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QUADRATIC VARIATION OF POTENTIALS AND HARMONIC FUNCTIONS

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Abstract. We prove the existence of a finite quadratic variation for stochastic processes $u(Y)$, where Y is Brownian motion on a Green domain of R^n, stopped upon reaching the Martin boundary, and u is a positive superharmonic function on the domain. As by-products we have results which are also of interest from a non-probabilistic point of view.

1. **Introduction and summary.** Let $X=\{X_t, 0\leqq t\leqq T<\infty\}$ be a stochastic process with a.e. continuous sample paths; let π_n be a sequence of partitions of $[0, T]$ given by $0=t_0^{(n)}<t_1^{(n)}<\cdots<t_{k_n}^{(n)}=T$. Let $S^2(X,\pi_n)=\sum_{j=0}^{k_n-1}(X_{t_{j+1}^{(n)}}-X_{t_j^{(n)}})^2$. Then it may be important to know the asymptotic behaviour of the random variable $S^2(X,\pi_n)$, if $\|\pi_n\|=\max_j(t_{j+1}^{(n)}-t_j^{(n)})\to 0$. If a limit exists in some sense it is called the quadratic variation of X.

This problem was studied first by P. Lévy in the case where X is Brownian motion on $[0, T]$. Lévy obtained L_1-convergence of $S^2(X,\pi_n)$ if $\|\pi_n\|\to 0$ and a.e. convergence if either $\{\pi_n\}$ is monotone and $\|\pi_n\|\to 0$ or if $\sum\|\pi_n\|<\infty$. In this case the limit is the constant T. Since under very general assumptions continuous martingales can be obtained as images of Brownian motion under a random time change [2] it is suggestive to expect similar results for continuous martingales. In [7], the problem of quadratic variation was studied for L_2-martingales which can be written as stochastic integrals with respect to Brownian motion. In [4] L_1-convergence of $S^2(X,\pi_n)$ was proved for the wider class of continuous L_2-martingales X, if $\|\pi_n\|\to 0$. In that paper it was also pointed out that the increasing process A of the Doob decomposition of the submartingale X^2 plays an important role. The most recent work is a paper by Millar [6] which discusses convergence of $S^2(X,\pi_n)$ in probability of L_1-bounded martingales. This includes of course martingales for which X^2 may fail to be a submartingale because $EX_t^2=\infty$. Under stronger assumptions Millar obtains convergence of $S(X,\pi_n)$ in L_p-norm.

It is the purpose of this paper to discuss the quadratic variation of a process $\{u(Y_t), t\geqq 0\}$ where

(1) Y is Brownian motion on some Green domain $\Omega\subseteq R^n$, stopped upon reaching $\partial\Omega$, the Martin boundary of Ω,

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(2) u is either a potential on Ω or the difference of two positive harmonic functions on Ω, and is extended to $\partial\Omega$ by its fine boundary function u^*.

We shall show that a quadratic variation of the process $u(Y)$ always exists in the sense that $S^2(u(Y), \pi_n)$ converges in probability for very general $\{\pi_n\}$, and we shall compute the value of the limit. Under somewhat stronger assumptions we shall obtain convergence of $S(u(Y), \pi_n)$ a.e. and also in L_2-norm if u is a potential and in L_p-norm $(p>1)$ if u is harmonic. In §5 we derive a generalized version of a formula of K. Ito.

For our proofs we shall use results in [4], [6] for martingales and make use of the theory of additive functionals for n-dimensional Brownian motion. This theory is presented for general Markov processes e.g. in [1].

The statements on a.e. convergence of $S(u(Y), \pi_n)$ will be obtained from a lemma on martingales whose derivation we give in §2 and which strengthens a theorem in [7].

As by-products of our discussion we will have a few results which might also be interesting from a strictly nonprobabilistic point of view. Finally we remark that parts of our theorems have formulations in terms of general superharmonic functions.

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2. **Three lemmas for martingales.** Let $M=\{M_t, \mathfrak{F}_t, 0\leqq t\leqq\infty\}$ be a martingale for which almost all sample paths are continuous and $EM_\infty^2<\infty$ and $\mathfrak{F}_t=\mathfrak{F}_{t+}$. All processes will be assumed adapted to $\mathfrak{F}_t$. The submartingale M^2 is of class (D) and moreover, because of the continuity of the M-paths, regular in the sense of [5]. Therefore M^2 has a Doob decomposition:

$$M_t^2 = M_0^2+N_t+A_t, \qquad 0 \leqq t \leqq \infty,$$

where N_t is a continuous martingale and A_t is a continuous nondecreasing process and $N_0=A_0=0$. The decomposition is unique.

Let $\{\pi_n\}$ be a sequence of partitions given by

$$0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n}^{(n)} < \infty$$

such that (1) $t_{k_n}^{(n)}\to\infty$, (2) $\|\pi_n\|=\max_j (t_{j+1}^{(n)}-t_j^{(n)})\to 0$.

Let $S^2(M, \pi_n)=\sum_j (M_{t_{j+1}^{(n)}}-M_{t_j^{(n)}})^2$.

The following lemma is essentially contained in [4].

LEMMA 2.1. *If M, A, $\{\pi_n\}$ are as above, then $S^2(M, \pi_n)\to A_\infty$ in L_1-norm and hence in probability.*

LEMMA 2.2. *If M, A, π_n are as above then as $0\leqq s\leqq t\leqq\infty$,*
(a) $E(M_t-M_s)^2=E(A_t-A_s)$, (b) $E(M_t-M_s)^4\leqq 30E(A_t-A_s)^2$.

Proof. Assume first that $s=0$, $M_0=0$ a.e. In this case (a) follows from the definition of A and (b) is essentially contained in Theorem 6.2 in [6]. We shall give here a different derivation of (b): By [2] we know that we can redefine M in such a way that there is a one-dimensional Brownian motion Y for which A is a random time change and for which $Y_{A_t}=M_t$.

However:

$$\{\tfrac{1}{6}Y_t^4-tY_t^2+\tfrac{1}{2}t^2, \mathfrak{F}_t, t \geqq 0\}$$

is a martingale as is easily seen from the fact that the function

$$u(y, t) = \tfrac{1}{6}y^4-ty^2+\tfrac{1}{2}t^2$$

is a solution of $\frac{1}{2}u_{yy}+u_t=0$. Therefore, for any stopping time τ,

$$\{\tfrac{1}{6}Y_{t\wedge\tau}^4-(t \wedge \tau)Y_{t\wedge\tau}^2+\tfrac{1}{2}(t \wedge \tau)^2, \mathfrak{F}_t, t \geqq 0\},$$

is also a martingale. Hence

$$\tfrac{1}{6}EY_{t\wedge\tau}^4+\tfrac{1}{2}E(t \wedge \tau)^2 = E(t \wedge \tau)Y_{t\wedge\tau}^2$$

where the expectations are finite. Applying the Schwarz inequality, we obtain:

$$(EY_{t\wedge\tau}^4)^2-30EY_{t\wedge\tau}^4\cdot E(t \wedge \tau)^2+9\{E(t \wedge \tau)^2\}^2 \leqq 0,$$

which implies $EY_{t\wedge\tau}^4 \leqq 30E(t\wedge\tau)^2$. Letting $t \uparrow \infty$ we obtain by Fatou's Lemma and the monotone convergence theorem

$$EY_\tau^4 \leqq 30E\tau^2.$$

Letting $\tau=A_t$, (b) follows.

