LOCALLY CONVEX TOPOLOGICAL LATTICES

BY

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Abstract. The main theorem of this paper is: Suppose that L is a topological lattice of finite breadth n. Then L can be embedded in a product of n compact chains if and only if L is locally convex and distributive. With this result it is then shown that the concepts of metrizability and separability are equivalent for locally convex, connected, distributive topological lattices of finite breadth.

In [8] R. P. Dilworth proved that every distributive lattice of finite breadth n could be embedded (algebraically) in a product of n chains. Since finite breadth and distributivity are hereditary properties this result served to characterize distributive lattices of finite breadth. Dyer and Shields in [9] and Anderson in [4] (also see question 90 of [7]) asked if a result of a similar nature could be obtained for topological lattices, specifically: Can every compact, connected, metric, distributive topological lattice of breadth n be embedded in an n-cell? This question was answered affirmatively and more generally by Kirby A. Baker and the present author in [6]. For easier reference this result along with several consequences appears as Theorem 1.3 below.

The major result of this paper is more nearly the topological analogue to Dilworth's theorem. We characterize the class of those topological lattices which can be embedded in a finite product of compact chains as the class of locally convex, distributive topological lattices of finite breadth. The class of locally convex topological lattices is rather large. For example, it contains all compact topological lattices [15], all locally compact and connected topological lattices [3], and all discrete lattices. This being the case our result contains those of [6] and [8].

We also show that the set of separating points of a locally convex, connected topological lattice is very well behaved. This fact together with our main theorem allows us to prove that separability and metrizability are equivalent for locally convex, connected, distributive topological lattices of finite breadth.

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1. **Definitions and preliminary results.** A topological lattice is a Hausdorff topological space with a pair of continuous maps $\wedge$, $\vee : L \times L \to L$ such that

$(L, \wedge, \vee)$ is a lattice. Now, let L be such a lattice. $A \subseteq L$ is meet-redundant if there is a proper subset B of A such that $\bigwedge A = \bigwedge B$. A set which is not meet-redundant is meet-irredundant. We say that L has breadth n, written $\mathrm{Br}\,(L) = n$, if L has a meet-irredundant subset consisting of n elements, but every subset of L with $n+1$ elements is meet-redundant. A is said to be a representation of x_0 if $x_0 = \bigwedge A$. Let $x_0 \in L$ define $\rho(x_0)$ to be the largest cardinality of those meet-irredundant representations of x_0. For $A \subseteq L$, $\rho(A) = \max\{\rho(a); a \in A\}$. Note that $\rho(L) = \mathrm{Br}\,(L)$. For $a, b \in L$ with $a < b$ we define $[a, b]$ to be $\{x \in L; a \leq x \leq b\}$. A subset A of L is convex if whenever it contains $\{a, b\}$ and $a \leq b$ then it contains $[a, b]$. L is locally convex if its topology has a base of convex sets.

For convenience we shall use the following notation.

$\mathscr{L}_n$ is the class of all locally convex, distributive topological lattices of finite breadth n.

$\mathscr{C}_n$ is the class of all compact, distributive topological lattices of finite breadth n.

By a chain we shall mean a linearly ordered set with a topology at least as large as the interval topology. It is obvious that every chain is a topological lattice. By I we shall mean the real interval $[0, 1]$ with its usual topology and its usual lattice operations. I^n is the n-fold Cartesian product of I and $\mathrm{Hom}\,(L, I)$ is the set of continuous homomorphisms of the lattice L into I.

For connected chains the concepts of local convexity and "linearity" (in the sense of Jones [11]) are equivalent. Thus, Theorem 10 of [11] may be rephrased as follows.

THEOREM 1.1 (JONES [11]). *Suppose that L is a member of $\mathscr{C}_1$ which is also connected and metric. Then L is separable.*

L. W. Anderson in [2] showed that locally convex, connected topological lattices are locally connected. With this fact we may modify Theorem 10 of [17] to read

THEOREM 1.2 (ROY [17]). *Suppose that L is a locally convex, connected, metric topological lattice with no separating points such that every pair of points of L can be separated by a closed separable subset of L. Then L is separable.*

The results of [6] which we mentioned in the introduction are now given as

THEOREM 1.3 ([6]). *Let $L \in \mathscr{L}_n$. If L is either (1) compact or (2) locally compact and connected, then L can be embedded in a product of n compact chains. Moreover if L is also separable and metric then it can be embedded in I^n.*

It should be pointed out that part (2) above is derived from part (1) via Corollary 3 of [5].

2. Some examples.

We now give several examples which will be of use in the subsequent discussion.

EXAMPLE 2.1. To form the lattice L_1 we take the real interval $[0, 1]$ with its usual lattice operations and give it a new topology τ_1. τ_1 is that topology which is generated by the usual topology on $[0, 1]$ and a "new" open set Q, the set of rational numbers in $[0, 1]$. L_1 is a connected chain which is neither compact nor locally convex.

