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NONCOMPACT SIMPLICES

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Abstract. A bounded, but not necessarily closed, (Choquet) simplex in R^n with nonempty interior is the intersection of $n+1$ half-spaces. There is no bounded simplex with nonempty interior in an infinite dimensional Hausdorff real linear topological space.

0. **Introduction.** We suppose that X is a nonzero real linear space. We use the symbols t, u, v, w, x, y, z to represent elements of X and $\alpha, \beta, \gamma, \delta, \eta$ to denote real numbers $\geqq 0$. We say that $S\,(\subset X)$ is a *simplex* if S is nonempty and convex and whenever $(x+\alpha S) \cap (y+\beta S) \neq \varnothing$ then there exist z, γ such that $(x+\alpha S) \cap (y+\beta S) = z+\gamma S$. The main results of this paper are: *a bounded simplex in R^n with nonempty interior is the intersection of $n+1$ half-spaces* (*each of which can be open or closed*) and *there is no bounded simplex with nonempty interior in an infinite dimensional Hausdorff linear topological space*. These results are in §§2 and 3, respectively. In §1, we discuss a nontopological boundedness condition.

1. **The closure of a simplex is sometimes a simplex.**

1. Definition. If $\varnothing \neq A \subset X$ and $x \in X\backslash\{0\}$ we write

$$D(A, x) = \sup\{\alpha : \alpha x \in A-A\}.$$

($D(A, x)$ is the "diameter of A in the direction of x".)

2. Lemma. *Let X be a linear topological space, $S\,(\subset X)$ be a simplex and $z \in \operatorname{int} S$. If $t, t', u, u' \in \bar{S}$ are such that $t'-t=u'-u\neq 0$ and ϕ is a nonzero continuous linear functional on X such that $\phi(t)=\inf \phi(S)$, $\phi(u)=\sup \phi(S)$ then $D(S, t'-t)=\infty$.*

Proof. We suppose $0<\alpha<\frac{1}{2}$. Since $t, u \in \bar{S}$ and $z \in \operatorname{int} S$,

$$(1)\qquad (1-\alpha)t+\alpha z \in S$$

and

$$\alpha t+(1-2\alpha)u+\alpha z \in S$$

from which

$$(2)\qquad (1-\alpha)t+\alpha z \in (1-2\alpha)(t-u)+S$$

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hence there exist w, β such that

(3) $$S \cap [(1-2\alpha)(t-u)+S] = w+\beta S.$$

Now

$$S \supset w+\beta S \subset (1-2\alpha)(t-u)+S$$

hence

$$\begin{aligned}\inf \phi(S) &\leqq \phi(w)+\beta \inf \phi(S) \leqq \phi(w)+\beta \sup \phi(S)\\ &\leqq (1-2\alpha)\phi(t-u)+\sup \phi(S)\end{aligned}$$

from which

$$\beta \operatorname{diam} \phi(S) \leqq (1-2\alpha)[\phi(t)-\phi(u)]+\operatorname{diam} \phi(S) = 2\alpha \operatorname{diam} \phi(S).$$

Since $\operatorname{int} S \neq \varnothing$, $\operatorname{diam} \phi(S)>0$ and so

(4) $$\beta \leqq 2\alpha.$$

From (1), (2) and (3)

$$(1-\alpha)t+\alpha z \in w+\beta S.$$

Now $\phi(t') \geqq \inf \phi(S)=\phi(t)$ and so $\phi(t'-t) \geqq 0$. Similarly $\phi(u'-u) \leqq 0$. Hence, since $t'-t=u'-u$, $\phi(t'-t)=\phi(u'-u)=0$ and so $\phi(t')=\inf \phi(S)$ and $\phi(u')=\sup \phi(S)$. Further, $t'-u'=t-u$ and so, by an argument similar to that above,

$$(1-\alpha)t'+\alpha z \in w+\beta S.$$

Hence $\beta D(S, t'-t) \geqq 1-\alpha$ and so, from (4), $2\alpha D(S, t'-t) \geqq 1-\alpha$. Letting $\alpha \to 0$ gives the required result.

3. Definition. If $\varnothing \neq A \subset X$ we say that A is *D-bounded* if, for all $x \in X \backslash \{0\}$, $D(A, x)<\infty$.

