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COMMUTATORS MODULO THE CENTER IN A PROPERLY INFINITE VON NEUMANN ALGEBRA(1)

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1. **Introduction.** An element C in a von Neumann algebra $\mathscr{A}$ is said to be a commutator in $\mathscr{A}$ if there are elements A and B in $\mathscr{A}$ such that $C=AB-BA$. For finite homogeneous discrete algebras and for properly infinite factor algebras the set of commutators has been completely described [1]–[5], [10]. In each of these special cases any element C is a commutator modulo a central element depending on C. In this paper we show that given any element C in a properly infinite von Neumann algebra $\mathscr{A}$ there is an element C_0 in the center of $\mathscr{A}$ depending on C such that $C-C_0$ is a commutator in $\mathscr{A}$. The element C_0 is an arbitrary element in the intersection $\mathscr{K}_C$ of the center with the uniform closure of the convex hull of $\{U^*CU \mid U \text{ unitary in } \mathscr{A}\}$ [6, III, §5]. We then present a few facts about those elements C such that $0 \in \mathscr{K}_C$ or what is the same as far as determining commutators is concerned about those elements C such that $0 \in \mathscr{K}_{S^{-1}CS}$ for some invertible S in $\mathscr{A}$.

2. **Commutators.** Let $\mathscr{A}$ be a C^*-algebra with identity and let I be a closed two-sided ideal in $\mathscr{A}$. The image of the element $A \in \mathscr{A}$ in the factor algebra $\mathscr{A}(I) = \mathscr{A}/I$ under the canonical homomorphism of $\mathscr{A}$ onto $\mathscr{A}/I$ will be denoted by $A(I)$. If ζ is a maximal ideal of the center of $\mathscr{A}$, the smallest closed two-sided ideal in $\mathscr{A}$ containing ζ is denoted by $[\zeta]$. For simplicity we write $A([\zeta])$ as $A(\zeta)$. The set of maximal (respectively, primitive) ideals of $\mathscr{A}$ with the hull-kernel topology is called the strong structure space (respectively, structure space) of $\mathscr{A}$. If $\mathscr{A}$ is a von Neumann algebra, then the strong structure space $M(\mathscr{A})$ of $\mathscr{A}$ is homeomorphic with the spectrum of the center $\mathscr{Z}$ of $\mathscr{A}$ under the map $M \to M \cap \mathscr{Z}$ [13]. This means $M(\mathscr{A})$ is extremely disconnected.

PROPOSITION 1. *Let $\mathscr{A}$ be a properly infinite von Neumann algebra and let A be a fixed element of $\mathscr{A}$. The function $M \to \|A(M)\|$ of the strong structure space $M(\mathscr{A})$ of $\mathscr{A}$ into the real numbers is continuous.*

Proof. For every $\alpha \geqq 0$ we know that the set $X=\{M \in M(\mathscr{A}) \mid \|A(M)\| \leqq \alpha\}$ is closed. If $I=\bigcap X$, then $\|A(I)\| \leqq \alpha$ [8, Lemma 1.9] and so $\|A(M)\| \leqq \alpha$ for every $M \in M(\mathscr{A})$ containing I. Thus $X=\{M \in M(\mathscr{A}) \mid I \subset M\}$.

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Conversely, let $\alpha>0$; we show that

$$Y = \{M \in M(\mathscr{A}) \mid \|A(M)\| \geqq \alpha\}$$

is closed in $M(\mathscr{A})$. Let J be the strong radical of $\mathscr{A}$ and let $\mathscr{P}$ be the structure space of $\mathscr{A}(J)$. The set

$$Y' = \{K \in \mathscr{P} \mid \|A(J)(K)\| \geqq \alpha\}$$

is compact (but not necessarily closed) in $\mathscr{P}$ [16, 4.9.18]. If $\mathscr{P}'$ is the structure space of $\mathscr{A}$, then $M \to M(J)$ defines a homeomorphism of $\{M \in \mathscr{P}' \mid M \supset J\} = h(J)$ onto $\mathscr{P}$ [16, 2.6.6]. But if $M \in \mathscr{P}'$, then the intersection of M with the center of $\mathscr{A}$ is a maximal ideal. So $M \in h(J)$ implies M is of the form $J+[\zeta]$ for some maximal ideal ζ of the center. It is then clear that $h(J)$ is the set of maximal ideals of $\mathscr{A}$ [10, Proposition 2.3]. Furthermore, the topology of $h(J)$ and $M(\mathscr{A})$ coincide. This proves that Y is compact in $M(\mathscr{A})$ since it is the inverse image of Y' under the homeomorphism $M \to M(J)$ of $M(\mathscr{A})$ onto $\mathscr{P}$. Because $M(\mathscr{A})$ is homeomorphic to the spectrum of the center which is Hausdorff, every compact set of $M(\mathscr{A})$ is closed. Thus Y is a closed subset of $M(\mathscr{A})$. Q.E.D.

REMARK. If $\mathscr{A}$ is not properly infinite, Proposition 1 is certainly not true.

Let H be a Hilbert space and let A be a bounded linear operator on H. Let F be a projection on H. Define the numerical gauge $\eta_A(F)$ to be

$$\eta_A(F) = \text{lub}\,\{\|Ax-(Ax, x)x\| \mid x \text{ is a unit vector in } F(H)\}.$$

Let $\mathscr{W}_A(F)$ be the closure of the convex set

$$\{(Ax, x) \mid x \text{ a unit vector in } F(H)\}.$$

For every $\alpha \in \mathscr{W}_A(F)$ we have that

$$\|(A-\alpha)F\| \leqq 65\eta_A(F). \tag{1}$$

This can be obtained by a simple reworking of Lemma 2.3 [2].

Let $\mathscr{A}$ be a properly infinite von Neumann algebra with no σ-finite type III direct summands; then for each projection F in $\mathscr{A}$ and each element A in $\mathscr{A}$ define $\nu_A(F)$ to be

$$\nu_A(F) = \text{lub}\,\{\|AE-EAE\| E \in (J),\ E \leqq F\} \tag{2}$$

where (J) is the set of projections in the strong radical J of $\mathscr{A}$. For every irreducible representation ϕ of $\mathscr{A}$ on a Hilbert space such that $\phi(J)\neq(0)$ we have that

$$\eta_{\phi(A)}(\phi(F)) \leqq \nu_A(F) \tag{3}$$

[10, Proposition 3.1]. Define $\nu(A)$ to be

$$\nu(A) = \text{glb}\,\{\nu_A(F) \mid 1-F \in (J)\}.$$

Let $\mathscr{A}$ now be the product of σ-finite type III algebras; let

$$\nu(A) = \text{lub}\,\{\|AE-EAE\| \mid E \text{ a projection in } \mathscr{A}\}$$

for each $A \in \mathscr{A}$. If A is in the complement in $\mathscr{A}$ of the set of all elements of $\mathscr{A}$ equal to scalar (zero included) multiples of the identity modulo some maximal ideal of $\mathscr{A}$, then there is a $\nu > 0$ such that $\nu(AP) \geqq \nu$ for every nonzero central projection P since there is a projection E in $\mathscr{A}$ with $E \sim 1-E \sim 1$ such that $EA^*(1-E)AE \geqq \alpha E$ for some scalar $\alpha > 0$ [10, Theorem 3.7]. Also it is easy to see from Proposition 3.1 [10] that

$$\eta_{\Phi(A)}(1) \leqq \nu(A) \tag{4}$$

for every irreducible representation of $\mathscr{A}$.

