subseteq X$ is infinite, there exists a compact subset K of X such that $K \cap S$ is infinite.*

(ii) *sX is countably compact if and only if X is sequentially compact.*

(iii) *If X is completely regular, k_RX is pseudocompact if and only if for each infinite family of nonempty disjoint k_R-open sets $\{U^n\}$, there exists a compact set $K \subseteq X$ such that $K \cap U^n \neq \varnothing$ for infinitely many n.*

(iv) *If X is completely regular, s_RX is pseudocompact if and only if for each infinite family of nonempty disjoint s_R-open sets $\{U^n\}$, there exists a convergent sequence $S \subseteq X$ such that $S \cap U^n \neq \varnothing$ for infinitely many n.*

Proof. For (i) and (ii) recall that a T_1 space is countably compact if and only if each closed discrete subset is finite. A subset of kX which meets each compact subset in a finite set, or a subset of sX which meets each convergent sequence in a finite set, is closed and discrete. For (ii) and (iii), use complete regularity to construct a function $f\colon X\to R$ such that $f(X\backslash\bigcup_n U^n)=\{0\}$, while $f^{-1}(\{n\})\subseteq U^n$ and $f|_{U^n}$ is continuous, $n=1, 2, \ldots$. If each compact set (resp. each convergent sequence) meets only finitely many of the U^n, f is k-continuous (resp. squentially continuous). The converse direction is obvious.

We might mention that (i) and (iii) were proved (and used) in [15]. Suppose now that $X=\prod_{\alpha\in A} X_\alpha$. As examples in [8] show, the assumption that $k(X_\alpha)$ is countably compact for each α does not insure that X, much less kX, will be countably compact. However, if each X_α is strongly countably compact (each countable subset has compact closure) then X is strongly countably compact, and therefore kX is (strongly) countably compact. Regarding (ii), recall that $X=\prod_{\alpha\in A} X_\alpha$ is sequentially compact if and only if A is countable and each factor is sequentially compact.

A space Y such that $k_R Y$ is pseudocompact is said to be k-compact. This term was introduced in [4] where it is shown that k-compact spaces can, in the context of function spaces, serve as a satisfactory substitute for compact spaces. Nevertheless, the following analogue of Tychonoff's Theorem is surprising.

4.2 THEOREM. *Each product of completely regular k-compact spaces is k-compact.*

Proof. Let $X=\prod_{\alpha\in A} X_\alpha$ with each X_α k-compact and suppose X is not k-compact. Then there exists a countably infinite family, $\{U^n\}$, of nonempty disjoint k_R-open subsets of X which has no cluster point. Choose $x^n\in U^n$ and k_R-open V^n with $x^n\in V^n$ such that the k_R-closure of V^n is contained in U^n. Then by Corollary 2.2 there exist finite $F_n\subseteq A$ such that $\Sigma_{F_n}(x^n)\subseteq V^n$ where $\Sigma_B(x)$ denotes the set of points y in the Σ-subspace containing x such that $y_\alpha=x_\alpha$ for each α in B. Thus the k_R-closure of each $\Sigma_{F_n}(x^n)$ is contained in U^n so (as in the proof of Theorem 1.1) $\prod_{\alpha\in F_n}\{x^n_\alpha\}\times\prod_{\alpha\in A\backslash F_n} X_\alpha$ is contained in U^n. Choose $x'\in\prod_{\alpha\in A\backslash C} X_\alpha$ where $C=\bigcup_n F_n$ and set $X'=\prod_{\alpha\in C} X_\alpha\times\{x'\}$. Then each U^n meets X', so X' is not k-compact. Thus we may, without loss of generality, suppose that A is countable. say $X=\prod_{i\in N} X_i$.

For each n, choose k_R-open sets $V_i^n\subseteq X_i$, $1\leqq i\leqq n$, such that

$$\prod_{i=1}^{n} V_i^n\times\prod_{i>n}\{x_i^n\}\subseteq U^n.$$

This is possible since $k_R(\prod_{i=1}^n X_i)=\prod_{i=1}^n k_R X$ by [18, Theorem 4]. Now by Theorem 4.1, if U is any infinite family of nonempty disjoint k_R-open subsets of X_i, then there exists a compact subset S of X_i and an infinite subfamily U' of U such that each member of U' meets S. Thus, inductively, we may choose infinite sets $N_i\subseteq\{n : n\geqq i\}$ and compact subsets K_i' of X_i such that $N_i\subseteq N_{i-1}$ and, for each $n\in N_i$, $K_i'\cap V_i^n\neq\varnothing$. Choose $n_i\in N_i$, note that $\{n_i\}$ is infinite, and set