The general case can be reduced to the one just considered: If we define $M_t'=M_{t+s}-M_s$, $\mathfrak{F}_t'=\mathfrak{F}_{t+s}$, then $\{M_t', \mathfrak{F}_t', 0\leqq t\leqq\infty\}$ is a martingale for which $M_0'=0$ a.e. The fact that $\{M_t^2-A_t, \mathfrak{F}_t, 0\leqq t\leqq\infty\}$ is a martingale implies that $\{M_t'^2-A_t', \mathfrak{F}_t', 0\leqq t\leqq\infty\}$ is a martingale for $A_t'=A_{t+s}-A_s$. Hence the increasing process in the Doob decomposition of $M_t'^2$ is A_t'.

LEMMA 2.3. *Let M and A be as above. Let a.e. $A\cdot(\omega)$ be absolutely continuous with respect to Lebesgue measure and let $\int_0^\infty E(\dot{A}_t^2)\,dt<\infty$. If $\{\pi_n\}$ is a sequence of partitions such that $t_{k_n}^{(n)}\to\infty$ and $\sum \|\pi_n\|<\infty$, then $S^2(M, \pi_n)\to A_\infty$ a.e.*

Proof. (a) It is easy to see that $ES^2(M, \pi_n)=EA_{t_{k_n}^{(n)}}$.

(b) We show now that

$$E\{S^2(M, \pi_n)-A_{t_{k_n}^{(n)}}\}^2 \leqq 31 \sum_j E(A_{t_{j+1}^{(n)}}-A_{t_j^{(n)}})^2.$$

The left side equals

$$E\left[\sum_{j=0}^{k_n-1}\{(M_{t_{j+1}^{(n)}}-M_{t_j^{(n)}})^2-(A_{t_{j+1}^{(n)}}-A_{t_j^{(n)}})\}\right]^2$$

$$= \sum_{j=0}^{k_n-1} E\{(M_{t_{j+1}^{(n)}}-M_{t_j^{(n)}})^2-(A_{t_{j+1}^{(n)}}-A_{t_j^{(n)}})\}^2.$$

The latter equation holds because the summands are orthogonal. This is derived as follows. If $t_1<t_2<t_3<t_4$, then

$$E\{(M_{t_4}-M_{t_3})^2-(A_{t_4}-A_{t_3})\}\{(M_{t_2}-M_{t_1})^2-(A_{t_2}-A_{t_1})\}$$
$$= E[\{(M_{t_2}-M_{t_1})^2-(A_{t_2}-A_{t_1})\}E\{(M_{t_4}-M_{t_3})^2-(A_{t_4}-A_{t_3})|\mathfrak{F}_{t_3}\}].$$

But the conditional expectation is 0 because $M_t^2-A_t$ is a martingale. Now

$$E\{S^2(M,\pi_n)-A_{t_{k_n}^{(n)}}\}^2 \leqq \sum_{j=0}^{k_n-1} \{E(M_{t_{j+1}^{(n)}}-M_{t_j^{(n)}})^4+E(A_{t_{j+1}^{(n)}}-A_{t_j^{(n)}})^2\}$$

and by Lemma 2.2

$$\leqq 31 \sum_{j=0}^{k_n-1} E(A_{t_{j+1}^{(n)}}-A_{t_j^{(n)}})^2.$$

(c)
$$P\{|S^2(M,\pi_n)-A_{t_{k_n}^{(n)}}| > \varepsilon\} \leqq \frac{1}{\varepsilon^2} E\{S^2(M,\pi_n)-A_{t_{k_n}^{(n)}}\}^2$$
$$\leqq \frac{31}{\varepsilon^2} \sum_{j=1}^{k_n-1} E(A_{t_{j+1}^{(n)}}-A_{t_j^{(n)}})^2.$$

Now

$$(A_{t_{j+1}^{(n)}}-A_{t_j^{(n)}})^2 = \left(\int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \dot{A}_t\,dt\right)^2 \leqq (t_{j+1}^{(n)}-t_j^{(n)}) \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \dot{A}_t^2\,dt,$$

and therefore

$$P\{|S^2(M,\pi_n)-A_{t_{k_n}^{(n)}}| > \varepsilon\} \leqq \frac{31}{\varepsilon^2} \|\pi_n\| \int_0^{t_{k_n}^{(n)}} E(\dot{A}_t^2)\,dt.$$

Hence we have by the Borel-Cantelli lemma

$$|S^2(M,\pi_n)-A_{t_{k_n}^{(n)}}| \to 0 \quad \text{a.e.}$$

and because $t_{k_n}^{(n)} \to \infty$ and therefore $A_{t_{k_n}^{(n)}} \to A_\infty$ a.e., we get $S^2(M,\pi_n) \to A_\infty$ a.e.

REMARK. The preceding lemma remains of course true if

(1) M is a continuous L_2-martingale on the finite interval $[0, T]$ with a.e. absolutely continuous A_t and $\int_0^T E(\dot{A}_t^2)\,dt<\infty$.

(2) π_n is a partition of $[0, T]$ such that $\sum \|\pi_n\|<\infty$. By Doob [3] the martingales of (1) are exactly the processes obtained as stochastic integrals $\int_0^t f_s\,dY(s)$ where Y is a 1-dimensional Brownian motion and f is such that $\int_0^T Ef_t^4\,dt<\infty$, both defined on a possibly enlarged probability space. The relation between f and A is given by $\dot{A}_t=f_t^2$. For these stochastic integrals the assertion of Lemma 2.3 was proved in [7] under the stronger assumption that $\lim_{n\to\infty} \|\pi_n\| n^{2+\delta}=0$ for some $\delta>0$.

3. **Quadratic variation of potentials.** We introduce some notations to be used in this section and in §4 and §5. Let Ω be a Green domain $\subseteq R_n$ $(n\geqq 2)$, $\partial\Omega$ its Martin boundary and g its Green function. Let Y be Brownian motion on $\Omega \cup \partial\Omega$ stopped upon reaching $\partial\Omega$; P_x, E_x refer to this motion starting at x. As σ-fields $\mathfrak{F}_t$ (to be suppressed subsequently in the notation) we shall use $\mathfrak{G}_{t^+}$, where the $\mathfrak{G}_t$'s

are the σ-fields determined by the history of Y up to time t. Let ζ be the first entry time of Y into $\partial\Omega$. If B is any open set, let T_B denote the first entry time of Y into B. If f is an extended real-valued function on Ω, we let $\Omega_f=\{x; f(x)<\infty\}$. If f is defined on Ω and has fine boundary function f^*, we extend f to $\partial\Omega$ by f^*; and if $\{\pi_n\}$ is a sequence of partitions given by $0=t_0^{(n)}<t_1^{(n)}<\cdots<t_{k_n}^{(n)}<\infty$, we let

$$S^2(f, \pi_n) = S^2(f(Y), \pi_n) = \sum_{j=0}^{k_n-1} \{f(Y_{t_{j+1}^{(n)}})-f(Y_{t_j^{(n)}})\}^2.$$

Now let $p(x)=\int_\Omega g(x, y)\mu(dy)$ be a potential on Ω. We recall the following well-known properties (see e.g. [8], [9]):

(i) $p^*\equiv 0$.