EXAMPLE 2.2. For each nonnegative integer let $c_n = 1/2^n$. Let A_n be the line segment from $(1, c_{n-1})$ to $(0, c_n)$ and let B_n be the line segment from $(0, c_n)$ to $(1, c_n)$ for each positive integer n. Define L_2 to be $(\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} B_n) \cup \{(1, 0)\}$. Give L_2 the topology which it inherits from the plane and let it have the separating point ordering with $\bigvee L_2 = (1, 1)$ and $\bigwedge L_2 = (1, 0)$. (I.e. $x < y$ if and only if the component of $L_2 \backslash \{x\}$ which contains y also contains $(1, 1)$.) L_2 is a connected chain which is neither locally convex nor compact.

EXAMPLE 2.3. Define X to be the set of all points in the plane with nonnegative y-coordinates. We give X the topology generated by the usual topology and all sets formed by taking the interiors of circles tangent to the x-axis along with the points of tangency. Let C be the usual representation of the Cantor set on the interval from $(0, 0)$ to $(1, 0)$. Let $\{(a_i, b_i); i \in N\}$ be the set of pairs of x-coordinates of end points of complimentary intervals to C with $a_i < b_i$. For example, $(1/2, 2/3)$, $(1/9, 2/9)$ and $(7/9, 8/9)$ are in this set. Now form a set $\{x_i; i \in N\}$ where x_i is the mid-point of the interval $[a_i, b_i]$. Define A_i to be the line segment from $(a_i, 0)$ to $(x_i, 1)$ and define B_i to be the line segment from $(x_i, 1)$ to $(b_i, 0)$. Let $L_3 = C \cup (\bigcup_{i=1}^{\infty} A_i) \cup (\bigcup_{i=1}^{\infty} B_i)$. With the topology L_3 inherits from X, L_3 is a connected space. Give L_3 the separating point order with $\bigvee L_3 = (1, 0)$ and $\bigwedge L_3 = (0, 0)$. L_3 is a separable chain which is not locally convex and not metrizable.

EXAMPLES 2.4 AND 2.5. These examples are obtained from L_2 and L_3. We shall only describe L_2 since the description of L_3 is analogous. Replace each A_i and B_i in L_2 by connected, distributive two-dimensional topological lattices A_i' and B_i' so that $L_4 = (\bigcup_{i=1}^{\infty} A_i') \cup (\bigcup_{i=1}^{\infty} B_i') \cup \{(1, 0)\}$ is a cyclic chain in the sense of Whyburn [21] with $\{(x, c_i); i = 1, 2, \ldots$ and $x \in \{0, 1\}\} \cup \{(1, 0)\}$ being the set of separating points and end points.

EXAMPLE 2.6. Let L_6 be the set of nonnegative integers with the discrete topology. Define $1 \wedge x = 0 \vee x = x$, $0 \wedge x = 0$ and $1 \vee x = 1$ for all $x \in L_6$. For $x, y \in L_6 \backslash \{0, 1\}$ define $x \wedge y = x$ if $x = y$ and 0 otherwise. Define $\vee$ dually. L_6 is a modular, non-distributive, locally convex topological lattice of breadth two. Moreover L_6 cannot be embedded in a compact topological lattice. This is seen by an elementary limit argument.

EXAMPLE 2.7. In the plane take A to be the line segment from $(0, 0)$ to $(1, 0)$ and B to be the line segment from $(0, 1)$ to $(1, 1)$. Define L_7 to be $A \cup B$ and give L_7 the lexicographic order. Then with the interval topology L_7 is a locally convex, separable chain which is not metrizable.

3. **The main theorem.** The goal of this section is to prove our main result which is

THEOREM 3.1. *Suppose that L is a topological lattice of finite breadth n. Then L can be embedded (algebraically and topologically) in a product of n compact chains if and only if L is locally convex and distributive.*

It is apparent that the chains mentioned above can be taken to be connected. While the proof of the sufficiency is obvious we will need a sequence of lemmas to establish the necessity.

LEMMA 3.1. *Let L be a distributive topological lattice of finite breadth. Then* Hom (L, I) *separates points in L.*

Proof. We use essentially the same proof that is used in [18]. If L is a chain then since the topology on L contains the interval topology, L is normally ordered in the sense of Nachbin [15]. Then upon applying Theorem 1 of Chapter 2 of [15] and Lemma 2 of [5] our proof is complete. Note that connectivity is not required in the last mentioned reference.

The author is indebted to the referee for the next observation.

PROPOSITION 3.1. *Let $L \in \mathscr{L}_n$. If $0 \in L$ then L has a basis of convex sublattices at 0.*

Proof. Let U be a neighborhood of 0. Let V be a convex neighborhood of 0 in V. From the continuity of V there is a neighborhood W of 0 such that $V^n W \subseteq v$. $W \wedge L$ is then the desired neighborhood.