LEMMA 2. *Let $\mathscr{A}$ be a properly infinite von Neumann algebra such that*
(i) *either $\mathscr{A}$ has no σ-finite type* III *direct summands or*
(ii) *$\mathscr{A}$ is a product of σ-finite type* III *algebras.*
Let $\{P_n\}$ be a set of mutually orthogonal central projections of $\mathscr{A}$ of sum P. Then $\nu(AP) = \text{lub}_n\, \nu(AP_n)$ for every $A \in \mathscr{A}$.

Proof. Suppose $\mathscr{A}$ satisfies condition (i). Let (J) be the set of projections in the strong radical of $\mathscr{A}$. Then given $\varepsilon > 0$ there is a projection F with $1-F \in (J)$ such that

$$\nu(AP)+\varepsilon \geqq \text{lub}\,\{\|(1-E)APE\| \mid E \leqq F,\ E \in (J)\}.$$

But

$$\|(1-E)AEP_n\| \leqq \|(1-E)APE\|$$

for every P_n. So $\nu(AP_n) \leqq \nu(AP)+\varepsilon$ for every n. Since $\varepsilon > 0$ is arbitrary, we have that lub $\nu(AP_n) \leqq \nu(AP)$. Conversely, given that $\varepsilon > 0$ there is for each P_n a projection F_n with $1-F_n \in (J)$ such that

$$\text{lub}\,\{\|(1-E)AP_nE\| \mid E \leqq F_n,\ E \in (J)\} \leqq \nu(AP_n)+\varepsilon.$$

Setting $F = \sum F_nP_n$, we have that $P-F \in (J)$ [10, Corollary, Proposition 2.2] and that

$$\|(1-E)APE\| = \text{lub}_n\, \|(1-E)AP_nE\| \leqq \text{lub}\, \nu(AP_n)+\varepsilon$$

for every E in (J) with $E \leqq F$. Thus $\nu(AP) \leqq \text{lub}\, \nu(AP_n)+\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have that $\nu(AP) \leqq \text{lub}\, \nu(AP_n)$. This completes the proof for case (i).

The proof for case (ii) is similar.

LEMMA 3. *Let $\mathscr{A}$ be a properly infinite von Neumann algebra with no σ-finite type* III *direct summands. Let G be a projection in $\mathscr{A}$ such that the orthogonal complement $1-G$ of G is in the set of projections (J) of the strong radical of $\mathscr{A}$. Let (J') be the strong radical of the reduced algebra $G\mathscr{A}G$. For any element A in $\mathscr{A}$ we have that*

$$\nu'(GAG) = \text{glb}\,\{\text{lub}\,\{\|(1-E)GAGE\| \mid E \leqq F,\ E \in (J')\} \mid G-F \in (J')\}$$

is equal to $\nu(A)$.

Proof. First we show that $(J')=\{E\in(J)\mid E\leqq G\}$. Suppose $E\notin(J')$ for some $E\in(J)$ majorized by G. There is a nonzero projection P in the center of $G\mathscr{A}G$ such that $PE\sim P$ [10, §2]. But $P=QG$ for some nonzero Q in the center of $\mathscr{A}$ [6, I, §2, Corollary, Proposition 2]. Then $QE\sim QG\sim Q$ since $G\sim 1$ [cf. 10, §2]. So $E\notin(J)$. This is a contradiction. Hence $\{E\in(J)\mid E\leqq G\}\subset(J')$. Conversely, suppose $E\in(J')$. If Q is a central projection in $\mathscr{A}$ with $EQ\sim Q$, then $EQ\sim Q\sim QG$. This implies $Q=0$. So $(J')\subset\{E\in(J)\mid E\leqq G\}$.

Now let $\varepsilon>0$ be given. There is by relation (2) a projection F in $\mathscr{A}$ with $1-F\in(J)$ such that $\nu_A(F)\leqq\nu(A)+\varepsilon$. Let G' be the least upper bound of $1-G$ and $1-F$. Then $G'\in(J)$ and $1-G'\leqq F$ and $1-G'\leqq G$. We see that

$$G-(1-G') = G'-(1-G)\in(J')$$

by the first paragraph. So

$$\begin{aligned}\nu'(GAG) &\leqq \operatorname{lub}\{\|(1-E)GAGE\| \mid E\in(J'),\ E\leqq 1-G'\}\\ &= \operatorname{lub}\{\|G(1-E)AE\| \mid E\in(J),\ E\leqq 1-G'\}\\ &\leqq \operatorname{lub}\{\|(1-E)AE\| \mid E\in(J),\ E\leqq F\}\leqq \nu(A)+\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we have that $\nu'(GAG)\leqq\nu(A)$.

Conversely, let $\varepsilon>0$ be given; there is a projection $F\in G\mathscr{A}G$ with $G-F\in(J')$ and

$$\operatorname{lub}\{\|(G-E)AE\| \mid E\leqq F,\ E\in(J')\}\leqq\nu'(GAG)+\varepsilon.$$

The domain support G' of $(1-G)A$ is a projection in (J) since $G'\prec 1-G$ [6, III, §1, Proposition 2]; similarly, the domain support G'' of $G'F$ is a projection in (J) majorized by F. So $1-(F-G'')=(1-G)+(G-F)+G''$ is a projection in (J). Then

$$\nu(A)\leqq\operatorname{lub}\{\|(1-E)AE\| \mid E\leqq F-G'',\ E\in(J)\}.$$

But

$$GAGE = AE-(1-G)AE = AE-(1-G)AG'FE = AE-(1-G)AG'FG''E = AE.$$

So we see that

$$\begin{aligned}\nu(A) &\leqq \operatorname{lub}\{\|(1-E)GAGE\| \mid E\leqq F-G'',\ E\in(J')\}\\ &\leqq \operatorname{lub}\{\|(1-E)GAGE\| \mid E\leqq F,\ E\in(J')\}\leqq\nu'(GAG)+\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we have that $\nu(A)\leqq\nu'(GAG)$. Q.E.D.