(ii) $p=\sum_{k=1}^{\infty} p_k$, where the p_k are continuous potentials which may assume the value ∞. (This follows e.g. from a theorem of Kishi which states that there is a sequence $F_n \uparrow \subseteq \Omega$ of closed sets such that $\mu(F_n^c) \downarrow 0$ and $\int_{F_n} g(\cdot, y)\mu(dy)$ are continuous potentials converging pointwise to p.)

(iii) $\Omega-\Omega_p$ is a polar set.

(iv) The first order derivatives of p exist a.e. (w.r.t. Lebesgue measure) and are locally integrable. These derivatives are the derivatives of p in distribution sense. Also in distribution sense: $\Delta p=-2\mu$.

In view of (iii), $\Omega-\Omega_p$ is avoided by the Y-paths and we may and shall consider Y as a process with state space Ω_p. Let $\tau_n=T_{\{p>n\}}$. Then $\tau_n<\tau_{n+1}$ and if $x\in\Omega_p$, then P_x-a.e., $p(Y_t)$ is continuous (and finite) on $[0, \infty]$, which implies $P_x\{\tau_n\to\infty\} = 1$.

The following two propositions answer the question of Doob decomposibility of the supermartingale $p(Y)$.

Proposition 3.1. *If μ lives on a polar set, then for all n and all $x\in\Omega_p$, the processes $\{p(Y_{t\wedge\tau_n}), 0\leqq t\leqq\infty, P_x\}$ are martingales.*

Proof. We prove the theorem first for continuous p (possibly assuming the value ∞). Fix $x\in\Omega_p$. Since $\tau_1<\tau_2$ implies that $p(Y_{t\wedge\tau_1})=p(Y_{t\wedge\tau_2\wedge\tau_1})$, it is sufficient to prove the theorem for sufficiently large n. But for sufficiently large n, $x\in\{p<n\}$. The set $\{p<n\}$ is open and has μ-measure 0. This implies that p is harmonic on $\{p<n\}$ and hence $\{p(Y_{t\wedge\tau_n}), 0\leqq t\leqq\infty, P_x\}$ is a martingale.

For general p let $p=\sum p_k$ where the p_k are continuous and let $\tau_n^{(k)}=T_{\{p_k>n\}}$. Then for $x\in\Omega_p$,

$$\begin{aligned} E_x p(Y_{t\wedge\tau_n}) &= E_x \sum_k p_k(Y_{t\wedge\tau_n}) = \sum_k E_x p_k(Y_{t\wedge\tau_n}) = \sum_k E_x p_k(Y_{t\wedge\tau_n^{(k)}\wedge\tau_n}) \\ &= \sum_k p_k(x) = p(x). \end{aligned}$$

Proposition 3.2. *The following conditions are equivalent*:

(1) *μ does not charge any polar set.*

(2) $\lim_{n\to\infty} nP_x\{\tau_n<\infty\}=0$ *for all* $x\in\Omega_p$.

(2′) $\lim_{n\to\infty} nP_x\{\tau_n<\infty\}=0$ *for some* $x\in\Omega_p$.

(3) *There exists a continuous homogeneous additive functional A of Y such that* $E_x A_\infty = p(x)$ *for all* $x \in \Omega_p$.

If one, and hence all, of these conditions are satisfied then:

(a) *A is determined uniquely* (*up to equivalence*).

(b) *If we let* $M_t = p(Y_t) + A_t$, *then the processes* $\{M_t, 0 \leqq t \leqq \infty, P_x\}$ *are martingales for all* $x \in \Omega_p$.

(c) *If* $f \geqq 0$ *is a measurable function on* Ω, *then for* $x \in \Omega_p$,

$$E_x \int_0^\zeta f(Y_s)\, dA_s = \int_\Omega f(y) g(x, y) \mu(dy).$$

Proof. Let us say that p fulfills condition (R) if the restriction of p to Ω_p is a regular potential (of $(Y, 1)$) in the sense of [1], i.e. if $E_x p(Y_{T_n}) \to E_x p(Y_T)$ for all $x \in \Omega_p$ and stopping times $T_n \uparrow T$. By VI,T20 in [5] and the continuity of the $p(Y)$-paths, (2) and (R) are equivalent. By IV-T3.13, IV-T2.13, VI-T3.1 in [1], (R) implies (3), (a), (c). On the other hand (3) implies (R) by the remark preceding IV, D3.2 in [1] and (b) by a simple calculation. Since for Brownian motion polar sets and semipolar sets coincide, (1) implies (R) by VI,T3.5 in [1]. We want to show now that one of the equivalent conditions (R), (2), (3) implies (1). If the former conditions hold for p, we know from VI-T3.5 in [1] that μ does not charge any polar set $\subseteq \Omega_p$. In order to show that $\mu\{y; p(y) = \infty\} = 0$, it is sufficient to prove that

$$q(x) = \int_{\{y, p(y) = \infty\}} g(x, y) \mu(dy) \equiv 0.$$

But since $q \leqq p$, the fact that (2) holds for p implies that (2) holds for q. Hence (3) holds also for q. Let B be the continuous homogeneous additive functional of Y corresponding to q by (3). Since by Proposition 3.1, $\{q(Y_{t \wedge \tau_n}), 0 \leqq t \leqq \infty, P_x\}$ is a martingale for all n and $x \in \Omega_q$, we get $q(x) = E_x B_\infty = 0$ for $x \in \Omega_q$, hence $q \equiv 0$. We finish the proof of this proposition by showing that (2′) implies (2). Let $x_0 \in \Omega_p$, and assume that $\lim_{n \to \infty} n P_{x_0}\{\tau_n < \infty\} = 0$. Assume first that p is continuous (possibly infinite). If also $x_1 \in \Omega_p$, then for some n_1, $x_0, x_1 \in \{p < n_1\}$. For $n \geqq n_1$, the functions $nP_{\cdot}\{\tau_n < \infty\}$ are harmonic on the open set $\{p < n_1\}$ and we have by Harnack's inequality $nP_{x_1}\{\tau_n < \infty\} \leqq C \cdot nP_{x_0}\{\tau_n < \infty\}$ with $C < \infty$, which implies $\lim_{n \to \infty} nP_{x_1}\{\tau_n < \infty\} = 0$. In the general case let $p = \sum p_k$ where the p_k are continuous. Then $x_0 \in \Omega_{p_k}$ and if $\tau_n^{(k)} = T_{\{p_k > n\}}$, then $\lim_{n \to \infty} nP_{x_0}\{\tau_n^{(k)} < \infty\} = 0$. Therefore (2) and hence (3) hold for p_k. Let $A^{(k)}$ be the A corresponding to p_k by (3); then the processes $\{p_k(Y_t) + A_t^{(k)}, 0 \leqq t \leqq \infty, P_x\}$ are martingales for $x \in \Omega_{p_k}$. Since $\Omega_p \subseteq \bigcap_k \Omega_{p_k}$ we conclude by VII, T32(1) and VI, T20 in [5] that (2) holds for p.

