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ON THE GEOMETRIC MEANS OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES[1]

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Abstract. Let $f(z_1, \ldots, z_n)$ be an entire function of the $n\,(\geqq 2)$ complex variables $z_1, \ldots, z_n$, holomorphic for $|z_t| \leqq r_t$, $t=1, \ldots, n$. We have considered the case of only two complex variables for simplicity. Recently many authors have defined the arithmetic means of the function $|f(z_1, z_2)|$ and have investigated their properties. In the present paper, the geometric means of the function $|f(z_1, z_2)|$ have been defined and the asymptotic behavior of certain growth indicators for entire functions of several complex variables have been studied and the results are given in the form of theorems.

1. Let

$$f(z_1, \ldots, z_n) = \sum_{k_1, \ldots, k_n \geqq 0} a_{k_1 \ldots k_n} z_1^{k_1} \cdots z_n^{k_n}$$

be an entire function of the n $(\geqq 2)$ complex variables $z_1, \ldots, z_n$, holomorphic for $|z_t| \leqq r_t$, $t=1, \ldots, n$. Let us denote the maximum modulus of the function $f(z_1, \ldots, z_n)$ as

$$M(r_1, \ldots, r_n) = \max_{|z_t| \leqq r_t} |f(z_1, \ldots, z_n)| \qquad (t = 1, \ldots, n).$$

Here we consider the case of only two complex variables for simplicity. The results can easily be extended to several complex variables.

The geometric mean of $|f(z_1, z_2)|$ for $|z_t| \leqq r_t$ $(t=1, 2)$ has been defined as [4]

$$G(r_1, r_2) = \exp\left\{\frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| \, d\theta_1 \, d\theta_2\right\}. \tag{1.1}$$

Further, let us define

$$g_k(r_1, r_2) = \exp\left\{\frac{(k+1)^2}{(r_1 r_2)^{k+1}} \int_0^{r_1} \int_0^{r_2} (x_1 x_2)^k \log G(x_1, x_2) \, dx_1 \, dx_2\right\}, \tag{1.2}$$

where k is any positive number.

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The finite order ρ of an entire function $f(z_1, z_2)$ is defined as [2, p. 219]

$$\limsup_{r_1, r_2 \to \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} = \rho.$$

Similarly, we can define the lower order λ as

$$\liminf_{r_1, r_2 \to \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} = \lambda,$$

where $\lambda = \min (\lambda_1, \lambda_2)$ and

$$\lambda_1 = \liminf_{r_2 \to \infty} \liminf_{r_1 \to \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)},$$

$$\lambda_2 = \liminf_{r_1 \to \infty} \liminf_{r_2 \to \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)}.$$

In this paper we have investigated a few properties of the above defined mean values $G(r_1, r_2)$ and $g_k(r_1, r_2)$; the results are given in the form of theorems.

2. THEOREM 1. *Let $f(z_1, z_2)$ be an entire function of finite order ρ and lower order λ (nonintegral with respect to z_1 and z_2), then*

$$\text{(2.1)} \qquad \lim_{r_1, r_2 \to \infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \frac{\log \log g_k(r_1, r_2)}{\log (r_1 r_2)} = \begin{Bmatrix} \rho. \\ \lambda \end{Bmatrix}$$

Proof. From (1.1), it follows that $G(r_1, r_2)$ is an increasing function of (i) r_1 for given r_2, (ii) r_2 for a given r_1, and (iii) r_1 and r_2 both increasing.

Next from (1.1) we have

$$\text{(2.2)} \qquad \int_0^{r_1} \int_0^{r_2} (x_1 x_2)^k \log G(x_1, x_2)\, dx_1\, dx_2 \leqq \frac{(r_1 r_2)^{k+1}}{(k+1)^2} \log M(r_1, r_2).$$

Hence

$$\text{(2.3)} \qquad \lim_{r_1, r_2 \to \infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \frac{\log \log g_k(r_1, r_2)}{\log (r_1 r_2)} \leqq \lim_{r_1, r_2 \to \infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} = \begin{Bmatrix} \rho. \\ \lambda \end{Bmatrix}$$