THEOREM 4. *Let C be an element in a properly infinite von Neumann algebra $\mathscr{A}$ and let C_0 be an element in the intersection $\mathscr{K}_C$ of the center of $\mathscr{A}$ with the uniform closure of the convex hull of $\{U^*CU\mid U$ unitary in $\mathscr{A}\}$. Then $C-C_0$ is a commutator in $\mathscr{A}$.*

Proof. We first make a preliminary reduction. There is a maximal central projection P in $\mathscr{A}$ such that $(C-C_0)P$ is in the strong radical J of $\mathscr{A}$ [10, Corollary, Proposition 2.2]. However, the structure of the strong radical of $\mathscr{A}$ allows us to conclude that $(C-C_0)P$ is in the strong radical of $\mathscr{A}P$ [10, §2]. Since $(C-C_0)P$ is a commutator in $\mathscr{A}P$, it is necessary to prove that $(C-C_0)(1-P)$ is a commutator in $\mathscr{A}(1-P)$. It is easy to see that the uniform closure of the convex hull of $\{U^*(C-C_0)U \mid U \text{ unitary in } \mathscr{A}(1-P)\}$ contains 0. So without loss of generality we may assume that C is an element in $\mathscr{A}$ such that $0 \in \mathscr{K}_C$ and such that $CP \in J$ for some central projection P in $\mathscr{A}$ implies that $P=0$.

Now there is a central projection Q in $\mathscr{A}$ such that

(i) $\mathscr{A}Q$ has no σ-finite type III direct summands and

(ii) $\mathscr{A}(1-Q)$ is the product of σ-finite type III algebras.

It is clearly necessary to prove only that CQ and $C(1-Q)$ are commutators in $\mathscr{A}Q$ and $\mathscr{A}(1-Q)$ respectively. Here $0 \in \mathscr{K}_{CQ}$ and $0 \in \mathscr{K}_{C(1-Q)}$ is also true when these sets are formed relative to $\mathscr{A}Q$ and $\mathscr{A}(1-Q)$ respectively. In the ensuing paragraphs we shall assume that either $\mathscr{A}$ satisfies condition (i) or condition (ii).

Let $\|C\|=\alpha$. We construct by induction a sequence $\{P_n\}$ of mutually orthogonal central projections of sum 1 such that for each nonzero central projection P majorized by P_n the number $\nu(CP)$ lies in the real interval $[2^{-n}\alpha, 2^{-n+1}\alpha]$. The induction hypothesis may be stated as follows: let $P_0=0$; then P_n is the largest central projection majorized by $1-\sum\{P_k \mid 0 \leqq k \leqq n-1\}$ such that $\nu(CP) \in I_n$ for every nonzero central projection P majorized by P_n. Suppose we have constructed $P_1, P_2, \ldots, P_n$. We find P_{n+1}. We may assume $R=1-\sum\{P_k \mid 0 \leqq k \leqq n\}$ is nonzero. There is a maximal ideal M in the strong structure space $M(\mathscr{A})$ of $\mathscr{A}$ such that $CR(M)$ is not in the center of $\mathscr{A}(M)$. Indeed, suppose $CR(M)$ is in the center of $\mathscr{A}(M)$ for every $M \in M(\mathscr{A})$. Given $\varepsilon>0$ there are unitary operators $U_1, U_2, \ldots, U_m$ in $\mathscr{A}$ and positive scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$ of sum 1 such that $\|\sum \alpha_i U_i^* C U_i\| < \varepsilon$ since $0 \in \mathscr{K}_C$. Thus

$$\|CR(M)\| = \left\|\left(\sum \alpha_i U_i^* C U_i\right)(M)\right\| < \varepsilon$$

for every $M \in M(\mathscr{A})$. Because $\varepsilon>0$ is arbitrary, we have that $CR(M)=0$ for every $M \in M(\mathscr{A})$. This means that CR is in the strong radical of $\mathscr{A}$. This is contrary to the choice of C. Hence, we must conclude that $CR(M)$ is not in the center of $\mathscr{A}(M)$ for at least one M in $M(\mathscr{A})$. Then there is a projection E in $\mathscr{A}$ such that $\|(1-E)CRE(M)\| \neq 0$. By the continuity of $M' \to \|(1-E)CRE(M')\|$ on $M(\mathscr{A})$ (Proposition 1), there is an open and closed neighborhood X of M in $M(\mathscr{A})$ such that for every $M' \in X$ the element $CR(M')$ is not in the center of $\mathscr{A}(M')$. Let Q be the nonzero central projection of $\mathscr{A}$ which determines X by the relation $X=\{M' \in M(\mathscr{A}) \mid Q \notin M'\}$ [13]. Then Q is majorized by R and $CQ(M')$ is not in the center of $\mathscr{A}Q(M')$ for every M' in the strong structure space $M(\mathscr{A}Q)$ of the algebra $\mathscr{A}Q$. The latter is true because $M' \to M'Q$ defines a homeomorphism of

$$\{M' \in M(\mathscr{A}) \mid Q \notin M'\} \quad \text{onto } M(\mathscr{A}Q) \text{ [16, Theorem 2.6.6].}$$

Then by 3.1 and 3.7 [10] there is a $\nu>0$ such that $\nu(CQ')\geqq\nu$ for every nonzero central projection Q' majorized by Q. It is clearly immaterial whether $\nu(CQ')$ is evaluated in $\mathscr{A}Q$ or in $\mathscr{A}$. Let m be the smallest integer for which there is a nonzero central projection Q majorized by R such that $\nu(CQ')\geqq 2^{-m}\alpha$ for every nonzero central projection Q' majorized by Q. We then have that $\nu(CQ')\in I_m$ for every such Q'. In fact by the choice of m the projection Q is easily seen to be the least upper bound of a set $\{Q_i\}$ of nonzero mutually orthogonal central projections which satisfy $2^{-m}\alpha\leqq\nu(CQ_i)<2^{-m+1}\alpha$. By Lemma 2 we may conclude that $\nu(CQ)\leqq 2^{-m+1}\alpha$. So for every nonzero central projection Q' majorized by Q we have that $\nu(CQ')\leqq\nu(CQ)\leqq 2^{-m+1}\alpha$. Now if $n+1<m$ set P_{n+1} equal to zero. If $m\leqq n+1$, then $m=n+1$. Indeed suppose that $m<n+1$; the projection $P_m+Q\neq P_m$, and for any nonzero central projection Q' majorized by P_m+Q we have that

$$\nu(CQ') = \text{lub}\,\{\nu(CP_mQ'),\ \nu(CQQ')\}\in I_m.$$

This contradicts the definition of P_m. Therefore $m=n+1$. Now we argue as follows. Let $\{Q_n\}$ be a maximal set of nonzero mutually orthogonal central projections majorized by R such that $\nu(CQ')\in I_{n+1}$ for every nonzero central projection Q' majorized by some Q_i. Let $P_{n+1}=\sum Q_i$. It is clear that $P_{n+1}\leqq R$. Because $\nu(CQ)=\text{lub}_i\ \nu(CQQ_i)$ for any nonzero central projection Q majorized by P_{n+1} (Lemma 2) and since at least one projection QQ_i is nonzero, we have that $\nu(CQ)\geqq 2^{-(n+1)}\alpha$. On the other hand $Q\leqq R$ and so by the induction hypothesis there is a set $\{R_i\}$ of nonzero mutually orthogonal central projections of sum Q such that $\nu(CR_i)<2^{-n}\alpha$ for each R_i. Thus $\nu(CQ)\leqq 2^{-n}\alpha$ (Lemma 2). This proves that $\nu(CQ)\in I_{n+1}$. It is clear that P_{n+1} is the largest central projection majorized by R such that $\nu(CQ)\in I_{n+1}$ for every nonzero central projection majorized by P_{n+1}.