Remark. If the conditions of Proposition 3.2 hold for p, then (c) and the "energy formula," VII, T23 in [5] imply

$$E_x A_\infty^2 = 2E_x \int_0^\infty p(Y_s)\, dA_s = 2E_x \int_0^\zeta p(Y_s)\, dA_s = 2 \int_\Omega p(y) g(x, y) \mu(dy).$$

On the other hand if $p_1(x)=2\int p(y)g(x,y)\mu(dy)\not\equiv\infty$, then conditions (1)–(3) of Proposition 3.2 hold, since $p_1(x_0)<\infty$ implies $\lim_{n\to\infty} n^2P_{x_0}\{\tau_n<\infty\}=0$. This can be seen as follows. If $p_1(x_0)<\infty$, then clearly $x_0\in\Omega_p$. By the Frostman maximum principle, which says that the supremum of a potential equals its supremum on the support of the corresponding measure, we have for $x\in\Omega$,

$$p(x) \leqq \int_{n/2<p} g(x,y)\mu(dy)+\frac{n}{2}.$$

Hence we get for $x\in\{p>n\}$, $1\leqq(2/n)\int_{n/2<p} g(x,y)\mu(dy)$, which implies for $x\in\Omega$, $P_x\{\tau_n<\infty\}\leqq(2/n)\int_{n/2<p} g(x,y)\mu(dy)$, because $P_x\{\tau_n<\infty\}$ is the equilibrium potential at x of the open set $\{p>n\}$. We conclude

$$n^2P_x\{\tau_n < \infty\} \leqq 4\int_{n/2<p} p(y)g(x,y)\mu(dy).$$

But the right side converges to 0 for $x=x_0$ because $\int_\Omega p(y)g(x_0,y)\mu(dy)<\infty$, which implies also $\mu\{y;p(y)=\infty\}=0$.

As a consequence of Propositions 3.1 and 3.2 we obtain the following

PROPOSITION 3.3. *If $p(x)=\int g(x,y)\mu(dy)$ is a potential on Ω, then there is a uniquely determined continuous homogeneous additive functional A of Y (considered as process with state space Ω_p) such that, for all n and all $x\in\Omega_p$, the processes*

$$\{p(Y_{t\wedge\tau_n})+A_{t\wedge\tau_n},\ 0 \leqq t \leqq \infty,\ P_x\}$$

are martingales. Moreover $E_xA_\infty\leqq p(x)$.

Proof. By VI, P3.6 in [1], there is a unique decomposition $\mu=\mu_1+\mu_2$ where μ_1 does not charge any polar set and μ_2 lives on a polar set. The proposition follows from the two preceding propositions.

REMARK. This proposition is the analogue of a theorem of K. Ito and S. Watanabe about the representation of a nonnegative supermartingale as the sum of a local martingale and an increasing process.

We shall prove now two lemmas which will be useful.

LEMMA 3.4. *If $p(x)=\int g(x,y)\mu(dy)$ is a potential on Ω, then*

$$2\int_\Omega p(y)g(x,y)\mu(dy) = p^2(x)+\int_\Omega |\operatorname{grad} p|^2(y)g(x,y)\,dy.$$

Proof. Assume first that p is bounded. Then p^2 is a distribution and we obtain (in distribution sense)

$$\Delta(p^2) = 2|\operatorname{grad} p|^2-4p\mu.$$

(It is not difficult to see that the application of the product rule in the preceding differentiation is legitimate. In particular, $|\operatorname{grad} p|^2$ is locally integrable, because for a bounded potential p_0 whose measure μ_0 has compact support, the energy

$\int_\Omega |\text{grad}\, p_0|^2(x)\, dx = \text{const} \int_\Omega p_0(x)\mu_0(dx)$ is finite.) If we now let $v = p_1 - p^2$ with $p_1(x) = 2\int p(y)g(x, y)\mu(dy)$, then $\Delta v = -2|\text{grad}\, p|^2$ (in distribution sense), which implies that $v = u$ a.e. (Lebesgue measure), where u is superharmonic. Since v is bounded, so is u, and hence the potential part of u, namely

$$p_2(x) = \int |\text{grad}\, p|^2(y) g(x, y)\, dy.$$

Now let $w = p^2 + p_2$. Then $\Delta w = -4p\mu$. Since w is lower semicontinuous and bounded and $w^* \equiv 0$, we have $w = p_1$.

In the general case let

$$p_n(x) = p(x) \wedge n = \int g(x, y)\mu_n(dy) \qquad (n \geqq 3).$$

Then $p_n \uparrow p$, and it is sufficient to prove:

$$\lim_{n\to\infty} \int |\text{grad}\, p_n|^2 g(x, y)\, dy = \int |\text{grad}\, p|^2(y) g(x, y)\, dy, \tag{1}$$

$$\lim_{n\to\infty} \int p_n(y) g(x, y)\mu_n(dy) = \int p(y) g(x, y)\mu(dy). \tag{2}$$

(1) is proved as follows: $\text{grad}\, p$ and $\text{grad}\, p_n$ exist for all $x \notin N$ where $N \subseteq \Omega$ is a set of Lebesgue measure 0; moreover if $x \notin N$, then $|\text{grad}\, p_n|^2(x) \leqq |\text{grad}\, p_{n+1}|^2(x) \leqq |\text{grad}\, p|^2(x)$, equality occurring for sufficiently large n. (1) follows from the monotone covergence theorem.

(2) follows for any sequence $p_n(x) = \int g(x, y)\mu_n(dy) \uparrow p(x)$ from the remark after Proposition 3.2 and VII, T62 and T63 in [5].

Remark. The preceding lemma and the proof of (2) imply that for any sequence $p_n(x) = \int g(x, y)\mu_n(dy) \uparrow p(x)$, (1) is also true, for all $x \in \Omega_p$. (1) and (2) can be interpreted as analogues of a classical theorem on the convergence of the energy of potentials.