The proof of Theorem 3.1 depends upon the embedding theorem of S. Mrówka [14]. The next definition and Lemma 3.3 recast certain concepts used in [14] into terms more appropriate for the present situation.

DEFINITION. Let L be a topological lattice. The set $\{(f_i, F_i); i = 1, \ldots, m\}$ is said to *-separate the point p of L and the closed subset K of L if m is finite and each f_i is a continuous homomorphism of L into a compact, distributive topological lattice F_i such that one, and hence both, of the following two equivalent conditions holds.

(i) $f_1 \times \cdots \times f_n(p) \notin f_1 \times \cdots \times f_m(K)^*$,

(ii) K can be written as $K_1 \cup \cdots \cup K_m$ where for each $i, f_i(p) \notin f_i(K_i)^*$.

The proof that these two conditions are equivalent will be omitted since it is straightforward.

LEMMA 3.2. *Suppose that $L \in \mathscr{L}_n$ satisfies the property that every point p and every closed subset K of L not containing p can be *-separated. Then L can be embedded in a product of n compact chains.*

Proof. Suppose that $\mathscr{F} = \{(f_\alpha, F_\alpha); \alpha \in \Gamma\}$ is the class of all pairs (f_α, F_α) where F_α is a compact, distributive topological lattice and f_α is a continuous homomorphism of L into F_α. From Lemma 3.1, $\mathscr{F}$ separates the points of L and hence is nonempty. Define $f: L \to \times_{\alpha \in \Gamma} F_\alpha$ by $f(x)_\alpha = f_\alpha(x)$. With the usual algebraic and topological structure $\times_{\alpha \in \Gamma} F_\alpha$ is a compact topological lattice and since $\mathscr{F}$ separates

points f is a continuous monomorphism. Since every point p and every closed set K not containing p can be *-separated, it follows from Mrówka's embedding theorem [14] that f is an iseomorphism. $f(L)^*$ is a compact, distributive topological lattice and by a simple limit argument it can be shown that $\mathrm{Br}\,(\varphi(L)^*) = \mathrm{Br}\,(L)$. By Theorem 1.3 above we may conclude that $\varphi(L)^*$ and hence L can be embedded in a product of n compact chains.

Theorem 3.1 will be proved by induction on breadth. The next lemma will be the first step in the induction argument.

LEMMA 3.3. *Suppose that $L \in \mathscr{L}_1$. Then L can be embedded in a compact chain (i.e., a member of $\mathscr{C}_1$).*

Proof. Suppose that $p \in L$ and K is a closed subset of L which does not contain p. Since L is linearly ordered $K = K_0 \cup K_1$ where $K_0 = (p \wedge L) \cap K$ and $K_1 = (p \vee L) \cap K$. No generality is lost by assuming that $K_0 \neq \varnothing \neq K_1$. If there is $q \in L$ such that $p < q \leq k$ for every $k \in K_1$, choose $\varphi_1 \in \mathrm{Hom}\,(L, I)$ so that $\varphi_1(p) = 1/2$ and $\varphi_1(q) = 1$. (Such a map exists by Lemma 3.1.) If no such point q exists define $\varphi_1 : L \to I$ by $\varphi_1(p \wedge L) = 1/2$ and $\varphi_1((K_1 \vee L)^*) = 1$. Thus in either situation there is a homomorphism φ_1 of L into I such that $\varphi_1(p) = 1/2$ and $\varphi_1(K_1) = 1$. In a similar fashion one can construct a homomorphism φ_0 of L into I such that $\varphi(K_0) = 0$ and $\varphi(p) = 1/2$. Combining these two homomorphisms in the appropriate way we obtain a homomorphism $\varphi : L \to I$ such that $\varphi(p) \notin \varphi(K)^*$. Thus by Lemma 3.2 L can be embedded in a compact chain.

The next lemma is the key result in our discussion.

LEMMA 3.4. *Suppose that Theorem 3.1 holds for every lattice in $\bigcup_{m=1}^{n-1} \mathscr{L}_m$. Let $L \in \mathscr{L}_n$ and suppose that L has a minimum element 0. If K is a closed subset of L which does not contain 0 then 0 and K can be *-separated.*

Proof. First we show that we may assume that $K = (K \vee L)^*$. Suppose that such an assumption cannot be made. This implies that there are nets $\{k_\alpha; \alpha \in \Gamma\}$ and $\{x_\alpha; \alpha \in \Gamma\}$ in K and L respectively such that the net $\{k_\alpha \vee x_\alpha; \alpha \in \Gamma\}$ converges to 0. Since L is locally convex and since $k_\alpha \vee x_\alpha \geq k_\alpha \geq 0$ for each $\alpha \in \Gamma$ the net $\{k_\alpha; \alpha \in \Gamma\}$ must converge to 0. However this is not possible because K is closed and $0 \notin K$. Hence we shall assume that $K = (K \vee L)^*$.