Suppose that the sequence $\{P_n\}$ with the required properties has been constructed by induction. We show that $\sum P_n=1$. If $R=1-\sum P_n$, then for each $n=1, 2,\ldots$ we may conclude that $\nu(CR)\leqq 2^{-n}\alpha$ by performing the construction of the previous paragraph. This means that $\nu(CR)=0$. The results of the previous paragraph show that $R=0$ by our choice of C. Hence $\sum P_n=1$.

Let Z be the spectrum of the center of $\mathscr{A}$ and let $X_n=\{\zeta\in Z \mid P_n\notin\zeta\}$ for $n=1, 2,\ldots$. Now suppose that $\mathscr{A}$ has no σ-finite type III direct summands. For each $n=1, 2,\ldots$ there is a projection F_n in $\mathscr{A}P_n$ with $P_n-F_n\in J$ such that $\nu_{CP_n}(F_n)\leqq 2\nu(CP_n)$ by definition. For each $\zeta\in Z$ there is an irreducible representation $\psi_\zeta=\psi$ of $\mathscr{A}$ on a Hilbert space whose kernel is the smallest closed two-sided ideal $[\zeta]$ of $\mathscr{A}$ which contains ζ [11, Theorem 4.7]. Then ψ does not annihilate J. Indeed J contains a projection E of central support 1, cf. [10, §2]. The two-valued continuous function $\zeta'\to\|E(\zeta')\|$ [9, Lemma 10] on Z assumes the value 0 on an open and closed set given by $\{\zeta'\in Z \mid R\notin\zeta'\}$ where R is a central projection. Then $\|(E(1-R)-E)(\zeta')\|=0$ for every $\zeta'\in Z$. Since $\bigcap\{[\zeta'] \mid \zeta'\in Z\}=(0)$ cf. [9, §4, remarks preceding Lemma 9], we have that $E(1-R)=E$. But this means that $R=0$ since the central support of E is 1. This proves that $E(\zeta')\neq 0$ for every $\zeta'\in Z$

and in particular $E(\zeta)\neq 0$. So ψ does not annihilate J. Thus relations (1) and (3) imply that

$$\|(\psi(CP_n)-\beta)\psi(F_n)\| \leqq 65\eta_{\psi(CP_n)}(\psi(F_n)) \leqq 65\nu_{CP_n}(F_n) \leqq 130\nu(CP_n) \tag{5}$$

whenever $\beta \in \mathscr{W}_{\psi(CP_n)}(\psi(F_n))=\mathscr{W}$ and $\zeta \in X_n$. We show that $0 \in \mathscr{W}$. Indeed, given $\varepsilon>0$, there is a set $U_1, U_2, \ldots, U_m$ of unitary elements in $\mathscr{A}$ and positive scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$ of sum 1 such that $\|\sum \alpha_i U_i^* C U_i\| < \varepsilon$. If G_i is the range projection of $U_i^*(P_n-F_n)$ for $i=1, 2, \ldots, m$, then G_i is a projection in J and $G=\text{lub}\, G_i$ is a projection in J. Thus the projection P_n-G is equivalent to P_n. Let x be a unit vector in the subspace determined by $\psi(P_n-G)$. For each U_i we have that $\psi(U_i)x = y_i$ is in the orthogonal complement of the subspace determined by $\psi(P_n-F_n)$ and thus $\psi(F_n)y_i=y_i$. Therefore

$$\sum \alpha_i(\psi(CP_n)y_i, y_i) \in \mathscr{W}$$

since $\mathscr{W}$ is convex. But we have that

$$\left|\sum (\alpha_i\psi(CP_n)y_i, y_i)\right| \leqq \|\psi\| \left\|\sum \alpha_i U_i^* C U_i\right\| \|x\|^2 < \varepsilon.$$

Because $\mathscr{W}$ is closed and because $\varepsilon>0$ is arbitrary, we see that $0 \in W$. The relation (5) now becomes

$$\|CP_nF_n(\zeta)\| = \|CF_n(\zeta)\| = \|\psi_\zeta(CP_n)\psi_\zeta(F_n)\| \leqq 130\nu(CP_n),$$

for every $\zeta \in X_n$. The orthogonal complement of the projection $F=\sum F_n$ is in J since $P_n(1-F) \in J$ for every $n=1, 2, \ldots$ [10, Proposition 2.2]. For every nonzero central projection P majorized by P_n we have that

$$\|CFP\| = \text{lub}\,\{\|(CF(\zeta)\| \mid P \notin \zeta, \zeta \in Z\} \leqq 130\nu(CP_n) \leqq 260\nu(CP).$$

So for any central projection P we have that

$$\begin{aligned} \|CFP\| &= \text{lub}\,\{\|CFPP_n\| \mid n = 1, 2, \ldots\} \\ &\leqq \text{lub}\,\{260\nu(CPP_n) \mid n = 1, 2, \ldots\} \\ &\leqq 260\nu(CP) \end{aligned} \tag{6}$$

by Lemma 2.

Now for an algebra which is the product of σ-finite type III algebras, we may show that $\|CP\| \leqq 260\nu(CP)$ for any central projection P. The proof is entirely similar to that just given except that relation (4) replaces relations (1) and (3).

Now let us suppose that $\mathscr{A}$ has no σ-finite type III direct summands. Let $D=FCF$ and let $\mathscr{B}$ be the von Neumann algebra $F\mathscr{A}F$. By setting $Q_n=P_nF$ we obtain a sequence $\{Q_n\}$ of mutually orthogonal central projections in $\mathscr{B}$ of sum F such that $\nu(DQ) \in I_n$ for any nonzero central projection Q in $\mathscr{B}$ majorized by Q_n. Here $\nu(DQ)$ is evaluated in $\mathscr{B}$ and Lemma 3 is employed. By relation (6) we see that