Lemma 3.5. *Let $p(x) = \int g(x, y)\mu(dy)$ be a potential on Ω and assume that*

$$p_1(x) = 2\int p(y) g(x, y)\mu(dy) \not\equiv \infty$$

or equivalently that $p_2(x) = \int |\text{grad}\, p|^2(y) g(x, y)\, dy \not\equiv \infty$. Let A be the continuous homogeneous additive functional associated with p by Proposition 3.2. If we let $M_t = p(Y_t) + A_t$, then for all $x \in \Omega_{p_1} = \Omega_p \cap \Omega_{p_2}$, the processes

$$\left\{M_t^2 - \int_0^{t\wedge\zeta} |\text{grad}\, p|^2(Y_s)\, ds,\ 0 \leqq t \leqq \infty, P_x\right\}$$

are martingales.

Proof. By the remark following Proposition 3.2, A is well defined by Proposition 3.2. Moreover, $E_x M_\infty^2 = E_x A_\infty^2 = p_1(x) < \infty$ if $x \in \Omega_{p_1}$. The functional

$$\int_0^{t\wedge\zeta} |\mathrm{grad}\, p|^2(Y_s)\, ds$$

is also well defined, because $\mathrm{grad}\, p$ exists a.e. on Ω (w.r.t. Lebesgue measure); moreover,

$$E_x \int_0^{t\wedge\zeta} |\mathrm{grad}\, p|^2(Y_s)\, ds \leqq p_2(x) < \infty \quad \text{if } x \in \Omega_{p_2}.$$

We have $M_t^2 = p_1(Y_t) - p_2(Y_t) + 2A_t p(Y_t) + A_t^2$. Now, for $x \in \Omega_{p_1}$,

(1) $\{p_1(Y_t) + 2\int_0^t p(Y_s)\, dA_s, 0 \leqq t \leqq \infty, P_x\}$ is a martingale because $2\int_0^t p(Y_s)\, dA_s$ is a continuous homogeneous additive functional of Y and $2E_x \int_0^\infty p(Y_s)\, dA_s = E_x A_\infty^2 = p_1(x)$.

(2) $\{p_2(Y_t) + \int_0^{t\wedge\zeta} |\mathrm{grad}\, p|^2(Y_s)\, ds, 0 \leqq t \leqq \infty, P_x\}$ is a martingale.

(3) $\{A_t p(Y_t) + \frac{1}{2}A_t^2 - \int_0^t p(Y_s)\, dA_s, 0 \leqq t \leqq \infty, P_x\}$ is a martingale, because

$$\begin{aligned} E_x\left\{\tfrac{1}{2}A_\infty^2 - \int_0^\infty p(Y_s)\, dA_s \,|\, \mathfrak{F}_t\right\} &= \tfrac{1}{2}A_t^2 + A_t E_{Y_t} A_\infty + \tfrac{1}{2}E_{Y_t} A_\infty^2 - \int_0^t p(Y_s)\, dA_s - E_{Y_t}\int_0^\infty p(Y_s)\, dA_s \\ &= \tfrac{1}{2}A_t^2 + A_t p(Y_t) - \int_0^t p(Y_s)\, dA_s. \end{aligned}$$

The lemma follows from (1), (2), (3).

Theorem 3.6. *Let $p(x) = \int g(x, y)\mu(dy)$ be a potential on Ω, let*

$$S^2(\omega) = \int_0^\zeta |\mathrm{grad}\, p|^2(Y_s)\, ds$$

and let

$$E_x S^2 = \int |\mathrm{grad}\, p|^2(y) g(x, y)\, dy = p_2(x).$$

(a) *If $x \in \Omega_p$, then $S^2 < \infty$, P_x-a.e.*

(b) *If $x \in \Omega_p$, then $S^2(p, \pi_n) \to S^2$ in P_x-probability for every sequence $\{\pi_n\}$ such that $\|\pi_n\| \to 0$ and $t_{k_n}^{(n)} \to \infty$.*

(c) *If $x \in \Omega_p \cap \Omega_{p_2}$, then $S^2(p, \pi_n) \to S^2$ in $L_1(P_x)$-norm for every sequence $\{\pi_n\}$ such that $\|\pi_n\| \to 0$ and $t_{k_n}^{(n)} \to \infty$.*

(d) *If $x \in \Omega_p \cap \Omega_{p_2}$ and if $\int |\mathrm{grad}\, p|^4 g(x, y)\, dy < \infty$, then $S^2(p, \pi_n) \to S^2$, P_x-a.e. for every sequence $\{\pi_n\}$ such that $\sum \|\pi_n\| < \infty$, $t_{k_n}^{(n)} \to \infty$.*

Proof. Let A be the continuous homogeneous additive functional associated with p by Proposition 3.3. Let $M_t = p(Y_t) + A_t$. Then

$$\begin{aligned} S^2(p, \pi_n) &= \sum_{j=0}^{k_n-1} (M_{t_{j+1}^{(n)}} - M_{t_j^{(n)}})^2 + \sum_{j=0}^{k_n-1} (A_{t_{j+1}^{(n)}} - A_{t_j^{(n)}})^2 \\ &\quad -2\sum_{j=0}^{k_n-1} (M_{t_{j+1}^{(n)}} - M_{t_j^{(n)}})(A_{t_{j+1}^{(n)}} - A_{t_j^{(n)}}). \end{aligned}$$

If $x \in \Omega_p$, and $\|\pi_n\| \to 0$, then

$$\sum_{j=0}^{k_n-1} (A_{t_{j+1}^{(n)}} - A_{t_j^{(n)}})^2 \to 0 \quad P_x\text{-a.e.}$$

and

$$\sum_{j=0}^{k_n-1} (M_{t_{j+1}^{(n)}} - M_{t_j^{(n)}})(A - A_{t_{j+1}^{(n)}}) \to 0 \quad P_x\text{-a.e.}$$

This follows from the fact that the first sum is majorized by $A_\infty \cdot \sup_j (A_{t_{j+1}^{(n)}} - A_{t_j^{(n)}})$ and the absolute value of the second one is majorized by $A_\infty \cdot \sup_j |M_{t_{j+1}^{(n)}} - M_{t_j^{(n)}}|$ and P_x-a.e., both $A_.(\omega)$ and $M_.(\omega)$ are continuous and hence uniformly continuous on $[0, \infty]$.

If $x \in \Omega_p \cap \Omega_{p_2}$ and $\|\pi_n\| \to 0$, then also

$$E_x\left\{\sum_{j=0}^{k_n-1} (A_{t_{j+1}^{(n)}} - A_{t_j^{(n)}})^2\right\} \to 0$$

and

$$E_x\left|\sum_{j=0}^{k_n-1} (M_{t_{j+1}^{(n)}} - M_{t_j^{(n)}})(A_{t_{j+1}^{(n)}} - A_{t_j^{(n)}})\right| \to 0.$$

This follows from the fact that the integrands converge to 0 P_x-a.e. and are majorized by A_∞^2 and $A_\infty(A_\infty + \sup_t p(Y_t))$ respectively; from p. 142 in [5] we have $E_x\{\sup_t p(Y_t)\}^2 \leqq 4E_x A_\infty^2 = 4p_1(x)$.