From Proposition 3.1 there is a neighborhood W of 0 which is a sublattice of L such that $W \cap K = \varnothing$. Note that since W is convex and $0 \in W$ it follows that W is an ideal of L. Choose $x_0 \in W$ so that $\rho(x_0) = \rho(W)$. We may assume that $x_0 = 0$, for otherwise, since $0 \notin K$ we could map L onto $x_0 \vee L$ and proceed with the same proof. If $\rho(0) = 1$ then W must be linearly ordered. If $W \neq \{0\}$ choose $q \in W \backslash \{0\}$ and choose $\varphi \in \mathrm{Hom}\,(L, I)$ so that $\varphi(0) \neq \varphi(q)$. If $W = \{0\}$ then define $\varphi : L \to I$ by $\varphi(0) = 0$ and $\varphi(L \backslash \{0\}) = 1$. In both situations we have succeeded in defining a continuous homomorphism of L into I such that $\varphi(0) \notin \varphi(K)^*$. Thus 0 and K can be *-separated.

Now suppose that $\rho(0) = m > 1$. Let $\{x_1, \ldots, x_m\}$ be a meet-irredundant representation of 0. Define a set $\{y_1, \ldots, y_m\}$ by letting $y_i = \bigwedge_{j \neq i} x_j$. It follows readily that $\{y_1, \ldots, y_m\}$ is join-irredundant (see the proof of Theorem 1 of [4]) and $y_i \wedge y_j = 0$ if and only if $i \neq j$.

One of the following two cases must then hold.

Case (i). Suppose that the set $\{x_1, \ldots, x_m\}$ can be chosen so that $\{y_1, \ldots, y_m\} \subseteq W$. Define a map $\theta : L \to \times_{i=1}^{m} [0, y_i]$ by $\theta(t) = (t \wedge y_1, \ldots, t \wedge y_m)$. θ is a continuous homomorphism. We now claim that $\theta(0) \notin \theta(K)^*$. Suppose, on the contrary, that $\theta(0) \in \theta(K)^*$. This implies that there is a net $\{k_\alpha ; \alpha \in \Gamma\}$ of members of K such that the net $\{k_\alpha \wedge y_i ; \alpha \in \Gamma\}$ converges to 0 for $i = 1, \ldots, m$. For $\alpha \in \Gamma$ we propose to show that some element of the set $\{y_1, \ldots, y_m, k_\alpha\}$ is join-redundant. Suppose that this is not so. Form a set $\{z_1, \ldots, z_{m+1}\}$ by defining $z_i = (\bigvee_{i \neq j} y_j) \vee k_\alpha$ for $1 \leq i \leq m$ and $z_{m+1} = \bigvee_{i=1}^{m} y_i$. $\{z_1, \ldots, z_{m+1}\}$ is meet-irredundant and

(1) $$z_1 \wedge \cdots \wedge z_{m+1} = (z_1 \wedge \cdots \wedge z_m) \wedge (y_1 \vee \cdots \vee y_m).$$

W is a sublattice of L and $\{y_1, \ldots, y_m\} \subseteq W$ so we must have $y_1 \vee \cdots \vee y_{m+1} \in W$. Then since W is also an ideal of L by (1) above $z_1 \wedge \cdots \wedge z_{m+1} \in W$. Thus we have found an element of W, $z_1 \wedge \cdots \wedge z_{m+1}$, such that $\rho(z_1 \wedge \cdots \wedge z_{m+1}) = m + 1 > \rho(W)$. This is not possible because of the way in which ρ was defined so $\{y_1, \ldots, y_m, k_\alpha\}$ must be join-redundant. If k_α were redundant in this set then we would have $k_\alpha \leq y_1 \vee \cdots \vee y_m$, i.e., $k_\alpha \in W$. But of course this is not possible since $W \cap K = \varnothing$. Then because m is finite we may assume that y_1 is join-redundant in $\{y_1, \ldots, y_m, k_\alpha\}$ for each $\alpha \in \Gamma$. This implies that

$$\begin{aligned} y_1 &= y_1 \wedge (y_2 \vee \cdots \vee y_m \vee k_\alpha) \\ &= (y_1 \wedge y_2) \vee \cdots \vee (y_1 \wedge y_m) \vee (y_1 \wedge k_\alpha) \\ &= 0 \vee (y_1 \wedge k_\alpha) = y_1 \wedge k_\alpha. \end{aligned}$$

However we than have

$$y_1 = \lim y_1 = \lim (y_1 \wedge k_\alpha) = 0$$

which contradicts the fact that $y_1 \neq 0$. Thus $\theta(0) \notin \theta(K)^*$.