$\{\|2^n DQ_n\|\}$ is a bounded sequence and hence $B=\sum 2^n DQ_n$ defines an element of $\mathscr{B}$ such that $\nu(BQ)\geqq\alpha$ for every nonzero central projection Q in $\mathscr{B}$. Indeed,

$$\begin{aligned}\nu(BQ) &= \text{lub}\,\{\nu(BQQ_n) \mid n = 1, 2, \ldots\}\\ &= \text{lub}\,\{2^n\nu(DQQ_n) \mid n = 1, 2, \ldots\} \geqq \alpha\end{aligned}$$

since at least one projection QQ_n is nonzero. There is an invertible element S in $\mathscr{B}$ and a projection G in $\mathscr{B}$ with $F\sim G\sim F-G$ such that $U^*S^{-1}BSU=0$ and $V^*S^{-1}BSV$ is a commutator in $\mathscr{B}$. Here U and V are partial isometries in B such that $U^*U=V^*V=F$, $UU^*=G$ and $VV^*=F-G$ [10, Theorem 3.6]. By multiplying both $U^*S^{-1}BSU$ and $V^*S^{-1}BSV$ by the central element $\sum 2^{-n}Q_n$ we see that $U^*S^{-1}DSU=0$ and $V^*S^{-1}DSV$ is a commutator in $\mathscr{B}$. Now let $T=S+(1-F)$ in $\mathscr{A}$; the element T is invertible with inverse $T^{-1}=S^{-1}+(1-F)$ where S^{-1} still denotes the inverse of S in $\mathscr{B}$. Let W be a partial isometry in $\mathscr{A}$ with domain support 1 and range support F. Then $V_1=VW$ is a partial isometric operator of domain support 1 and range support $F-G$. Then it is easy to see that $V_1^*T^{-1}CTV_1 =W^*V^*S^{-1}DSVW$ is a commutator in $\mathscr{A}$. We have that $1\sim F\sim G\prec G+(1-F)$. Thus there is a partial isometry U_1 in $\mathscr{A}$ with domain support 1 and range support $G+(1-F)$. Then

$$U_1^*T^{-1}CTU_1 = U_1^*T^{-1}CT(1-F)U_1+U_1^*(1-F)T^{-1}CTGU_1$$

is an element of the strong radical of $\mathscr{A}$ and therefore, is a commutator in $\mathscr{A}$ [10, Theorem 2.5]. We have proved that there is an isomorphism of $\mathscr{A}$ onto the algebra $\mathscr{A}_2$ of 2×2 matrices over $\mathscr{A}$ which carries $T^{-1}CT$ into the matrix (B_{ij}) where B_{11} and B_{22} are commutators in $\mathscr{A}$. But this matrix is a commutator in the algebra $\mathscr{A}_2$. Indeed, let $B_{11}=S_{11}T_{11}-T_{11}S_{11}$ and $B_{22}=S_{22}T_{22}-T_{22}S_{22}$ for S_{11}, S_{22}, T_{11}, T_{22} in $\mathscr{A}$. We may assume that S_{11} and S_{22} have disjoint spectra since $B_{11}=(S_{11}+\beta)T_{11}-T_{11}(S_{11}+\beta)$ for a scalar β. There is an operator T_{12} and an operator T_{21} in $\mathscr{A}$ such that $S_{11}T_{12}-T_{12}S_{22}=B_{12}$ and $S_{22}T_{21}-T_{21}S_{11}=B_{21}$ [12]. Setting $S_{21}=S_{12}=0$, we find by direct calculation that $(S_{ij})(T_{ij})-(T_{ij})(S_{ij})=(B_{ij})$ in $\mathscr{A}_2$. This proves (B_{ij}) is a commutator in $\mathscr{A}_2$ and $T^{-1}CT$ is a commutator in $\mathscr{A}$.

Now if $\mathscr{A}$ is the product of σ-finite type III agebras, the preceding paragraph allows us to conclude that there is an invertible S in $\mathscr{A}$ such that $S^{-1}CS$ may be identified with the 2×2 matrix (B_{ij}) over $\mathscr{A}$ with $B_{11}=0$ and B_{22} a commutator in $\mathscr{A}$. So (B_{ij}) is a commutator in the 2×2 matrices over $\mathscr{A}$ and $S^{-1}CS$ is a commutator in $\mathscr{A}$. Q.E.D.

3. **Elements C with $0\in\mathscr{K}_C$.** The construction of Theorem 4 actually depended upon choosing a central element C_0 corresponding to a given element C in a properly infinite von Neumann algebra $\mathscr{A}$ such that $ES^{-1}(C-C_0)SE=0$ for some invertible S in $\mathscr{A}$ and some projection E in $\mathscr{A}$ equivalent to 1. The next proposition clarifies this choice.

PROPOSITION 5. *If C is an element in a properly infinite von Neumann algebra $\mathscr{A}$ such that $ECE=0$ for some projection E in $\mathscr{A}$ equivalent to 1, then 0 is an element of the intersection $\mathscr{K}_C$ of the center of $\mathscr{A}$ with the uniform closure of the convex hull of the set $\{U^*CU \mid U$ unitary in $\mathscr{A}\}$.*

Proof. There are projections E' and E'' such that $E'\sim E''\sim E$ and $E'+E''=E$ [6, III, §8, Theorem 1, Corollary 2]. Then $1\sim E'\leqq E'+(1-E)\leqq 1$ implies that $E'+(1-E)\sim 1$. So there is no loss of generality in supposing that $E\sim 1-E\sim 1$. The operator $U=E-(1-E)$ is unitary in $\mathscr{A}$ and

$$2^{-1}(C+U^*CU) = (1-E)C(1-E).$$

Now let $E_1, E_2,\dots, E_n$ be orthogonal projections of sum E such that $E_1\sim\cdots\sim E_n\sim E$. There are unitary operators $U_1, U_2,\dots, U_n$ in $\mathscr{A}$ such that $(1-E)U_j$ has domain support E_j for $j=1, 2,\dots, n$ since $E_j\sim 1-E$ and $1-E_j\sim E$ for $j=1, 2, \dots, n$. So we have that

$$\left\|\sum \{n^{-1}U_j^*(1-E)C(1-E)U_jx \mid j = 1, 2,\dots, n\}\right\|^2 \leqq n^{-2}\|C\|^2 \sum \|E_jx\|^2 \leqq n^{-2}\|C\|^2\|x\|^2$$

for every x in the Hilbert space. Thus

$$\left\|\sum n^{-1}U_j^*(1-E)C(1-E)U_j\right\| \leqq n^{-1}\|C\|.$$

This means that 0 is an element of the uniform closure of the convex hull of $\{V^*CV \mid V$ unitary in $\mathscr{A}\}$ because n is arbitrary. Hence $0\in\mathscr{K}_C$. Q.E.D.