We therefore have to discuss

$$S^2(M, \pi_n) = \sum_{j=0}^{k_n-1} (M_{t_{j+1}^{(n)}} - M_{t_j^{(n)}})^2.$$

Proof of (a) and (b). Let $M_t^{(n)} = M_{t\wedge\tau_n}$ for $0 \leqq t \leqq \infty$ where $\tau_n = T_{\{p>n\}}$. Then by Proposition 3.3, $\{M_t^{(n)}, 0 \leqq t \leqq \infty, P_x\}$ is a martingale for $x \in \Omega_p$. We shall prove that $S^2(M^{(n)}, \pi_k) \to \int_0^{\tau_n\wedge\zeta} |\text{grad } p|^2(Y_s)\, ds$ in P_x-probability. This is trivial if $x \in \{p > n\}$; so we may assume that $x \in \{p \leqq n\}$. In view of Lemma 2.1 it is sufficient to prove that

$$\left\{M_t^{(n)2} - \int_0^{t\wedge\tau_n\wedge\zeta} |\text{grad } p|^2(Y_s)\, ds,\ 0 \leqq t \leqq \infty, P_x\right\}$$

is a martingale. This is seen as follows: Let $p_n = p \wedge n$, and let $A^{(n)}$ be the continuous homogeneous additive functional associated with p_n, and let $N_t^{(n)} = p_n(Y_t) + A_t^{(n)}$. Then by Lemma 3.5,

$$\left\{N_t^{(n)2} - \int_0^{t\wedge\zeta} |\text{grad } p_n|^2(Y_s)\, ds,\ 0 \leqq t \leqq \infty, P_x\right\}$$

is a martingale, hence

$$\left\{N_{t\wedge\tau_n}^{(n)2} - \int_0^{t\wedge\tau_n\wedge\zeta} |\text{grad } p_n|^2(Y_s)\, ds,\ 0 \leqq t \leqq \infty, P_x\right\}$$

is a martingale. But $N^{(n)}_{t\wedge\tau_n}=M^{(n)}_t$, P_x-a.e., because P_x-a.e., $p_n(Y_{t\wedge\tau_n})=N^{(n)}_{t\wedge\tau_n}-A^{(n)}_{t\wedge\tau_n}$, $p_n(Y_{t\wedge\tau_n})=p(Y_{t\wedge\tau_n})=M^{(n)}_t-A_{t\wedge\tau_n}$ and the decomposition of $p_n(Y_{t\wedge\tau_n})$ is unique. Also

$$\int_0^{t\wedge\tau_n\wedge\zeta}|\text{grad}\, p_n|^2(Y_s)\,ds=\int_0^{t\wedge\tau_n\wedge\zeta}|\text{grad}\, p|^2(Y_s)\,ds.$$

Hence $\{M^{(n)2}_t-\int_0^{t\wedge\tau_n\wedge\zeta}|\text{grad}\, p|^2(Y_s)\,ds,\ 0\leqq t\leqq\infty,\ P_x\}$ is a martingale, and

(X) $$S^2(M^{(n)},\pi_k)\to\int_0^{\tau_n\wedge\zeta}|\text{grad}\, p|^2(Y_s)\,ds$$

in P_x-probability as $k\to\infty$, if $x\in\Omega_p$.

For the remainder of the proof we use an argument of the proof of Theorem 6.2 in [6]. If $x\in\Omega_p$, then

(XX) $$P_x\{|S^2(M,\pi_k)-S^2(M^{(n)},\pi_k)|>\varepsilon\}\leqq P_x\{\tau_n<\infty\}\to 0$$

uniformly in k. Because of

$$\begin{aligned}P_x\{|S^2(M,\pi_{k_2})-S^2(M,\pi_{k_1})|>\varepsilon\}\leqq\ & P_x\{|S^2(M,\pi_{k_2})-S^2(M^{(n)},\pi_{k_2})|>\varepsilon/3\}\\ &+P_x\{|S^2(M^{(n)},\pi_{k_2})-S^2(M^{(n)},\pi_{k_1})|>\varepsilon/3\}\\ &+P_x\{|S^2(M^{(n)},\pi_{k_1})-S^2(M,\pi_{k_1})|>\varepsilon/3\}\end{aligned}$$

we conclude that there is a real-valued random variable ξ such that $S^2(M,\pi_k)\to\xi$ in P_x-probability as $k\to\infty$. From (X) and (XX) we obtain

$$P_x\left\{\left|\xi-\int_0^{\tau_n\wedge\zeta}|\text{grad}\, p|^2(Y_s)\,ds\right|>\varepsilon\right\}\leqq P_x\{\tau_n<\infty\}\to 0$$

and therefore $\xi=\int_0^\zeta|\text{grad}\, p|^2(Y_s)\,ds=S^2$. This proves (a) and (b).

Proof of (c) and (d). If $\Omega_{p_1}=\Omega_p\cap\Omega_{p_2}\neq\varnothing$, then μ does not charge any polar set and $\{M_t,\ 0\leqq t\leqq\infty,\ P_x\}$ is a martingale for $x\in\Omega_p$. Now (c) follows from Lemmas 2.1 and 3.5, (d) from Lemmas 2.3 and 3.5.

REMARK. The random variable S^2 is in general not a constant on Y-paths converging to a fixed boundary point. For example let $\Omega=R_n(n>2)$ and let p be the potential of a measure living on a polar set $\subseteq R_n$. Here $\partial\Omega=\{\infty\}$, $E_xS^2=p_2(x)\equiv\infty$, whereas, for $x\in\Omega_p$, $S^2<\infty$ P_x-a.e.

4. **Quadratic variation of harmonic functions.** We start by introducing some notations. Let $\Omega_n\uparrow\Omega$, $\overline{\Omega}_n$ compact $\subseteq\Omega$; let ζ_n be the life-time of Y in Ω_n.

If h is harmonic on Ω and $r\geqq 1$, let $\|h\|_r^r(x)=\sup_{\Omega_n\ni x}E_x|h(Y_{\zeta_n})|^r$. The following facts are well known:

(1) The value of $\|h\|_r(x)$ is independent of the particular sequence $\{\Omega_n\}$; moreover $\|h_r\|(x)<\infty$ for all x or $\|h\|_r(x)\equiv\infty$.

(2) $\|h\|_1<\infty$ iff h is the difference of two positive harmonic functions. In this case h has a fine boundary function h^*.

(3) If $r>1$, then $\|h\|_r<\infty$ iff h is the Dirichlet solution corresponding to an $\mathscr{L}_r$-boundary function h^* ($\mathscr{L}_r$ w.r.t. harmonic measure ω on $\partial\Omega$). In this case:

$$\|h\|_r^r(x) = E_x|h^*(Y_\zeta)|^r = \int_{\partial\Omega} |h^*(y)|^r\omega(x, dy).$$

REMARK. If h is harmonic on Ω, then $\|h\|_2<\infty$ iff $\int_\Omega |\text{grad } h|^2(y)g(x,y)\,dy<\infty$ for some $x\in\Omega$ (and hence for all $x\in\Omega$). This follows from

$$\|h\|_2^2(x)-h^2(x) = \int_\Omega |\text{grad } h|^2(y)g(x,y)\,dy.$$

The preceding equation follows from the Riesz decomposition of the restriction of the subharmonic function h^2 to Ω_n, where $\Omega_n \uparrow \Omega$, $\bar{\Omega}_n$ compact $\subseteq\Omega$.