$$K_i = K_i'\cup\{(x^{n_j})_i : j<i\}.$$

Then each K_i is compact, so $K=\prod_i K_i$ is a compact subset of X. To complete the proof (that is, to contradict the assumption that $\{U^n\}$ has no cluster point) it suffices to show that $K \cap U^{n_j} \neq \varnothing$ for each j. For a fixed index j, define y^j as follows: For $i>j$, $(y^j)_i=(x^{n_j})_i$; for $i \leqq j$ $(y^j)_i \in K_i \cap V_i^{n_j}$ (this intersection is not empty since $n_j \in N_j \subseteq N_i$). Now $y^j \in K$ since, by the construction of K_i, $(y^j)_i =(x^{n_j})_i \in K_i$ for $j<i$. Finally, $y^j \in U^{n_j}$, in fact, $y^j \in \prod_{i=1}^{n_j} V_i^{n_j} \times \prod_{i>n_j} \{(x^{n_j})_i\}$, since $j \subseteq n_j$ (because $n_j \in N_j \subseteq \{n : n \geqq j\}$).

We naturally call a space Y s-compact if $s_R Y$ is pseudo-compact.

4.3 THEOREM. *Let $X=\prod_{\alpha \in A} X_\alpha$ where each X_α is completely regular and s-compact. If* card (A) *is not sequential, X is s-compact.*

Proof. Since card (A) is not sequential, the reduction to the case with A countable goes through as in the proof of Theorem 4.2. Paralleling that proof, one constructs points x^n in $U^n \cap K$ where K is a product of countably many convergent sequences. Since K is compact and first countable, it is sequentially compact so $\{x^n\}$ have a convergent subsequence, as desired.

For our applications we need results slightly stronger than 4.2 and 4.3. The proofs of these two theorems easily generalize to yield:

4.4 THEOREM. *Let $X=\prod_\alpha X_\alpha$ where each X_α is completely regular, let $Y_\alpha \subseteq X_\alpha$ and let $Y=\prod_\alpha Y_\alpha$.*

(i) *If each Y_α is pseudocompact as a subspace of $k_R(X_\alpha)$, then Y is pseudocompact as a subspace of $k_R(X)$.*

(ii) *If* card A *is not sequential and each Y_α is pseudocompact as a subspace of $s_R(X_\alpha)$, then Y is pseudocompact as a subspace of $s_R(X)$.*

5. Sequentially continuous and k-continuous functions on products.

The problem of determining when finite products of k-spaces, k_R-spaces, sequential spaces or s_R-spaces will retain these properties has been studied in [14], [18], [20], and the papers referenced therein. In particular, the results of [18] show that for X_0 as in the discussion following 2.2 and, for $n \geqq 1$, X_n a copy of the integers, $\prod_{n=0}^\infty X_n$ is not a k_R-space even though each finite subproduct is sequential (and $\prod_{n=1}^\infty X_n$ is first countable).

Turning to infinite products, we begin with some additional terminology. Call a space Y a k_3-space if each k-continuous function on Y with regular (or equivalently T_3) range is continuous, and an s_N-space if each integer valued sequentially continuous function on Y is continuous.

5.1 THEOREM. *Let $X=\prod_{\alpha \in A} X_\alpha$ where each X_α is not indiscrete.*

(i) *If X is sequential, then* card (A) *is countable.*

(ii) *If X is an s_R-space, then* card (A) *is not sequential.*

(iii) *If X is an s_N-space, then* card (A) *is not measurable.*

Furthermore, if each X_α is of type $\mathscr{C}^*$, the converse of each of these statements is true.

Proof. Theorems 1.1, 2.3, 2.4 and 3.2.