Let $\mathscr{A}$ be a properly infinite von Neumann algebra. If we could prove that $0\in\bigcup\{\mathscr{K}_{S^{-1}CS} \mid S$ invertible in $\mathscr{A}\}$ for every commutator C in $\mathscr{A}$, then we would have a complete characterization of the set commutators. This characterization is certainly valid for factor algebras. Indeed C is a commutator in the properly infinite factor algebra $\mathscr{A}$ if and only if $\mathscr{A}$ is not a nonzero scalar multiple of the identity modulo the unique maximal ideal M of $\mathscr{A}$. If $C\in M$, then $0\in\mathscr{K}_C$ [10, Proposition 2.4]. If C is not a scalar multiple of the identity modulo M, then the canonical form of Brown and Pearcy [2] in conjunction with the preceding proposition shows $0\in\mathscr{K}_{S^{-1}CS}$ for some invertible S in $\mathscr{A}$. The characterization though is at odds with a conjecture that the set of commutators in $\mathscr{A}$ is the complement (F') in $\mathscr{A}$ of the set of all elements equal to a nonzero scalar multiple of the identity modulo some maximal ideal of $\mathscr{A}$ [4]. In fact let $\{P_n\}$ be a sequence of nonzero mutually orthogonal central projections of sum 1. (This presupposes that $\mathscr{A}$ has a sufficiently large center.) Then let E_n be a projection in $\mathscr{A}P_n$ such that $E_n\sim P_n\sim P_n-E_n$ $(n=1, 2,\dots)$. Let $C=\sum(n^{-1}P_n+n^{-2}E_n)$. We have that

$$\bigcup_n \{M\in M(\mathscr{A}) \mid P_n\notin M\}$$

is dense in the strong structure space $M(\mathscr{A})$ of $\mathscr{A}$ by the remarks at the beginning of §2. Then $C(M)=n^{-1}\cdot 1(M)+n^{-2}E_n(M)$ for every M with $P_n\notin M$ and clearly

$C(M)$ is not a scalar (zero included) multiple of the identity. But $S^{-1}CS(M)$ is not a scalar multiple of the identity for every M with $P_n \notin M$. If for example $0 \in \mathcal{K}_{S^{-1}CS}$, then by the proof of Theorem 4 we would be able to find an invertible T and a projection E equivalent to 1 with $ET^{-1}CTE=0$. Thus

$$\|n^{-1}EP_n\| = n^{-2}\|ET^{-1}E_nTE\| \leqq n^{-2}\|T^{-1}\| \|T\|$$

for each $n=1, 2, \ldots$. This is obviously impossible.

It might be well to remark that there is no canonical matrix form with 0 on the diagonal in the sense of Brown and Pearcy [2] for operators of class (F').

LEMMA 6. *Let C be an element in a von Neumann algebra $\mathcal{A}$. Let D_1 and D_2 be elements in $\mathcal{K}_C$ and let A be a central element of $\mathcal{A}$ with $0 \leqq A \leqq 1$. Then $AD_1 + (1-A)D_2 \in \mathcal{K}_C$.*

Proof. First let A be a projection in the center of $\mathcal{A}$. There are unitary operators $U_1, U_2, \ldots, U_n$ (respectively $V_1, V_2, \ldots, V_m$) and positive scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ (respectively $\beta_1, \beta_2, \ldots, \beta_m$) of sum 1 such that

$$\left\|\sum \alpha_i U_i^* C U_i - D_1\right\| < \varepsilon \quad \text{(respectively, } \left\|\sum \beta_i V_i^* C V_i - D_2\right\| < \varepsilon).$$

Here $\varepsilon > 0$ is a preassigned constant. Then $U_i' = U_i A + (1-A)$ $(1 \leqq i \leqq n)$ and $V_i' = V_i(1-A) + A$ $(1 \leqq i \leqq m)$ are unitary in $\mathcal{A}$ with the property

$$\left\|\sum \alpha_i \beta_j V_j'^* U_i'^* C U_i' V_j' - (D_1 A + D_2(1-A))\right\| < \varepsilon.$$

Since $\mathcal{K}_C$ is closed, we have that $AD_1 + (1-A)D_2 \in \mathcal{K}_C$.

Suppose that the restriction that A is a projection is removed. Let Z be the spectrum of the center of $\mathcal{A}$ and let $D^\frown(\zeta)$ denote the Gelfand transform of the central element D evaluated at $\zeta \in Z$. Since $\mathcal{K}_C$ is convex, the set

$$\mathcal{K}_\zeta = \{D^\frown(\zeta) \mid D \in \mathcal{K}_C\}$$

is convex and so

$$A^\frown(\zeta) D_1^\frown(\zeta) + (1-A)^\frown(\zeta) D_2(\zeta) \in \mathcal{K}_\zeta$$

for every $\zeta \in Z$. Thus there is for each $\varepsilon > 0$ a finite set $P_1, P_2, \ldots, P_n$ of orthogonal central projections of sum 1 and corresponding elements $B_1, B_2, \ldots, B_n$ in $\mathcal{K}_C$ such that

$$\|(AD_1 + (1-A)D_2 - B_j)P_j\| < \varepsilon$$

for $j = 1, 2, \ldots, n$. Since $\sum B_j P_j \in \mathcal{K}_C$ by the first paragraph and since $\varepsilon > 0$ is arbitrary, we see that $AD_1 + (1-A)D_2$ is in the closed set $\mathcal{K}_C$. Q.E.D.

The next proposition corresponds to a result of C. R. Putnam [14]. We use many of his calculations cf. [15, 1.5.1].

PROPOSITION 7. *Let A, B, C be elements in a properly infinite von Neumann algebra $\mathcal{A}$. If A is seminormal (i.e. if either $\pm(AA^* - A^*A)$ is positive) and if $C = AB - BA$, then $0 \in \mathcal{K}_{S^{-1}CS}$ for some invertible S in $\mathcal{A}$.*

Proof. There is a projection P in the center of $\mathscr{A}$ such that CP is in the strong radical of $\mathscr{A}P$ and

$$\{M \in M(\mathscr{A}) \mid C(1-P)(M) \neq 0\}$$

is dense in the subset $\{M \in M(\mathscr{A}) \mid 1-P \notin M\}$ of the strong structure space $M(\mathscr{A})$ of $\mathscr{A}$. Since $\mathscr{K}_C = \mathscr{K}_{CP} + \mathscr{K}_{C(1-P)} = \mathscr{K}_{C(1-P)}$ due to the fact that $\mathscr{K}_{CP} = \{0\}$ [10, Proposition 2.4] and since $\{M \in M(\mathscr{A}) \mid 1-P \notin M\}$ is identified with the strong structure space of $\mathscr{A}(1-P)$, we may assume that $\{M \in M(\mathscr{A}) \mid C(M) \neq 0\}$ is dense in $M(\mathscr{A})$.

As we have previously argued we may assume that either $\mathscr{A}$ has no σ-finite type III direct summands or that $\mathscr{A}$ is the product of σ-finite type III algebras.