4. Theorem. *If X is a linear topological space, S $(\subset X)$ is a D-bounded simplex and* $\operatorname{int} S \neq \varnothing$ *then $\bar{S}$ is a simplex.*

Proof. It is immediate that $\operatorname{int} S$ is a D-bounded simplex. Further $\bar{S}=\overline{\operatorname{int} S}$ so we can suppose, without loss of generality, that S is open. Suppose $T=(x+\alpha\bar{S}) \cap (y+\beta\bar{S}) \neq \varnothing$. If $T=\{z\}$ then $T=z+0\bar{S}$ and the result follows. So we suppose that $T \ni v$, $T \ni w$ where $v \neq w$. This implies that $\alpha>0$, $\beta>0$. We shall show that

(5) $$(x+\alpha S) \cap (y+\beta S) \neq \varnothing.$$

Indeed, if (5) is false then there exists a nonzero continuous linear functional ϕ on X such that

$$\inf \phi(x+\alpha S) = \phi(v) = \sup \phi(y+\beta S).$$

We suppose, without loss of generality, that $\beta \leqq \alpha$. Then, from Lemma 2 with

$$t = \frac{v-x}{\alpha}, \qquad u = \frac{v-y}{\beta}, \qquad t' = \frac{w-x}{\alpha}, \qquad u' = \frac{1}{\beta}\left\{\left(1-\frac{\beta}{\alpha}\right)v+\frac{\beta}{\alpha} w-y\right\},$$

$D(S, t'-t)=\infty$ contradicting our hypothesis that S is D-bounded. Hence (5) is true. Since S is a simplex, there exist z, γ such that

$$(x+\alpha S) \cap (y+\beta S) = z+\gamma S. \tag{6}$$

It follows by considering gauge functionals that if A, B are convex and open in X and $A \cap B \neq \varnothing$ then $\overline{A \cap B}=\bar{A} \cap \bar{B}$. Hence from (6),

$$(x+\alpha \bar{S}) \cap (y+\beta \bar{S}) = z+\gamma \bar{S}.$$

Thus $\bar{S}$ is a simplex.

5. Remark. Let $X=R^2$ and $S=(0, 1) \times R$. Then S is a simplex but $\bar{S}$ is not a simplex. This example, suggested by D. Randtke, shows that some boundedness condition on S is essential in Theorem 4. In the next theorem we explore the properties of D-boundedness.

6. Theorem. (a) *If X is a Hausdorff linear topological space, $\varnothing \neq A \subset X$ and A is bounded then A is D-bounded.*

(b) *If $\varnothing \neq A \subset R^n$ and A is convex and D-bounded then A is bounded.*

Proof. (a) Let $x \in X\backslash\{0\}$, U be a balanced neighborhood of 0 such that $U \not\ni x$ and V be a neighborhood of 0 such that $V-V \subset U$. If $\alpha x \in V-V$ then $\alpha x \in U$ and so $\alpha<1$. Hence $D(V, x) \leqq 1$. But A is absorbed by V hence $D(A, x)<\infty$.

(b) We first translate A to contain 0 and then replace R^n by the subspace spanned by A. We may, therefore, suppose that $\operatorname{int} A \ni z$, say. If A is unbounded then, from the argument used in [2, p. 370], $\bar{A}$ contains an infinite half-line l. Then $A \supset \frac{1}{2}(z+l)$ and so A is not D-bounded.

7. Remark. We give an example of a D-bounded open simplex in $C[0, 1]$. Let

$$S = \left\{x : x \in C[0, 1],\ x(t) > 0 \text{ for all } t \in [0, 1],\ \int x < 1\right\}.$$

S is clearly open. S is D-bounded, for if $x \in C[0, 1]\backslash\{0\}$ and $y, y+\alpha x \in S$ then

$$0 \leqq y+\alpha x \tag{7}$$

and

$$\int (y+\alpha x) \leqq 1; \tag{8}$$

if, for some $t \in [0, 1]$, $x(t)<0$ then, from (7), $\alpha|x(t)| \leqq y(t)$ hence $D(S, x)<\infty$; if, on the other hand, for all $t \in [0, 1]$, $x(t) \geqq 0$ then $\int x>0$ and, from (8), $\alpha \int x \leqq 1-\int y$ and, again, $D(S, x)<\infty$. Finally, S is a simplex, for if $(x+\alpha S) \cap (y+\beta S) \neq \varnothing$ then

$$(x+\alpha S) \cap (y+\beta S) = x \vee y+\left[\left(\int x+\alpha\right) \wedge \left(\int y+\beta\right)-\int x \vee y\right] S.$$

We observe that S is unbounded. We shall see in Theorem 14 that this is, in fact, forced by the other conditions on S.

The above example emerged from a conversation with M. Rosenfeld.