5.2 THEOREM. *Let $X=\prod_{\alpha\in A} X_\alpha$ where each X_α is a completely regular s_R-space. If all but one of the X_α are pseudocompact, then X is an s_R-space if and only if* card (A) *is not sequential.*

Proof. Suppose that for $\alpha\neq\alpha_0$, X_α is pseudocompact. Let $f\colon X\to R$ be sequentially continuous, and let $F\subseteq A$ be finite. We will be applying Theorem 1.2, and an examination of the proof of that theorem shows that we may suppose that $\alpha_0\in F$. Now for $X'=s_R(\prod_{\alpha\in A\setminus F} X_\alpha)$ and $X''=\prod_{\alpha\in F} X_\alpha$, X'' is a sequential space by [18, Theorem 4] and X' is pseudocompact by Theorem 4.3. Hence by [17, Corollary 1] $\pi_F\colon X'\times X''\to X''$ maps zero sets to zero sets. Now by [18, Theorem 4], $X'\times X''$ is itself an s_R-space, so $f\colon X'\times X''\to R$ is continuous. It follows that $\pi_F\circ f^{-1}$ is closed. Hence by Theorem 1.2, X is an s_R-space.

Theorem 5.2 is best in the sense that if infinitely many of the X_α are not pseudocompact, then there exists an s_R-space X_{α_0} such that $\prod_{\alpha\in A\cup\{\alpha_0\}} X_\alpha$ is not an s_R-space. (This follows by [18, Theorem 4].) However, if we restrict every factor, a somewhat sharper result is possible.

5.3 THEOREM. *Let $X=\prod_{\alpha\in A} X_\alpha$ where each X_α is a completely regular s_R-space. If each X_α is locally pseudocompact, then X is an s_R-space if and only if* card (A) *is not sequential.*

Proof. Let $f\colon X\to R$ be sequentially continuous, let $U\subseteq R$ be open, let $x\in f^{-1}(U)$ and let V be a neighborhood of $f(x)$ such that cl $V\subseteq U$. For each α let S_α be a closed pseudocompact neighborhood of x_α. As in the proof of Theorem 5.2 (the necessary result on projections is given in [17, Theorem 2]) the restriction of f to $S=\prod_{\alpha\in A} S_\alpha$ is continuous, so we may suppose that S is contained in $f^{-1}(V)$. For $F\subseteq A$ let $\Sigma_F(S)=\{x\in X$: for each α in F, $x_\alpha\in S_\alpha$, and $x_\alpha\notin S_\alpha$ for only finitely many α in $A\}$. It suffices to show that $\Sigma_F(S)$ is contained in $f^{-1}(\text{cl } V)$ for some finite F (since then cl $\Sigma_F(S)\subseteq f^{-1}(U)$ is a neighborhood of X). Suppose not; then we can construct a sequence $\{V^n\}$ of k_R-open sets such that

$$V^n\subseteq \prod_{\alpha\in A'} S_\alpha\times\prod_{\alpha\in F_n}(X_\alpha\setminus S_\alpha)\times\prod_{\alpha\in A''} X_\alpha$$

where $A'=\bigcup_{i=1}^{n-1} F_i$, $A''=A\setminus(A'\cup F_n)$ and F_n is finite and disjoint from A', such that $V^n\cap f^{-1}(V)=\varnothing$ for each n. Choose $x^n\in V^n$ and set $S'_\alpha=S_\alpha\cup\{x^n_\alpha\}$. Each S'_α is pseudocompact (since $S'_\alpha\setminus S_\alpha$ contains at most one point) so $\prod_\alpha S'_\alpha=S'$ is pseudocompact when topologized as a subspace of $s_R(X)$ (by Theorem 4.4). Hence $\{V^n\cap S'\}$ must have a k_R-cluster point, and clearly that cluster point is in S. This contradicts the fact that $f^{-1}(U)$ is k_R-open and completes the proof.

Recall that each T_0 topological group is completely regular Hausdorff. Call a group G an s-group (resp. k-group) if each sequentially continuous (resp. k-continuous) homomorphism from G to a T_0 group is continuous. By Theorems 1.3, 2.2, and 3.2 we have:

5.4 THEOREM. *Let* $G=\prod_{\alpha\in A} G_\alpha$, *let* $G_0=\{x\in G : \text{card}\,(\delta(x,e))$ *is not sequential*$\}$ (*where e is the identity of G*) *and let H be any subgroup of* G_0 *which is invariant under projections. If each* G_α *is an s-group, then H is an s-group.*

Note that (also by Theorems 1.3 and 3.2) the property "each sequentially continuous homomorphism whose range has a base at the identity consisting of open subgroups is continuous" is preserved under products with nonmeasurably many factors. The corresponding result for homomorphisms whose range contains no small subgroups (for instance, any Lie group) also holds, by an easy adaptation of the proof of [19, Lemma π_2]. For some much more interesting results concerning sequentially continuous homomorphisms, see [19].