Further, if $f(z_1, z_2)$ is analytic in $|z_1| \leqq R_1$, $|z_2| \leqq R_2$, $z_j = r_j e^{i\theta_j}$ and $\zeta_j = R_j e^{i\phi_j}$ $(j = 1, 2)$, then

$$\text{(2.4)} \qquad \log |f(z_1, z_2)| \leqq \int_0^{2\pi} \int_0^{2\pi} P(z_1, \zeta_1) P(z_2, \zeta_2) \log |f(\zeta_1, \zeta_2)|\, d\phi_1\, d\phi_2,$$

where

$$P(z_s, \zeta_s) = \frac{1}{2\pi} \left\{ \frac{R_s^2 - r_s^2}{R_s^2 - 2 r_s R_s \cos (\phi_s - \theta_s) + r_s^2} \right\} \qquad (s = 1, 2).$$

Using (2.4) for an entire function $f(z_1, z_2)$, we get

$$\text{(2.5)} \qquad \log |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| \leqq \frac{R_1 + r_1}{R_1 - r_1} \frac{R_2 + r_2}{R_2 - r_2} \log G(R_1, R_2).$$

Taking $R_1=\alpha r_1$, $R_2=\alpha r_2$ $(\alpha>1)$, we have from (2.5)

$$\log M(r_1, r_2) \leqq \{(\alpha+1)/(\alpha-1)\}^2 \log G(\alpha r_1, \alpha r_2).$$

Therefore

$$\begin{aligned}\log g_k(\alpha r_1, \alpha r_2) &\geqq \{(\alpha-1)/(\alpha+1)\}^2\{1-1/\alpha^{k+1}\}^2 \log M(r_1/\alpha, r_2/\alpha)\\ &= H\{\log M(r_1/\alpha, r_2/\alpha)\},\end{aligned}$$

where $H>0$ and is independent of r_1 and r_2. Hence

$$\lim_{r_1,r_2\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log\log g_k(r_1, r_2)}{\log(r_1r_2)} \geqq \lim_{r_1,r_2\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log\log M(r_1, r_2)}{\log(r_1r_2)} = \begin{Bmatrix}\rho\\ \lambda\end{Bmatrix}. \tag{2.6}$$

Combining (2.3) and (2.6), we get (2.1).

3. Let $\phi(r_1, r_2)$ be a "slowly changing" function; that is, $\phi(r_1, r_2)>0$ and is continuous for $r_1>r_1^0$, $r_2>r_2^0$ and for every constant l, $m>0$, $\phi(lr_1, mr_2)\sim\phi(r_1, r_2)$ as r_1 or r_2, or r_1 and r_2, tend to infinity.

Also let

$$\lim_{r_1,r_2\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log g_k(r_1, r_2)}{(r_1r_2)^\rho\phi(r_1, r_2)} = \begin{Bmatrix}p\\ q\end{Bmatrix} \qquad (0<q\leqq p<\infty), \tag{3.1}$$

and

$$\lim_{r_1,r_2\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log G(r_1, r_2)}{(r_1r_2)^\rho\phi(r_1, r_2)} = \begin{Bmatrix}c\\ d\end{Bmatrix} \qquad (0<d\leqq c<\infty). \tag{3.2}$$

THEOREM 2. *If $f(z_1, z_2)$ be an entire function of finite nonzero order ρ, then*

$$\{(k+1)/(k+\rho+1)\}^2 d \leqq q \leqq p \leqq \{(k+1)/(k+\rho+1)\}^2 c; \tag{3.3}$$

$$c \leqq p\{1+\rho/(k+1)\}^2\{1+(k+1)/\rho\}^{2\rho/(k+1)}; \tag{3.4}$$

$$c+\{\rho(2k+3\rho+2)/(k+\rho+1)^2\}d \leqq p\{1+\rho/(k+1)\}^2\{1+(k+1)/\rho\}^{2\rho/(k+1)}. \tag{3.5}$$

Proof. For $0<\eta<1$, from (1.2) we have

$$\begin{aligned}&\log g_k(r_1+\eta r_1, r_2+\eta r_2) \\ &= \frac{(k+1)^2}{(r_1r_2)^{k+1}(1+\eta)^{2(k+1)}}\int_0^{r_1+\eta r_1}\int_0^{r_2+\eta r_2}(x_1x_2)^k\log G(x_1, x_2)\,dx_1\,dx_2\end{aligned} \tag{3.6}$$