Br $([0, y_i]) < $ Br (L) by Lemma 1 of [5]. Hence by hypothesis $[0, y_i]$ can be embedded in a compact topological lattice. Then since $\theta(0) \notin \theta(K)^*$ there is a continuous homomorphism ψ of L into a compact topological lattice such that $\psi(0) \notin \psi(K)^*$. Thus in this case 0 and K can be *-separated. We now turn to the alternative situation.

Case (ii). Suppose that it is not possible to choose $\{x_1, \ldots, x_m\}$ so that $\{y_1, \ldots, y_m\} \subseteq W$. We may choose $\{x_1, \ldots, x_m\}$ so that the number of elements of $\{y_1, \ldots, y_m\}$ which are in W is maximum. Also we may assume that $y_1 \notin W$. Suppose that there is an element $y \in ([0, y_1] \backslash \{0, y_1\}) \cap W$. Define $\{y_1', \ldots, y_m'\}$ and $\{x_1', \ldots, x_m'\}$ as follows: $y_1' = y$, $y_i' = y_i$ for $i \neq 1$ and $x_i' = \bigvee_{j \neq i} y_j'$ for each i. By elementary calculation it is easily shown that

(a) $\{x'_1, \ldots, x'_m\}$ is meet-irredundant.

(b) $\bigwedge_{i=1}^{m} x'_i = 0$.

(c) The set of y's corresponding to $\{x'_1, \ldots, x'_m\}$ is $\{y'_1, \ldots, y'_m\}$.

This leads us to the conclusion that the number of elements in $\{y'_1, \ldots, y'_m\} \cap W$ is larger by one than the number of elements in $\{y_1, \ldots, y_m\} \cap W$. This is not possible so we must have $[0, y_1] \cap W = \{0\}$. K may be written as the union of two closed sets K_0 and K_1 where $K_0 = \{k \in K; y_1 \wedge k = 0\}$ and $K_1 = \{k \in K; y_1 \wedge k \in (L \backslash W)^*\}$. $\mathrm{Br}\,([0, y_1]) < \mathrm{Br}\,(L)$ and $0 \notin (y_1 \wedge K_1)^*$, so by hypothesis there is a continuous homomorphism τ of L into a compact topological lattice X such that $\tau(0) \notin \tau(K_1)^*$. Now let $A = \{x \in L; y_1 \wedge x = 0\}$. From Lemma 1 of [5], $\mathrm{Br}\,(A) < n = \mathrm{Br}\,(L)$. Note that K_0 is a closed subset of A which does not contain 0. By hypothesis and Lemma 3.2, A can be embedded in a product of $r = \mathrm{Br}\,(A)$ compact chains $C_1 \times \cdots \times C_r$. For $i = 1, \ldots, r$ let φ_i be the map formed by composing the embedding map of A into $C_1 \times \cdots \times C_r$ with the natural projection of $C_1 \times \cdots \times C_r$ onto C_i. From this it follows that K_0 may be written as the union of r sets $M_1 \cup \cdots \cup M_r$ where $\varphi_i(0) \notin \varphi_i(M_i)^*$ (in C_i). Choose $m_i \in M_i$ for $i = 1, \ldots, r$ and define $\sigma_i: L \to [0, m_i]$ by $\sigma_i(x) = m_i \wedge x$. We claim that $0 \notin \sigma_i(M_i)^*$. Suppose that $0 \in \sigma_i(M_i)^*$. Then there is a net $\{x_\alpha; \alpha \in \Gamma\}$ of elements of M_i such that the net $\{m_i \wedge x_\alpha; \alpha \in \Gamma\}$ converges to 0. This implies that the net $\{\varphi_i(m_i \wedge x_\alpha); \alpha \in \Gamma\}$ converges to $\varphi_i(0) \in C_i$. Since C_i is a chain this leads to the conclusion that $\{\varphi_i(m_i \wedge x_\alpha); \alpha \in \Gamma\} \subseteq \varphi_i(M_i)$. However $\varphi_i(0) \notin \varphi_i(M_i)^*$. Thus we have a contradiction and our claim is valid. To complete our proof we note that then $\{(\tau, X), (\varphi_1 \circ \sigma_1, C_1), \ldots, (\varphi_k \circ \sigma_k, C_k)\}$ *-separates 0 and K.

One more lemma is still needed before we can prove our theorem.

LEMMA 3.5. *Suppose that L is a locally convex topological lattice. Let K be a closed subset of L and let p be a point in $L \backslash K$. Then there is a convex neighborhood W of p such that K can be written as the union of two closed sets K_0 and K_1 with $(p \wedge K_0) \cup (p \vee K_1) \subseteq L \backslash W$.*

Proof. Suppose that our lemma is not true. Let $\mathscr{U} = \{U_\alpha; \alpha \in \Gamma\}$ be a convex local base at p. There must be a net $\{k_\alpha; \alpha \in \Gamma\}$ of elements of K such that for each $\alpha \in \Gamma$, $\{p \wedge k_\alpha, p \vee k_\alpha\} \subseteq U_\alpha$. Thus $\lim p \vee k_\alpha = p = \lim p \wedge k_\alpha$. Since each U_α is convex and $p \wedge k_\alpha \leq k_\alpha \leq p \vee k_\alpha$ for each $\alpha \in \Gamma$, we must have $\lim k_\alpha = p$. But K is closed and $p \notin K$. Thus we have a contradiction. Hence our lemma is proved.