2. **Simplices in R^n.** We suppose throughout this section that $X=R^n$. If $S(\subset X)$ is a bounded simplex then, by using the argument sketched in the proof of Theorem 6 (b), we can suppose that int $S\neq \varnothing$ and, to avoid trivial cases, that $n>1$.

If $A\subset X$ we write conv A for the convex hull of A. If $A=\{x_1,\ldots,x_k\}$ is a finite set we write

$$\mathrm{conv}_+\ A = \left\{\sum_{i=1}^{k} \alpha_i x_i : \alpha_i > 0,\ \sum_{i=1}^{k} \alpha_i = 1\right\}.$$

If S is as above then, from Theorem 6 (a) and Theorem 4, $\bar{S}$ is a compact simplex hence [1, Remarks following Definition 5] and [3, Proposition 9.11, p. 75] there exists affinely independent $v_0,\ldots,v_n\in X$ such that $\bar{S}=\mathrm{conv}\,\{v_0,\ldots,v_n\}$. (We shall return to this result in Theorem 12.) Since S is convex, $S\supset \mathrm{conv}_+\,\{v_0,\ldots,v_n\}$.

8. Lemma. *We suppose that X, S, $v_0,\ldots,v_n$ are as above and, further, that $v_0=0$. Then $v_1,\ldots,v_n$ form a basis of X. We write $\psi_1,\ldots,\psi_n$ for the dual basis of X and ψ for $\psi_1+\cdots+\psi_n$. We define the lattice ordering $\leqq$ (with lattice operations $\vee$ and $\wedge$) on X by "$x\leqq y$" means that "for all $i=1,\ldots,n$, $\psi_i(x)\leqq\psi_i(y)$". We write $w=n^{-1}(v_1+\cdots+v_n)\in\bar{S}$.*

(a) *If $x+\alpha S\supset z+\delta S$ then $z\geqq x$ and $\psi(z)+\delta\leqq\psi(x)+\alpha$.*

(b) *If $\eta>0$, $z\geqq x$ and $\psi(z)+\eta<\psi(x)+\alpha$ then $z+\eta w\in x+\alpha S$.*

(c) *If $(x+\alpha S)\cap(y+\beta S)=T\neq\varnothing$ then $T=x\vee y+\gamma S$, where*

$$\gamma = \min\{\psi(x)-\psi(x\vee y)+\alpha,\ \psi(y)-\psi(x\vee y)+\beta\}.$$

(d) *If $y\in S$, $z\geqq 0$, $\psi(y)=\psi(z)=1$ and*

$$\{i : 1\leqq i\leqq n,\ \psi_i(z)=0\}\subset\{i : 1\leqq i\leqq n,\ \psi_i(y)=0\}$$

then $z\in S$.

(e) *If $x\in\alpha S$ and $y\in\beta S$ then $x\wedge y\in\gamma S$, where*

$$\gamma = \min\{\psi(x\wedge y)-\psi(x)+\alpha,\ \psi(x\wedge y)-\psi(y)+\beta\}.$$

Proofs. (a) follows by taking the images under ψ_i and ψ and the inf and sup, respectively. In (b) the conditions imply that $z-x+\eta w\in\alpha$ int $S\subset\alpha S$.

(c) Since S is a simplex; there exist z, δ such that $T=z+\delta S$. From (a),

$$z\geqq x,\qquad z\geqq y,\qquad \psi(z)+\delta\leqq\psi(x)+\alpha,\qquad \psi(z)+\delta\leqq\psi(y)+\beta \tag{9}$$

and so $z\geqq x\vee y$. If we had $z\neq x\vee y$ then, for some i such that $1\leqq i\leqq n$ and for some $\eta>0$, $\psi_i(x\vee y)+\eta<\psi_i(z)$ hence $\psi(x\vee y)+\eta<\psi(z)$. From (9),

$$\psi(z)\leqq\min\{\psi(x)+\alpha,\ \psi(y)+\beta\}$$

thus, from (b), $x\vee y+\eta w\in(x+\alpha S)\cap(y+\beta S)=T$ hence $x\vee y+\eta w\geqq z$. Letting $\eta\to 0$ gives $x\vee y\geqq z$, hence $z=x\vee y$, as required. Returning to (9) we now see that $0\leqq\delta\leqq\gamma$. If $\gamma=0$ then clearly $\gamma=\delta$. If $\gamma>0$ then, for any η such that $0<\eta<\gamma$,

$\psi(x\vee y)+\eta<\min\{\psi(x)+\alpha, \psi(y)+\beta\}$ and so, as above, $x\vee y+\eta w\in z+\delta S=x\vee y+\delta S$. Hence $\eta w\in\delta S$ which implies that $\eta\leqq\delta$. Letting $\eta\to\gamma$ gives that $\gamma\leqq\delta$. Hence $\gamma=\delta$, as required.