$$\begin{aligned}&< \frac{A}{(r_1r_2)^{k+1}}+\left(\frac{r_1^0}{r_1}\right)^{k+1}\left\{\frac{\log G(r_1^0, r_2)}{(1+\eta)^{2(k+1)}}+\frac{((1+\eta)^{k+1}-1)\log G(r_1^0, r_2+\eta r^2)}{(1+\eta)^{2(k+1)}}\right\}\\ &\quad+\left(\frac{r_2^0}{r_2}\right)^{k+1}\left\{\frac{\log G(r_1, r_2^0)}{(1+\eta)^{2(k+1)}}+\frac{((1+\eta)^{k+1}-1)\log G(r_1+\eta r_1, r_2^0)}{(1+\eta)^{2(k+1)}}\right\}\\ &\quad+\frac{(k+1)((1+\eta)^{k+1}-1)}{(1+\eta)^{2(k+1)}}\left\{\frac{1}{r_1^{k+1}}\int_{r_1^0}^{r_1}x_1^k\log G(x_1, r_2+\eta r_2)\,dx_1\right.\\ &\qquad\left.+\frac{1}{r_2^{k_2+1}}\int_{r_2^0}^{r_2}x_2^k\log G(r_1+\eta r_1, x_2)\,dx_2\right\}\end{aligned} \tag{3.7}$$

$$+\frac{(c+\varepsilon)(k+1)^2}{(r_1r_2)^{k+1}(1+\eta)^{2(k+1)}}\int_{r_1^0}^{r_1}\int_{r_2^0}^{r_2}(x_1x_2)^{k+\rho}\phi(x_1,x_2)\,dx_1\,dx_2$$

$$+\frac{((1+\eta)^{k+1}-1)^2}{(1+\eta)^{2(k+1)}}\log G(r_1+\eta r_1, r_2+\eta r_2),$$

where A is a constant,

$$\sim \frac{A}{(r_1r_2)^{k+1}}+\left(\frac{r_1^0}{r_1}\right)^{k+1}\left\{\frac{\log G(r_1^0,r_2)}{(1+\eta)^{2(k+1)}}+\frac{((1+\eta)^{k+1}-1)\log G(r_1^0, r_2+\eta r_2)}{(1+\eta)^{2(k+1)}}\right\} \tag{3.8}$$

$$+\left(\frac{r_2^0}{r_2}\right)^{k+1}\left\{\frac{\log G(r_1,r_2^0)}{(1+\eta)^{2(k+1)}}+\frac{((1+\eta)^{k+1}-1)\log G(r_1+\eta r_1, r_2^0)}{(1+\eta)^{2(k+1)}}\right\}$$

$$+\left\{\frac{k+1}{k+\rho+1}\right\}\frac{(c+\varepsilon)(r_1r_2)^\rho((1+\eta)^{k+1}-1)}{(1+\eta)^{2(k+1)}}$$

$$\cdot\{\phi(r_1, r_2+\eta r_2)+\phi(r_1+\eta r_1, r_2)\}$$

$$+\left\{\frac{k+1}{k+\rho+1}\right\}^2\frac{(c+\varepsilon)(r_1r_2)^\rho\phi(r_1,r_2)}{(1+\eta)^{2(k+1)}}$$

$$+\frac{((1+\eta)^{k+1}-1)^2}{(1+\eta)^{2(k+1)}}\log G(r_1+\eta r_1, r_2+\eta r_2)$$

from [3, Lemma 5].

Hence, dividing both the sides by $\{(r_1+\eta r_1)^\rho(r_2+\eta r_2)^\rho\phi(r_1+\eta r_1, r_2+\eta r_2)\}$ and taking limit, we get

$$\limsup_{r_1,r_2\to\infty}\left\{\frac{\log g_k(r_1,r_2)}{(r_1r_2)^\rho\phi(r_1,r_2)}\right\}\leqq c\left[\left\{\frac{k+1}{k+\rho+1}\right\}^2\frac{1}{(1+\eta)^{2(k+\rho+1)}}\right.$$

$$\left.+2\left\{\frac{k+1}{k+\rho+1}\right\}\frac{((1+\eta)^{k+1}-1)}{(1+\eta)^{2(k+\rho+1)}}+\frac{((1+\eta)^{k+1}-1)^2}{(1+\eta)^{2(k+1)}}\right]. \tag{3.9}$$