Now for each nonzero real scalar α and each unitary element U in the center $\mathscr{Z}$ of $\mathscr{A}$ let

$$F(\alpha, U) = |\alpha|^{1/2}UA + |\alpha|^{-1/2}B^*.$$

Then

$$\operatorname{sgn}(\alpha)(FF^* - F^*F) = D(\alpha) + \operatorname{sgn}(\alpha)S(U)$$

where

$$D(\alpha) = \alpha(AA^* - A^*A) - \alpha^{-1}(BB^* - B^*B)$$

and $S(U) = UC + U^*C^*$. The set $\{|D| = (D^*D)^{1/2} \mid D \in \mathscr{K}_C\}$ is monotonely decreasing in Z. Indeed, let D_1 and D_2 be elements of $\mathscr{K}_C$; there is a central projection P such that $|D_1|P \leqq |D_2|P$ and $|D_2|(1-P) \leqq |D_1|(1-P)$. But $D_1P + D_2(1-P) \in \mathscr{K}_C$ and $|D_1P + D_2(1-P)| = |D_1|P + |D_2|(1-P)$ is majorized by both $|D_1|$ and $|D_2|$. This proves that $\{|D| \mid D \in \mathscr{K}_C\}$ is monotonely decreasing. Let D_0 be the greatest lower bound of this set [6, Appendix II]. Suppose D_1 is a positive central element which majorizes $D(\alpha)$ for some α. We show that $D_0 \leqq D_1$. If not, there is an $\varepsilon > 0$ and a nonzero central projection P such that $D_0P \geqq (D_1 + \varepsilon)P$. By reducing to $\mathscr{A}P$ we may assume that $P = 1$. Let $U_1, U_2, \ldots, U_n$ be unitary elements in $\mathscr{Z}$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive scalars of sum 1; then for any central element R in the sphere of radius $2^{-1}\varepsilon$ about 0 we have that

$$\left|\left(\sum \alpha_i U_i\right)D_1 + R\right|^2 \leqq \left(\sum \alpha_i D_1 + |R|\right)^2 = (D_1 + |R|)^2 \leqq (D_1 + 2^{-1}\varepsilon)^2.$$

Hence $(D^*Dx, x) \leqq ((D_1 + 2^{-1}\varepsilon)^2 x, x)$ for any D in the strong closure $\mathscr{K}''$ of the convex hull of the set

$$\{UD_1 + R \mid U, R \text{ in } \mathscr{Z}, U \text{ unitary}, \|R\| \leqq 2^{-1}\varepsilon\}$$

and for any x in the Hilbert space H of $\mathscr{A}$. On the other hand we see that $(D^*Dx, x) \geqq ((D_1 + \varepsilon)^2 x, x)$ for any D in the strong closure $\mathscr{K}'$ of $\mathscr{K}_C$ and any x in H because $R^*R \geqq (D_1 + \varepsilon)^2$ for every R in $\mathscr{K}_C$. By the standard separation theorem there is a nonzero strongly continuous functional f on $\mathscr{Z}$ such that

$$\text{lub}\{\operatorname{Re} f(R) \mid R \in \mathscr{K}'\} \leqq \text{glb}\{\operatorname{Re} f(R) \mid R \in \mathscr{K}''\}. \tag{7}$$

Here $\operatorname{Re}\beta$ denotes the real part of the complex number β. Indeed the element 0 is not in the strong closure of $\mathscr{K}'-\mathscr{K}''$. Since f is also weakly continuous on $\mathscr{Z}$ [6, I, §3, Theorem 1 (i)], there is a unitary U in $\mathscr{Z}$ and a nonzero vector x in H such that $f(R)=(RUx, x)$ for every $R\in\mathscr{Z}$ [17] and [6, III, §1, Corollary, Theorem 4]. Now let $F=F(\alpha, \operatorname{sgn}(\alpha)U)$. We have that

$$\operatorname{sgn}(\alpha)(FF^*-F^*F) = D(\alpha)+S(U) \leqq D_1+S(U). \tag{8}$$

Let D be an arbitrary element in $\mathscr{K}_{\operatorname{sgn}(\alpha)(FF^*-F^*F)}=\mathscr{K}$. There is an element S in $\mathscr{K}_{S(U)}$ such that $D\leqq D_1+S$ [6, III, §5, Problem 2a]. We may find a T in $\mathscr{K}_{UC-U^*C^*}$ such that

$$2^{-1}(S+T)\in\mathscr{K}_{UC} \quad\text{and}\quad 2^{-1}(S-T)\in\mathscr{K}_{U^*C^*}$$

[6, III, §5, Problem 2a]. We then have that $2^{-1}U^*(S+T)$ and $2^{-1}U^*(S-T)^*$ are elements of $\mathscr{K}_C$. The latter is true because $\mathscr{K}_{C^*}=\{R^* \mid R\in\mathscr{K}_C\}$. From relation (7) we obtain that both $\operatorname{Re}(2^{-1}(S+T)x, x)$ and $\operatorname{Re}(2^{-1}(S-T)x, x)$ are majorized by $\operatorname{glb}\{\operatorname{Re}f(R) \mid R\in\mathscr{K}''\}$. Thus

$$(Sx, x) = \operatorname{Re}(Sx, x) \leqq 2\operatorname{glb}\{\operatorname{Re}f(R) \mid R\in\mathscr{K}''\}$$

since S is clearly selfadjoint. But $-U^*D_1-2^{-1}\varepsilon U^*$ is an element of $\mathscr{K}''$. So

$$2^{-1}(Sx, x) \leqq \operatorname{Re}(U(-U^*D_1-2^{-1}\varepsilon U^*)x, x) = -((D_1+2^{-1}\varepsilon)x, x). \tag{9}$$

Therefore,

$$(Dx, x) \leqq (D_1x, x)-\varepsilon(x, x)-2(D_1x, x) \leqq -\varepsilon(x, x)$$

by relation (8). Using reasoning similar to that which we used to prove that $\{|R| \mid R\in\mathscr{K}_C\}$ is monotonely decreasing, we may prove that $\mathscr{K}$ is monotonely increasing. Setting $R_0=\operatorname{lub}\mathscr{K}$, we see that $(R_0x, x)\leqq-\varepsilon(x, x)$. We show that this is impossible by showing $(R_0x, x)\geqq 0$. Indeed, in proving this then we may certainly assume that F is invertible and that $\alpha>0$. Because F is invertible, there is a unitary operator V in $\mathscr{A}$ obtained from the polar decomposition of F [6, Appendix III] such that $V^*FF^*V=F^*F$. If $R\in\mathscr{K}_{F^*F}$, then there are unitary operators $U_1, U_2, \ldots, U_n$ in $\mathscr{A}$ and positive scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ of sum 1 such that

$$\left\|\sum \alpha_i U_i^*F^*FU_i-R\right\| < \varepsilon'$$

for any preassigned constant $\varepsilon'>0$. But this means that

$$\left\|\sum \alpha_i(VU_i)^*FF^*(VU_i)-R\right\| < \varepsilon'.$$

Because $\varepsilon'>0$ is arbitrary we have that $R\in\mathscr{K}_{FF^*}$. By symmetry it is then clear that $\mathscr{K}_{FF^*}=\mathscr{K}_{F^*F}$. Now for any $\varepsilon'>0$ there is an element R_1 in $\mathscr{K}_{FF^*}$ such that

$$(R_1x, x) \geqq (R_2x, x)-\varepsilon'(x, x)$$

where R_2 is the least upper bound of the monotonely increasing set $\mathscr{K}_{FF^*}$. But there is an element $R \in \mathscr{K}_{F^*F}$ such that $R_1 - R \in \mathscr{K}$ [6, III, §5, Problem 2a]. However we have that

$$(R_0x, x) \geqq ((R_1 - R)x, x) \geqq (R_2x, x) - (Rx, x) - \varepsilon'(x, x) \geqq -\varepsilon'(x, x)$$

since $R_2 \geqq R$. Because $\varepsilon' > 0$ is arbitrary, we see that $(R_0x, x) \geqq 0$. This is a contradiction. We must conclude that $D_0 \leqq D_1$.