(d) If $\alpha>0$ is sufficiently small then $z-\alpha y\geqq 0$ and $z-\alpha y+\alpha \operatorname{int} S\subset \operatorname{int} S$ hence $(z-\alpha y+\alpha S)\cap S\neq\varnothing$. From (c), $(z-\alpha y+\alpha S)\cap S=(z-\alpha y+\alpha S)$, hence $z-\alpha y+\alpha S\subset S$. In particular, $z=z-\alpha y+\alpha y\in S$.

(e) We have $0\in(-x+\alpha S)\cap(-y+\beta S)$. The result is immediate from (c).

9. Lemma. *We suppose that* $X, S, v_0,\ldots,v_n$ *are as in the discussion preceding Lemma* 8. *We write* $\mathscr{N}=\{F:\varnothing=F\subset\{0,\ldots,n\}\}$ *and, if* $F\in\mathscr{N}$,

$$[F]=\operatorname{conv}_+\{v_i : i\in F\}.$$

Also we write $\mathscr{F}=\{F : F\in\mathscr{N}, [F]\cap S\neq\varnothing\}$.

(a) $S=\bigcup\{[F] : F\in\mathscr{F}\}$.

(b) $G\in\mathscr{F}$, $F\in\mathscr{N}$ *and* $G\subset F$ *imply* $F\in\mathscr{F}$.

(c) *If* $F, G\in\mathscr{F}$ *and* $F\cap G\neq\varnothing$ *then* $F\cap G\in\mathscr{F}$.

Proofs. (a) We first observe that $S\subset\bar{S}=\bigcup\{[F] : F\in\mathscr{N}\}$ hence

$$S\subset\bigcup\{[F] : F\in\mathscr{F}\}.$$

Now we suppose $F\in\mathscr{F}$. If $F=\{0,\ldots,n\}$ then $[F]=\operatorname{int} S\subset S$. If $F\subsetneqq\{0,\ldots,n\}$ we assume, without loss of generality, that $F\not\ni 0$ and we translate S so that $v_0=0$. There exists y, say, such that $y\in[F]\cap S$. It follows from Lemma 8 (d) that, if $z\in[F]$ then $z\in S$, i.e. $[F]\subset S$. Hence $\bigcup\{[F] : F\in\mathscr{N}\}\subset S$.

(b) As in (a) we suppose, without loss of generality, that $F\not\ni 0$, $v_0=0$ and $y\in[G]\cap S$. From Lemma 8 (d) again, $[F]\subset S$, hence $F\in\mathscr{F}$.

(c) This time we reduce the problem to the case $F\cap G\ni 0$ and $v_0=0$. We write $x=\sum\{v_i : i\in F\}$ and $y=\sum\{v_i : i\in G\}$. Then, from (a), $x\in(\psi(x)+1)S$ and $y\in(\psi(y)+1)S$. From Lemma 8 (e), $x\wedge y\in(\psi(x\wedge y)+1)S$. But

$$x\wedge y=\sum\{v_i : i\in F\cap G\}$$

hence $F\cap G\in\mathscr{F}$.

10. Theorem. *If S is a bounded simplex in R^n and* $\operatorname{int} S\neq\varnothing$ *then S is the intersection of $n+1$ half-spaces.*

Proof. We use the notation of Lemma 9. If there exists $F, G\in\mathscr{F}$ such that $F\cap G=\varnothing$ then, using (b) and (c) of Lemma 9, for each $i\in\{0,\ldots,n\}$, $\{i\}=(F\cup\{i\})\cap(G\cup\{i\})$ hence $\{i\}\in\mathscr{F}$ and so $\mathscr{F}=\mathscr{N}$. Hence, from Lemma 9 (a), $S=\bar{S}$ which is the intersection of $n+1$ closed half-spaces.

If, on the other hand, for all $F, G\in\mathscr{F}$, $F\cap G\neq\varnothing$ then, from (b) and (c) of Lemma 9 again, $\mathscr{F}$ is a filter of subsets of $\{0,\ldots,n\}$ and so there exists $F_0\in\mathscr{N}$ such that $\mathscr{F}=\{F : F\in\mathscr{N}, F\supset F_0\}$. From Lemma 9 (a)

$$S=\{\lambda_0v_0+\cdots+\lambda_nv_n : \lambda_i>0 \text{ for all } i\in F_0,\ \lambda_i\geqq 0 \text{ for all } i\in\{0,\ldots,n\}\backslash F_0, \lambda_0+\cdots+\lambda_n=1\},$$

which is the intersection of m open and $n+1-m$ closed half-spaces, where m is the cardinality of F_0.