Since η is arbitrary, we get

$$p\leqq\{(k+1)/(k+\rho+1)\}^2c. \tag{3.10}$$

Next, from (3.6), we have

$$\log g_k(r_1+\eta r_1, r_2+\eta r_2) \tag{3.11}$$

$$>\frac{(d-\varepsilon)(k+1)^2}{(r_1r_2)^{k+1}(1+\eta)^{2(k+1)}}\int_{r_1^0}^{r_1}\int_{r_2^0}^{r_2}(x_1x_2)^{k+\rho}\phi(x_1,x_2)\,dx_1\,dx_2$$

$$+\frac{(d-\varepsilon)(k+1)((1+\eta)^{k+1}-1)}{(1+\eta)^{2(k+1)}}\left\{\frac{r_2^\rho}{r_1^{k+1}}\int_{r_1^0}^{r_1}x_1^{k+\rho}\phi(x_1,r_2)\,dx_1\right.$$

$$\left.+\frac{r_1^\rho}{r_2^{k+1}}\int_{r_2^0}^{r_2}x_2^{k+\rho}\phi(r_1,x_2)\,dx_2\right\}$$

$$+\frac{((1+\eta)^{k+1}-1)^2}{(1+\eta)^{2(k+1)}}\log G(r_1,r_2)$$

$$\sim \frac{(d-\varepsilon)(r_1r_2)^\rho\phi(r_1, r_2)}{(1+\eta)^{2(k+1)}}\left[\left\{\frac{k+1}{k+\rho+1}\right\}^2+2\left\{\frac{k+1}{k+\rho+1}\right\}\{(1+\eta)^{k+1}-1\}\right]$$

$$+\left[\frac{\{(1+\eta)^{k+1}-1\}}{(1+\eta)^{2(k+1)}}\right]\log G(r_1, r_2),$$

using [3, Lemma 5].

Hence

$$q \geqq \frac{d}{(1+\eta)^{2(k+\rho+1)}}\left[\left\{\frac{k+1}{k+\rho+1}\right\}+\{(1+\eta)^{k+1}-1\}\right]^2. \tag{3.12}$$

Since η is arbitrary, we get

$$q \geqq \{(k+1)/(k+\rho+1)\}^2 d. \tag{3.13}$$

Combining (3.10) and (3.13), the inequality (3.3) follows. Also, from (3.11)

$$\{(1+\eta)^{k+1}-1\}^2 c \leqq p(1+\eta)^{2(k+\rho+1)}-d\left[\left\{\frac{k+1}{k+\rho+1}\right\}^2+2\left\{\frac{k+1}{k+\rho+1}\right\}\{(1+\eta)^{k+1}-1\}\right],$$

and so for all $\eta>0$

$$c \leqq p(1+\eta)^{2(k+\rho+1)}/\{(1+\eta)^{k+1}-1\}^2. \tag{3.14}$$

The right-hand side of this inequality has the least value when $(1+\eta)^{k+1}=\{1+(k+1)/\rho\}$, therefore

$$c \leqq p\{1+\rho/(k+1)\}^2\{1+(k+1)/\rho\}^{2\rho/(k+1)}.$$

Also, from (3.14) we get

$$c+\frac{\rho(2k+3\rho+2)}{(k+\rho+1)^2}\,d \leqq p\{1+\rho/(k+1)\}^2\{1+(k+1)/\rho\}^{2\rho/(k+1)},$$

which completes the proof of the theorem.

4. Theorem 3. *Let $f(z_1, z_2)$ be an entire function of finite order ρ and if*

$$\lim_{r_1,r_2\to\infty} \begin{Bmatrix}\sup\\ \inf\end{Bmatrix} \frac{\log g_k(r_1, r_2)}{(r_1r_2)^\rho\phi(r_1, r_2)} = \begin{Bmatrix}p,\\ q\end{Bmatrix} \tag{4.1}$$

and

$$\log G(r_1,r_2) \sim \beta(r_1r_2)^\rho\phi(r_1, r_2), \tag{4.2}$$

for large values of r_1 and r_2, where β is a constant and $0<q\leqq p<\infty$ and $\phi(r_1, r_2)$ as defined in §3. *Then*

(i) *$f(z_1, z_2)$ is of regular growth*;

(ii) $(k+\rho+1)^2p=(k+1)^2\beta=(k+\rho+1)^2q$; *and*

(iii) $\lim_{r_1,r_2\to\infty} \log g_k(r_1, r_2)/\log G(r_1, r_2)=\{(k+1)/(k+\rho+1)\}^2$.

Proof. (i) Taking the logarithm of (4.2), the result follows.