Finally, we may now complete the proof of our theorem.

Proof of Theorem 3.1. From Lemma 3.3 our theorem holds for lattices in $\mathscr{L}_1$. Assume that the theorem holds for lattices in $\bigcup_{m=1}^{n-1} \mathscr{L}_m$. Let $L \in \mathscr{L}_n$. Let $p \in L$ and let K be a closed subset of L such that $p \notin K$. Let W, K_0 and K_1 be those sets obtained by Lemma 3.5. It follows that $p \notin (p \wedge K_0)^* \cup (p \vee K_1)^*$. So by Lemma 3.4 there are continuous homomorphisms $f_1, \ldots, f_k$ of L into compact lattices $F_1, \ldots, F_k$ such that

$$f_1 \times \cdots \times f_k(p) \notin f_1 \times \cdots \times f_k((p \vee K_1)^*)^*.$$

And from a dual formulation of Lemma 3.4 there are continuous homomorphisms $g_1, \ldots, g_j$ of L into compact lattices $C_1, \ldots, C_j$ such that

$$g_1 \times \cdots \times g_j(p) \notin g_1 \times \cdots \times g_j((p \wedge K_0)^*)^*.$$

Let $\sigma: L \to p \vee L$ be defined by $\sigma(x) = p \vee x$ and let $\mu: L \to p \wedge L$ be defined by $\mu(x) = p \wedge x$. It then follows that

$$\{(f_1 \circ \sigma, F_1), \ldots, (f_k \circ \sigma, F_k), (g_1 \circ \mu, \mathscr{C}_1), \ldots, (g_j \circ \mu, \mathscr{C}_j)\}$$

*-separates p and K. Therefore, by Lemma 3.2, L can be embedded in a product of n compact chains.

In [12] J. D. Lawson gives an example of a compact, distributive topological lattice which has no continuous homomorphism onto I. Thus Theorem 3.1 cannot be extended to lattices of infinite breadth. Also Example 2.6 above shows that distributivity cannot be deleted from the list of hypotheses of Theorem 3.1. It should be noted that Theorem 3.1 answers partially but not completely the following two questions.

(1) Can every distributive topological lattice of finite breadth n be embedded in a product of n chains?

(2) Which topological lattices can be embedded in a compact topological lattice?

Our next theorem is the metric analog of Theorem 3.1.

THEOREM 3.2. *Suppose that $L \in \mathscr{L}_n$ and suppose that L is separable and metrizable. Then L can be embedded in I^n.*

Proof. From Theorem 3.1 there are homomorphisms $\varphi_1, \ldots, \varphi_n$ and compact chains $C_1, \ldots, C_n$ such that $\varphi_1 \times \cdots \times \varphi_n$ embeds L into $C_1 \times \cdots \times C_n$. Let $D_i = \varphi_i(L)$. D_i is a separable metric chain for each i. We claim that D_i can be embedded in I for each i. Let $M_i = \{x \in L; x \vee L \text{ is open in } L\}$. Since D_i is separable and metric it satisfies the second axiom of countability. Hence M_i is countable. Let K_i be a countable dense subset of D_i. $M_i \cup K_i$ is then a countable order-dense subset of D_i. Thus from page 200 of [7] we can embed D_i in I. It then follows easily that L can be embedded in I^n.

Theorem 3.2 yields immediately the following corollary. This result complements the dimension-theoretic results which appear in [6] and [12].

COROLLARY 3.2. *Suppose that L is a separable, metrizable member of $\mathscr{L}_n$. Then $\mathrm{Br}\,(L) \geq \dim L$. Moreover, if L is connected then $\mathrm{Br}\,(L) = \dim L$.*

It is obvious what changes Theorem 5.1 and Theorem 5.2 below make in Theorem 3.2 and Corollary 3.2.

4. Separating points in locally convex, connected topological lattices. For a connected topological lattice L we define a set $S(L)$ to be

$$\{x \in L; L = (x \vee L) \cup (x \wedge L)\}.$$

$S(L)$ consists of the set of separating points of L along with those universal bounds which L might possess (see Theorem 1 of [2]). A point $x \in S(L)$ is a local maximum (minimum) if x is an isolated point in $x \vee L$ ($x \wedge L$). $M(S(L))$ is the set of local maxima of $S(L)$ while $m(S(L))$ is the set of local minima. If $x \in m(S(L))$ and $y \in M(S(L))$ such that $x < y$ and $[x, y] \cap S(L) = \{x, y\}$, then $\langle x, y \rangle$ is said to be a gap and x and y are called adjacent gap points. If C is a compact, metric chain then it is well known that C can have at most a countable number of gaps. For example see Corollary 3.3 of [6].