11. REMARKS. We leave to the reader the proof of the converse of Theorem 10, that any *bounded* set with nonempty interior that is the intersection of $n+1$ half-spaces is a simplex. We observe that there are exactly $n+2$ affinely different bounded simplices with nonempty interior in R^n, distinguished by the minimum dimension of an "open face" (i.e., a set $[F]$ for $F \in \mathcal{N}$). On the other hand there are just 3 topologically different such simplices. Finally, any such simplex contains exactly 0, 1 or $n+1$ of its vertices.

3. **Bounded simplices with nonempty interior.** In this section we return to the general notation of §1.

12. THEOREM. *If S is a linearly compact* (*see* [2]) *simplex in X, $x \in S$, $\eta > 0$ and $x - \eta(S-x) \subset S$, then S is the convex hull of a finite affinely independent set containing at most $1+1/\eta$ points.*

Proof. Let C be the cone $\{(\alpha y, \alpha) : \alpha \geqq 0, y \in S\}$ in $E \times R$. From [2], C induces a lattice ordering on $L = C - C$. The map ϕ: $(x, \alpha) \to \alpha$ is a positive linear functional on L, and $f \in L, f \geqq 0$ and $\phi(f) = 0$ imply that $f = 0$. If $f \in L$ we write $\|f\| = \phi(|f|)$. $\|\cdot\|$ is a norm on L with respect to which L is a normed lattice and $\|\cdot\|$ is additive on the positive elements of L. Hence the completion, $\tilde{L}$, of L is an abstract L-space.

We write $g = (1+1/\eta)(x, 1)$. If $f \in L, f \geqq 0$ then there exists $\alpha \geqq 0$, $y \in S$ such that $f = (\alpha y, \alpha)$. Then

$$\begin{aligned} \phi(f)g - f &= \alpha(1+1/\eta)(x, 1) - (\alpha y, \alpha) \\ &= (\alpha/\eta)(x - \eta(y-x), 1) \geqq 0 \end{aligned} \tag{10}$$

by hypothesis, from which g is an order unit for $\tilde{L}$. From [4, V. 8.6, Corollary 1, p. 249], $\tilde{L}$ is finite dimensional and hence, for some $n \geqq 1$, $L = \tilde{L} \cong l_n^1$.

We suppose that $f_1, \ldots, f_n$ are the elements of L that correspond to the basic unit vectors of l_n^1. For each $i = 1, \ldots, n$ there exists $x_i \in S$ such that $f_i = (x_i, 1)$. It follows from the linear independence of $\{f_i\}$ that $\{x_i\}$ are affinely independent and from $L = \operatorname{lin}\{f_i\}$ that $S = \operatorname{conv}\{x_i\}$.

There exist $\alpha_1, \ldots, \alpha_n \geqq 0$, $\sum \alpha_i = 1$ such that $(x, 1) = \sum \alpha_i f_i$. We suppose that $i \in \{1, \ldots, n\}$ has been chosen so that $\alpha_i \leqq 1/k$. From (10), $f_i \leqq \phi(f_i)g = \|f_i\| g = g = (1+1/\eta) \sum \alpha_i f_i$ hence, projecting along the vector f_i, $1 \leqq (1+1/\eta)\alpha_i \leqq (1+1/\eta)1/k$. Thus $k \leqq 1+1/\eta$, as required.

13. REMARKS. The constant $1+1/\eta$ above is the best possible in the sense that if S is the convex hull of $n+1$ affinely independent points and x is the barycenter of S then $x - n^{-1}(S-x) \subset S$. This observation is due to D. Randtke. Theorem 12 generalizes the results used at the beginning of §2.

14. THEOREM. *There is no bounded simplex with nonempty interior in an infinite dimensional Hausdorff linear topological space.*

Proof. If X is a Hausdorff linear topological space and S ($\subset X$) is a bounded simplex with $x \in \operatorname{int} S$ then, from Theorem 6 (a) and Theorem 4, $\bar{S}$ is a simplex. $\bar{S}$ is linearly compact and, for some $\eta > 0$, satisfies the condition of Theorem 12. It now follows from Theorem 12 that $\dim X < \infty$.

15. Remark. The above result should be compared with Remark 7.

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