(ii) From (1.2), we have

$$\text{(4.3)}\quad \log g_k(r_1, r_2) < \frac{A}{(r_1r_2)^{k+1}} + \left\{\frac{r_1^0}{r_1}\right\}^{k+1} \log G(r_1^0, r_2) + \left\{\frac{r_2^0}{r_2}\right\}^{k+1} \log G(r_1, r_2^0)$$

$$+\frac{\beta(k+1)^2}{(r_1 r)^{k+1}} \int_{r_1^0}^{r_1} \int_{r_2^0}^{r_2} (x_1x_2)^{k+\rho}\phi(x_1, x_2)\, dx_1\, dx_2$$

from (4.2) and A is a constant,

$$\sim \frac{A}{(r_1r_2)^{k+1}} + \left\{\frac{r_1^0}{r_1}\right\}^{k+1} \log G(r_1^0, r_2) + \left\{\frac{r_2^0}{r_2}\right\}^{k+1} \log G(r_1, r_2^0)$$

$$+\beta\left\{\frac{k+1}{k+\rho+1}\right\}^2 (r_1r_2)^\rho\phi(r_1, r_2)$$

by repeated application of [3, Lemma 5].

Taking the limit, we get

$$\text{(4.4)}\quad \lim_{r_1, r_2\to\infty} \frac{\log g_k(r_1, r_2)}{(r_1r_2)^\rho\phi(r_1, r_2)} \leqq \beta\left\{\frac{k+1}{k+\rho+1}\right\}^2.$$

Also, from (1.2), we have

$$\text{(4.5)}\quad \begin{aligned} \log g_k(r_1, r_2) &> \frac{(k+1)^2}{(r_1r_2)^{k+1}} \int_{r_1^0}^{r_1} \int_{r_2^0}^{r_2} (x_1x_2)^k \log G(x_1, x_2)\, dx_1\, dx_2 \\ &\sim \beta\left\{\frac{k+1}{k+\rho+1}\right\}^2 (r_1r_2)^\rho\phi(r_1, r_2) \quad \text{from (4.2).} \end{aligned}$$

Hence

$$\text{(4.6)}\quad \lim_{r_1, r_2\to\infty} \frac{\log g_k(r_1, r_2)}{(r_1r_2)^\rho\phi(r_1, r_2)} \geqq \beta\left\{\frac{k+1}{k+\rho+1}\right\}^2.$$

Combining (4.4) and (4.6), the result follows.

(iii) Further, if we divide by $\log G(r_1, r_2)$, a positive increasing function, the inequalities (4.3) and (4.5) respectively, we get

$$\text{(4.7)}\quad \frac{\log g_k(r_1, r_2)}{\log G(r_1, r_2)} < \frac{1}{\log G(r_1, r_2)} \left[\frac{A}{(r_1r_2)^{k+1}} + \left\{\frac{r_1^0}{r_1}\right\}^{k+1} \log G(r_1^0, r_2) + \left\{\frac{r_2^0}{r_2}\right\}^{k+1} \log G(r_1, r_2^0)\right] + \beta\left\{\frac{k+1}{k+\rho+1}\right\}^2 \frac{(r_1r_2)^\rho\phi(r_1, r_2)}{\log G(r_1, r_2)}$$

and

$$\text{(4.8)}\quad \frac{\log g_k(r_1, r_2)}{\log G(r_1, r_2)} > \beta\left\{\frac{k+1}{k+\rho+1}\right\}^2 \frac{(r_1r_2)^\rho\phi(r_1, r_2)}{\log G(r_1, r_2)}.$$

Since (4.2) holds, on proceeding to limits, the inequalities (4.7) and (4.8) lead to the result.

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References

1. S. Bochner and W. T. Martin, *Several complex variables*, Princeton Math. Series, vol. 10, Princeton Univ. Press, Princeton, N. J., 1948. MR **10**, 366.

2. S. K. Bose and D. Sharma, *Integral functions of two complex variables*, Compositio Math. **15** (1963), 210–226. MR **29** #270.

3. G. H. Hardy and W. W. Rogosinski, *Notes on Fourier series*. III: *Asymptotic formulae for the sums of certain trigonometrical series*, Quart. J. Math. Oxford Ser. **16** (1945), 49–58. MR **7**, 247.

4. R. K. Srivastava, *Integral functions represented by Dirichlet series and integral functions of several complex variables*, Ph.D. Thesis, Lucknow Univ., Lucknow, India, 1964.

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