THEOREM 4.1. *Let h be harmonic on Ω and let $S^2=\int_0^\zeta |\text{grad } h|^2(Y_s)\,ds$.*

(a) *If h is the difference of two positive harmonic functions or equivalently if $\|h\|_1<\infty$, then for all $x\in\Omega$, $S^2<\infty$ P_x-a.e. and $S^2(h,\pi_n)\to S^2$ in P_x-probability for $\|\pi_n\|\to 0$ and $t_{k_n}^{(n)}\to\infty$.*

(b) *For $r>1$, there are positive finite numbers α_r and β_r, independent of Ω, h and x, such that*

$$\alpha_r E_x S^r \leqq \|h-h(x)\|_r^r(x) \leqq \beta_r E_x S^r.$$

If $\|h\|_r<\infty$ or equivalently $E_. S^r<\infty$ then for all $x\in\Omega$, $S(h,\pi_n)\to S$ in $L_r(P_x)$-norm for $\|\pi_n\|\to 0$ and $t_{k_n}^{(n)}\to\infty$.

(c) *If $E_xS^2=\int |\text{grad } h|^2(y)g(x,y)\,dy<\infty$ (or equivalently if $\|h\|_2<\infty$) and if $\int |\text{grad } h|^4(y)g(x,y)\,dy<\infty$, then for all $x\in\Omega$, $S^2(h,\pi_n)\to S^2$, P_x-a.e. for $\sum \|\pi_n\|<\infty$ and $t_{k_n}^{(n)}\to\infty$.*

REMARK. The integrals in (c) converge for all $x\in\Omega$ or for none.

Proof. (a) Let $\tau_n=T_{\{|h|>n\}}$. Because P_x-a.e., $h(Y_t)$ is finite and continuous on $[0,\infty]$, $P_x\{\tau_n<\infty\}\to 0$ for all $x\in\Omega$. Let $M_t^{(n)}=h(Y_{t\wedge\tau_n})$. We show first that, for all $x\in\Omega$, $S^2(M^{(n)},\pi_k)\to\int_0^{\zeta\wedge\tau_n} |\text{grad } h|^2(Y_s)\,ds$ in P_x-probability. This is trivial if $|h(x)|\geqq n$. We may assume therefore that $|h(x)|<n$. It is clear that

$$\{M_t^{(n)}, 0 \leqq t \leqq \infty, P_x\}$$

is a continuous bounded martingale. In view of Lemma 2.1 it is sufficient to show that for all $x\in\Omega$,

$$\text{(X)} \qquad \left\{M_t^{(n)2}-\int_0^{t\wedge\tau_n\wedge\zeta} |\text{grad } h|^2(Y_s)\,ds,\ 0 \leqq t \leqq \infty,\ P_x\right\}$$

is a martingale. Now if Ω_n is the component of the open set $\{|h|<n\}$ which contains x and if g_n is its Green function, then, for $y\in\Omega_n$,

$$E_y\int_0^{\tau_n\wedge\zeta} |\text{grad } h|^2(Y_s)\,ds = \int_{\Omega_n} |\text{grad } h|^2(z)g_n(y,z)\,dz$$
$$= H_{h^2}^{\Omega_n}(y)-h^2(y) < \infty,$$

where $H_{h^2}^{\Omega_n}$ is the harmonic function on Ω_n which has as boundary function the restriction of h^2. This implies that (X) is a martingale. To finish the proof of (a) we observe that for $x \in \Omega$,

$$P_x\{|S^2(M^{(n)}, \pi_k) - S^2(M, \pi_k)| > \varepsilon\} \leqq P_x\{\tau_n < \infty\} \to 0,$$

uniformly in k, and repeat the last argument in the proof of (a) and (b) of Theorem 3.6.

(b) If $\|h\|_r < \infty$, then $\{h(Y_t), 0 \leqq t \leqq \infty\}$ is a continuous martingale and

$$E_x|h(Y_\infty)|^r = E_x|h^*(Y_\zeta)|^r < \infty.$$

The L_r-convergence of $S^2(h, \pi_n)$ and the inequality for the moments follow from Theorem 6.2 in [6].

If $\|h\|_r = \infty$, the left inequality for the moments is trivial; but it is also not difficult to see that $\|h\|_r = \infty$ implies $E_x S^r = \infty$. Let $\Omega_n \uparrow \Omega$, $\bar{\Omega}_n$ compact $\subseteq \Omega$, $x \in \Omega_n$ and let ζ_n be the life-time of Y in Ω_n. Then h is bounded on Ω_n and

$$E_x|h(Y_{\zeta_n})|^r \leqq \beta_r E_x\left(\int_0^{\zeta_n} |\text{grad } h|^2(Y_s)\, ds\right)^{r/2}.$$

Letting $n \to \infty$, we get $\infty = E_x S^r$.

(c) Here the processes $\{h(Y_t), 0 \leqq t \leqq \infty, P_x\}$ and

$$\left\{h^2(Y_t) - \int_0^{t \wedge \zeta} |\text{grad } h|^2(Y_s)\, ds,\ 0 \leqq t \leqq \infty, P_x\right\}$$

are martingales for $x \in \Omega$, and the assertion follows from Lemma 2.3.

REMARK. If we let $r = 2n$ in (b) of the preceding theorem then we get estimates for harmonic functions which can be formulated in strictly nonprobabilistic language. Recall that the constants α_{2n} and β_{2n} are independent of Ω.

5. **Generalization of a formula by K. Ito.** We are now able to give a generalization of a classical formula by K. Ito. According to Ito we have for a function $u \in C^2(R_n)$, for all $x \in R_n$, P_x-a.e.

$$u(Y_t) - u(Y_0) = P_x\text{-}\int_0^t \text{grad } u(Y_s)\cdot dY_s + \frac{1}{2}\int_0^t (\Delta u)(Y_s)\, ds.$$

Here Y is Brownian motion on R_n. In the following we denote by Y again Brownian motion on a Green domain $\Omega \subseteq R_n$, stopped upon reaching $\partial\Omega$.