We now show that $D_0 = 0$. We may assume that $AA^* - A^*A \leqq 0$. Then for $\alpha > 0$ we have that

$$D(\alpha) \leqq -\alpha^{-1}(BB^* - B^*B) \leqq 2\alpha^{-1}\|B\|^2.$$

Thus we see that $D_0 \leqq 2\alpha^{-1}\|B\|^2$ for every $\alpha > 0$. Therefore $D_0 = 0$. If Q is a nonzero central projection in $\mathscr{A}$ and if $\varepsilon > 0$ there is a net $\{Q_n\}$ of mutually orthogonal central projections of sum Q such that each set $\mathscr{K}_{CQ_n}$ contains an element D_n of norm not exceeding ε. Indeed, if $\{Q_n\}$ is a maximal set of mutually orthogonal nonzero central projections majorized by Q with this property, then the assumption that $Q' = Q - \sum Q_n \neq 0$ gives a contradiction. Since glb $\{|D| \mid D \in \mathscr{K}_{CQ'}\} = 0$, there is a $D \in \mathscr{K}_{CQ'}$ such that $|D| \geqq 2^{-1}\varepsilon Q'$ is not true. This means that there is a nonzero central projection Q'' majorized by Q' such that $|D|Q'' \leqq \varepsilon Q''$. This contradicts the maximality of $\{Q_n\}$. Hence we have that $\sum Q_n = Q$.

Now suppose $\mathscr{A}$ has no σ-finite type III direct summands. In Theorem 4 we constructed a sequence $\{P_n\}$ of mutually orthogonal central projections of sum 1 and a projection F whose orthogonal complement $1 - F$ was in the strong radical such that $\|(C-R)FP\| \leqq 260\nu(CP)$ whenever P is a central projection majorized by P_n and whenever $R \in \mathscr{K}_{CP}$. Also either $P_n = 0$ or $\nu(CP) \in [2^{-n}\|C\|, 2^{-n+1}\|C\|]$ for every nonzero central projection P majorized by P_n. By the preceding paragraph there is a set $\{P_{nj}\}$ of mutually orthogonal central projections of sum P_n such that each set $\mathscr{K}_{CP_{nj}}$ contains an element D_{nj} of norm not exceeding $\nu(CP_n)$. Then for each nonzero central projection P majorized by P_n we have that

$$\begin{aligned} \|CFP\| &= \text{lub}_j \|CFPP_{nj}\| \leqq \text{lub} \|(C - D_{nj})PP_{nj}\| + \text{lub} \|D_{nj}PP_{nj}\| \\ &\leqq 260\nu(CP) + \nu(CP_n) \leqq 262\nu(CP) \end{aligned}$$

by relation (6). By the same reasoning as found in Theorem 4, we may find an invertible W in $\mathscr{A}$ such that $EW^{-1}CWE = 0$ for some projection E in $\mathscr{A}$ which is equivalent to 1. However, this means that $0 \in \mathscr{K}_{W^{-1}CW}$ by Proposition 5.

If A is the product of σ-finite type III algebras a similar proof holds. Q.E.D.

COROLLARY. *If F is an element in a properly infinite von Neumann algebra $\mathscr{A}$, then there is an invertible S in $\mathscr{A}$ such that $\mathscr{K}_{S^{-1}(F^*F - FF^*)S}$ contains* 0.

Proof. If $A = 2^{-1}i(F - F^*)$ and $B = 2^{-1}(F + F^*)$, then $2^{-1}i(FF^* - FF^*) = AB - BA$. Now Proposition 7 applies.

Added in proof (April 25, 1970). I have improved Proposition 7 by showing that $0 \in \mathscr{K}_C$.

BIBLIOGRAPHY

1. A. Brown, P. Halmos and C. Pearcy, *Commutators of operators on Hilbert space*, Canad. J. Math. **17** (1965), 695–708. MR **34** #3311.

2. A. Brown and C. Pearcy, *Structure of commutators of operators*, Ann. of Math. (2) **82** (1965), 112–127. MR **31** #2612.

3. ———, *Commutators in factors of type* III, Canad. J. Math. **18** (1966), 1152–1160. MR **34** #1864.

4. A. Brown, C. Pearcy and D. Topping, *Commutators and the strong radical*, Duke Math. J. **35** (1968), 853–859. MR **38** #1534.

5. D. Deckard and C. Pearcy, *On continuous matrix-valued functions on a Stonian space*, Pacific J. Math. **14** (1964), 857–869. MR **30** #2356.

6. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Cahiers Scientifiques, fasc. 25, Gauthier-Villars, Paris, 1957. MR **20** #1234.

7. ———, *Les C*-algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Gauthier-Villars, Paris, 1964. MR **30** #1404.

8. J. M. G. Fell, *The structure of algebras of operator fields*, Acta Math. **106** (1961), 233–280. MR **29** #1547.

9. J. Glimm, *A Stone-Weierstrass theorem for C*-algebras*, Ann. of Math. (2) **72** (1960), 216–244. MR **22** #7005.

10. H. Halpern, *Commutators in properly infinite von Neumann algebras*, Trans. Amer. Math. Soc. **139** (1969), 55–73.

11. ———, *Irreducible homomorphisms of a von Neumann algebra into its center*, Trans. Amer. Math. Soc. **140** (1969), 195–221. MR **39** #3322.

12. G. Lumer and M. Rosenblum, *Linear operator equations*, Proc. Amer. Math. Soc. **10** (1959), 32–41. MR **21** #2927.

13. Y. Misonou, *On a weakly central operator algebra*, Tôhoku Math. J. (2) **4** (1952), 194–202. MR **14**, 566.

14. C. R. Putnam, *On commutators of bounded matrices*, Amer. J. Math. **73** (1951), 127–131. MR **12**, 836.

15. ———, *Commutation properties of Hilbert space operators and related topics*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 36, Springer-Verlag, New York, 1967. MR **36** #707.

16. C. Rickart, *General theory of Banach algebras*, The University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR **22** #5903.

17. S. Sakai, *On linear functionals of W*-algebras*, Proc. Japan Acad. **34** (1958), 571–574. MR **21** #5915.

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