LEMMA 4.1. *Suppose that L is a locally convex, connected topological lattice. Then $S(L)$ is a closed linearly ordered sublattice of L such that if $S_0 \subseteq S(L)$ and S_0 is bounded above in L then $\bigvee S_0$ exists and is in $S(L)$.*

Proof. It is easily seen that $S(L)$ is a linearly ordered sublattice of L. Suppose that $\{s_\alpha; \alpha \in \Gamma\}$ is a net in $S(L)$ converging to s_0. For $x \in L$ and $\alpha \in \Gamma$ either $x \leq s_\alpha$ or $s_0 \geq x$. Hence $s_0 \in S(L)$ and $S(L)$ is closed.

Suppose that $S_0 \subseteq S(L)$ and suppose that S_0 has an upper bound x_0. If $\bigvee S_0$ exists and is in S_0 then we are done. So we suppose that if $\bigvee S_0$ exists then it is not in S_0, i.e. for each $s \in S$ there is $t \in S_0$ such that $s < t$. For $s \in S_0$, $L = A(s) \cup B(s) \cup \{s\}$ where $A(s) = s \wedge L \backslash \{s\}$ and $B(s) = s \vee L \backslash \{s\}$. Let $A_0 = \bigcup \{A(s); s \in S_0\}$ and let $B_0 = \bigcap \{B(s); s \in S_0\}$. Since $x_0 \in B(s)$, for every $s \in S_0$, $B_0 \neq \varnothing$. Then because $S_0 \subseteq A_0$ we have $L = A_0 \cup B_0$. A_0 is open, $A_0 \cap B_0 = \varnothing$ and L is connected so $A_0^* \cap B_0 \neq \varnothing$. Choose $\{a_\alpha; \alpha \in \Gamma\}$ a net of elements of A_0 converging to some point $s_0 \in B_0$. For each $\alpha \in \Gamma$ there is $s_\alpha \in S_0$ such that $a_\alpha \leq s_\alpha \leq b_0$. Then since L is locally convex there is a subnet of $\{s_\alpha; \alpha \in \Gamma\}$ which converges to s_0. It then follows that $s_0 = \bigvee S_0$ and since $S(L)$ is closed we have $s_0 \in S(L)$.

The next theorem lists several properties of $S(L)$ when L is locally convex. This result is similar to Theorem 2 of [20] in which A. D. Wallace proves that every compact, connected, metrizable topological lattice is a cyclic chain in the sense of Whyburn [21].

THEOREM 4.1. *Suppose that L is a locally convex, connected topological lattice. Then*

(i) *$S(L)$ is conditionally complete.*

(ii) *The topology which $S(L)$ inherits from L coincides with the interval topology on $S(L)$.*

(iii) *If $a, b \in S(L)$ with $a < b$ then $[a, b] \cap S(L)$ is compact.*

(iv) *$S(L)$ is locally compact.*

(v) *If L is metric then it possesses at most a countable number of gaps.*

(vi) *If L is metric then $S(L)$ is separable.*

Proof. Part (i) follows from Lemma 4.1 and p. 115 of [7]. Part (ii) is easily derived. Part (iii) follows from parts (i) and (ii) and p. 115 of [7]. Part (iv) follows directly from parts (ii) and (iii).

Now let $S(L)$ be a metric space with metric d. Suppose that $S(L)$ has uncountably many gaps. This implies that $m(S(L))$ must also be uncountable. For each $x \in m(S(L))$ define $\zeta(x)$ to be a positive integer n such that $B(x, 1/n) \cap (x \vee L) = \{x\}$. $B(x, 1/n) = \{y \in S(L); d(x, y) < 1/n\}$. Since $m(S(L))$ is uncountable there is a positive integer n such that $\zeta^{-1}(n)$ is uncountable. We may assume that $n = 1$. If both $\bigvee \zeta^{-1}(1)$ and $\bigwedge \zeta^{-1}(1)$ existed, then by part (iii) of this theorem