Turning now to k-continuous functions, note that by [7, Theorem 4.3] and Theorem 4.1, countably compact k-spaces are preserved under countable products, and by Proposition 2.3 spaces of type $\mathscr{C}$ are preserved under countable products.

5.5 PROPOSITION. *Suppose each* X_α *is a* T_1 *k-space. If* $\prod_{\alpha\in A} X_\alpha$ *is a k-space, then all but countably many of the* X_α *are countably compact, and the product of the countably compact factors is pseudocompact.*

Proof. A straightforward adaptation of the proof outlined in [11, p. 240] shows that a product of uncountably many infinite discrete spaces is not a k-space. Since a T_1-space is countably compact if and only if it contains no infinite closed discrete subspace, and since each closed subspace of a k-space is a k-space, it follows that all but countably many of the factors are countably compact. Now a product space is pseudocompact if each of its countable subproducts is pseudocompact, so the product of the countably compact factors is pseudocompact.

5.6 THEOREM. *Let* $X=\prod_{\alpha\in A} X_\alpha$.

(i) *If each* X_α *is of type* $\mathscr{C}$, *then X is a* k_3*-space.*

(ii) *If each* X_α *is a completely regular locally pseudocompact* k_R*-space, then X is a* k_R*-space.*

Proof. (i) Follows by Theorems 1.1 and 2.4, and (ii) follows by the obvious adaptation of the proof of 5.3.

5.7 THEOREM. *If H is a subgroup of a product of k-groups and H is invariant under projections, then H is a k-group.*

Proof. Theorems 1.3 and 2.2.

REFERENCES

1. A. V. Arhangel′skiĭ, *Bicompact sets and the topology of spaces*, Trudy Moskov. Mat. Obšč. **13** (1965), 3–55 = Trans. Moscow Math. Soc. **1965**, 1–62. MR **33** #3251.

2. W. W. Comfort, *On the Hewitt realcompactification of a product space*, Trans. Amer. Math. Soc. **131** (1968), 107–118. MR **36** #5896.

3. H. H. Corson, *Normality in subsets of product spaces*, Amer. J. Math. **81** (1959), 785–796. MR **21** #5947.

4. ———, *Compact subsets and the* C. O. Topology (to appear).

5. R. Engelking, *On functions defined on Cartesian products*, Fund. Math. **59** (1966), 221–231. MR **34** #3546.

6. I. Fleischer and S. P. Franklin, "On compactness and projections" in *Contributions to extension theory of topological structures*, Academic Press, New York, 1969, 77–79.

7. Z. Frolík, *The topological product of two pseudo-compact spaces*, Czechoslovak Math. J. **10 (85)** (1960), 339–349. MR **22** #7099.

8. ———, *The topological product of countably compact spaces*, Czechoslovak Math. J. **10 (85)** (1960), 329–338. MR **22** #8480.

9. A. Hager, *Projections of zero sets (and the fine uniformity on a product)*, Trans. Amer. Math. Soc. **140** (1969), 87–94.

10. H. J. Keisler and A. Tarski, *From accessible to inaccessible cardinals. Results holding for all accessible cardinal numbers and the problem of their extension to inaccessible ones*, Fund. Math. **53** (1963/64), 225–308. MR **29** #3385.

11. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955. MR **16**, 1136.

12. S. Mazur, *On continuous mappings on Cartesian products*, Fund. Math. **39** (1952), 229–338. MR **14**, 1107.

13. E. A. Michael, $\aleph_0$-*spaces*, J. Math. Mech. **15** (1966), 983–1002. MR **34** #6723.

14. ———, *Local compactness of Cartesian products of quotient maps and k-spaces*, Ann. Inst. Fourier Grenoble **53** (1969), 281–286.

15. N. Noble, *Countably compact and pseudocompact products*, Czechoslovak Math. J. **19 (94)** (1969), 390–397.

16. ———, *Products with closed projections*, Trans. Amer. Math. Soc. **140** (1969), 381–391.

17. ———, *A note on z-closed projections*, Proc. Amer. Math. Soc. **23** (1969), 73–76.

18. ———, *Products of quotient maps and spaces with weak topologies*, (to appear).

19. N. Varopoulos, *A theorem on the continuity of homomorphisms of locally compact groups*, Proc. Cambridge Philos. Soc. **60** (1964), 449–463. MR **29** #184.

20. D. Weddington, *k-spaces*, Doctoral Dissertation, University of Miami, Coral Gables, Fla., 1968.

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