THEOREM 5.1. (a) *If p is a potential on Ω and A the continuous homogeneous additive functional corresponding to p by Proposition* 3.3 *then, for $x \in \Omega_p$, P_x-a.e.,*

$$\text{(x)} \qquad p(Y_t) - p(Y_0) = P_x\text{-}\int_0^{t \wedge \zeta} \text{grad } p(Y_s)\, dY_s - A_t$$

(b) *If h is a harmonic function on Ω such that $\|h\|_1 < \infty$, then for $x \in \Omega$, P_x-a.e.,*

$$h(Y_t) - h(Y_0) = P_x\text{-}\int_0^{t \wedge \zeta} \text{grad } h(Y_s)\cdot dY_s.$$

An immediate consequence of this theorem is the following

COROLLARY 5.2. *Let u be the difference of two positive superharmonic functions on* Ω. *If* $|u(x)|<\infty$, *then* P_x*-a.e.*

$$u(Y_t)-u(Y_0) = P_x-\int_0^{t\wedge\zeta} \operatorname{grad} u(Y_s)\cdot dY_s - C_t$$

where C is the difference of two (*nonnegative*) *continuous homogeneous additive functionals of Y* (*with state space* $\{x\in\Omega\cup\partial\Omega;\ |u(x)|<\infty\}$).

Proof of Theorem 5.1(a).

(1) We assume first that $p\in C^2(\Omega)$. Let $\Omega_k\uparrow\Omega$, Ω_k compact $\subseteq\Omega$, $x\in\Omega_k$. Denote by ζ_k the life-time of Y in Ω_k. It is easy to see that the classical Ito formula

$$\text{(o)}\qquad p(Y_{t\wedge\zeta_k})-p(x) = P_x-\int_0^{t\wedge\zeta_k} \operatorname{grad} p(Y_s)\cdot dY_s+\frac{1}{2}\int_0^{t\wedge\zeta_k} \Delta p(Y_s)\,ds \quad P_x\text{-a.e.}$$

is valid. But by Theorem 3.6(a), $\int_0^\zeta |\operatorname{grad} p|^2(Y_s)\,ds<\infty$ P_x-a.e. Hence we obtain (x) with $A_t=-\frac{1}{2}\int_0^{t\wedge\zeta}\Delta p(Y_s)\,ds$ by taking in (o) the limits in P_x-probability as $k\to\infty$.

(2) Assume now that p is a bounded potential on Ω. We shall see first that for any nondecreasing sequence of potentials $p_n\uparrow p$, we have $\operatorname{grad} p_{n_k}\to\operatorname{grad} p$ a.e. (w.r.t. Lebesgue measure) for a subsequence p_{n_k}: Let B be any closed ball in Ω, and let p_n', p' be the potentials obtained from p_n, p by a "sweeping out" process with respect to B. The potentials p_n', p' are bounded, and the associated measures (living on B) have finite total mass. Therefore p_n', p' have finite energy; and since $p_n'\uparrow p'$, we get $\int_\Omega |\operatorname{grad}(p_n'-p')|^2(x)\,dx\to 0$. But on B, $p_n'=p_n$ and $p'=p$, and we get $\operatorname{grad} p_{n_k}\to\operatorname{grad} p$ a.e. in the interior of B, for some sequence n_k. The rest follows from a diagonal argument. Now let p_n be a nondecreasing sequence of potentials such that $p_n\in C^2(\Omega)$, $p_n\uparrow p$. We may assume that $\operatorname{grad} p_n\to\operatorname{grad} p$ a.e. on Ω. Now firstly, (x) is valid for p_n. Secondly, $A_t^{(n)}=-\frac{1}{2}\int_0^{t\wedge\zeta}\Delta p_n(Y_s)\,ds\to A_t$ in $L_2(P_x)$-norm because, by VII, T36 in [5], $E_x\{A_\infty^{(n)}-A_\infty\}^2\to 0$ and by Proposition (3.3) $A_t^{(n)}=E_x\{A_\infty^{(n)}|\mathfrak{F}_t\}-p_n(Y_t)$ and $A_t=E_x\{A_\infty|\mathfrak{F}_t\}-p(Y_t)$, P_x-a.e. We conclude therefore that $P_x-\int_0^{t\wedge\zeta}\operatorname{grad} p_n(Y_s)\cdot dY_s$ converges in $L_2(P_x)$-norm. If we denote by q the transition function of Y on Ω and let $g_t(x,y)=\int_0^t q_s(x,y)\,ds$, this implies

$$\begin{aligned}\sup_k \int_\Omega |\operatorname{grad}(p_{n+k}-p_n)|^2(y)g_t(x,y)\,dy &\\ = \sup_k E_x\left\{\int_0^{t\wedge\zeta}\operatorname{grad} p_{n+k}(Y_s)\cdot dY_s-\int_0^{t\wedge\zeta}\operatorname{grad} p_n(Y_s)\cdot dY_s\right\}^2 &\\ \to 0.&\end{aligned}$$

But since $\operatorname{grad} p_n\to\operatorname{grad} p$ a.e. on Ω, we obtain

$$\int_\Omega |\operatorname{grad}(p-p_n)|^2(y)g_t(x,y)\,dy\to 0.$$

The left side equals

$$E_x\left\{\int_0^{t\wedge\zeta} \operatorname{grad} p(Y_s)\cdot dY_s - \int_0^{t\wedge\zeta} \operatorname{grad} p_n(Y_s)\cdot dY_s\right\}^2.$$

We have therefore $\int_0^{t\wedge\zeta} \operatorname{grad} p_n(Y_s)\cdot dY_s \to \int_0^{t\wedge\zeta} \operatorname{grad} p(Y_s)\cdot dY_s$ in $L_2(P_x)$-norm. We conclude that (x) is valid for p.

(3) If finally p is an arbitrary potential on Ω, let $p_n = p \wedge n$. Then, for all $x \in \Omega$,

$$(+) \qquad p_n(Y_t) - p_n(x) = \int_0^{t\wedge\zeta} \operatorname{grad} p_n(Y_s)\cdot dY_s - A_t^{(n)} \quad P_x\text{-a.e.},$$

where $A^{(n)}$ is the continuous homogeneous additive functional corresponding to p_n by Proposition 3.2. Now for $x \in \Omega_p$, $\int_0^\zeta |\operatorname{grad} p|^2(Y_s)\, ds < \infty$ P_x-a.e., and we conclude from $|\operatorname{grad} p_n|^2 \uparrow |\operatorname{grad} p|^2$ a.e. (w.r.t. Lebesgue measure) that

$$\int_0^{t\wedge\zeta} \operatorname{grad} p_n(Y_s)\cdot dY_s \to \int_0^{t\wedge\zeta} \operatorname{grad} p(Y_s)\cdot dY_s$$

in P_x-probability. Moreover if the measure corresponding to p does not charge any polar set and if A is the continuous homogeneous additive functional corresponding to p by Proposition 3.2, then $A_t^{(n)} \to A_t$ in P_x-probability for $x \in \Omega_p$. On the other hand, if the measure corresponding to p lives on a polar set, then $A_t^{(n)} \to 0$ in P_x-probability for $x \in \Omega_p$. In either case we obtain (x) by taking in (+) the limit in P_x-probability as $n \to \infty$; for general p, we obtain (x) by using the decomposition in the proof of Proposition 3.3.

Proof of Theorem 5.1(b). The proof follows by the same argument as in the first part of the preceding proof.

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