$$[\bigwedge \zeta^{-1}(1), \bigvee \zeta^{-1}(1)] \cap S(L)$$

would be a compact metric space. This would contradict our assumption that $\zeta^{-1}(1)$ is uncountable. Thus we may assume that $\bigvee \zeta^{-1}(1)$ does not exist. We may also assume that there is $x_0 \in \zeta^{-1}(1)$ such that $\zeta^{-1}(1) \cap (x_0 \vee S(L))$ is uncountable. We define a function $f: \zeta^{-1}(1) \to \zeta^{-1}(1)$ by letting $f(x)$ be an element of $\zeta^{-1}(1) \cap (x \vee S(L))$ such that $f(x) \notin B(x, 1)$. Since $\bigvee \zeta^{-1}(1)$ does not exist it follows that f is well defined. Define an infinite sequence $\{x_1, x_2, x_3, \ldots\}$ in $\zeta^{-1}(1)$ by setting $x_1 = x_0$ and letting $x_n = f^{n-1}(x_0)$ for $n > 1$. This sequence is indeed infinite since $\zeta^{-1}(1) \cap (x_0 \vee S(L))$ is uncountable and each $[x_n, x_{n+1}]$ is compact. For the same reasons it follows that $\{x_1, x_2, x_3, \ldots\}$ is bounded above in $S(L)$. Let

$$x = \bigvee \{x_1, x_2, x_3, \ldots\}.$$

Since $[x_1, x]$ is compact and metrix x must be a limit point of $\{x_1, x_2, x_3, \ldots\}$. However this is not possible since $d(x, x_n) > 1$ for each n. This is a contradiction. Thus $S(L)$ can have at most a countable number of gaps. Hence part (v) is proved.

Let $S_1 = \{x \in S(L); x < y \text{ for all } y \in m(S(L)) \cup M(S(L))\}$ and let

$$S_2 = \{x \in S(L); x > y \text{ for all } y \in m(S(L)) \cup M(S(L))\}.$$

S_1 and S_2 are connected (if nonempty) locally convex, metric chains. Hence, by Theorem 1.1, S_1 and S_2 are separable if nonempty. From part (v) above the set of gaps in $S(L)$ is countable. Thus $m(S(L)) \cup M(S(L))$ is countable. Then $S(L) \backslash S_1 \cup S_2 = \bigcup_{i=1}^{\infty} \{[a_i, b_i] \cap S(L)\}$ where $a_i, b_i \in m(S(L)) \cup M(S(L))$ with $a_i < b_i$. Each $[a_i, b_i] \cap S(L)$ is a compact metric space by part (iii) above. Hence $[a_i, b_i] \cap S(L)$ is separable for each i. It then follows that $S(L) \backslash S_1 \cup S_2$ and hence $S(L)$ is separable.

The examples in §2, in particular Examples 2.4 and 2.5, show that local convexity is necessary in Lemma 4.1 and in Theorem 4.1.

5. Separability and metrizability in topological lattices.

The relationship between separability and metrizability in connected topological spaces has been studied by Alexandroff, Jones, Roberts, Treybig and Roy, among others. See [1], [10], [16] and [19]. In this section we obtain results of a similar nature.

In one direction we have

Theorem 5.1. *Suppose that $L \in \mathscr{L}_n$ and suppose also that L is separable and connected. Then L is metrizable. Moreover L can be embedded in I^n.*

Proof. Suppose that K is a countable dense subset of L. From Theorem 3.1 we may consider L to be a sublattice of $C = C_1 \times \cdots \times C_n$ where each C_i is a compact chain. Let π_i be the natural projection of C onto C_i and let $D_i = \pi_i(L)^*$. If $a_i = \bigwedge D_i$ and $b_i = \bigvee D_i$, then $K_i = \pi_i(K) \cup \{a_i, b_i\}$ is a countable dense subset of D_i. Then it follows from connectivity that $\mathscr{B} = \{[a, b] \backslash \{a, b\}; a, b \in K_i$ and $a < b\}$ is a countable base for the topology of D_i. Hence by Urysohn's Theorem D_i is metrizable. From Corollary 3.3 of [6], among other sources, D_i can be embedded into I. We may then conclude that L can be embedded in I^n.

Example 2.3 shows that local convexity is necessary in Theorem 3.1 while Example 2.7 shows that connectivity is necessary.

The next theorem is a partial converse to Theorem 5.1. The real interval $[0, 1]$ with the discrete topology and the usual lattice operations is a simple example which shows that connectivity cannot be deleted from our next result.

THEOREM 5.2. *Suppose that $L \in \mathscr{L}_n$ and suppose that L is connected and metric. Then L is separable.*

Proof. Our proof will be accomplished by means of induction on breadth. If L satisfies the hypothesis of our theorem and Br $(L) = 1$, then, by Theorem 1.1, L is separable.

Now suppose that our theorem holds for lattices of breadth less than n and suppose that L is a lattice which satisfies the hypothesis above and Br $(L) = n$. We consider two cases.

Case (i). Suppose that L does not have any separating points. From Theorem 3.1, L may be assumed to be a sublattice of $C = C_1 \times \cdots \times C_n$ where each C_i is a compact connected chain. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two distinct points in L. There is some i such that $x_i \neq y_i$. We may assume that $x_i < y_i$. Since C_i is connected there is $c_i \in C_i$ such that $x_i < c_i < y_i$. Let $A = \{(a_1, \ldots, a_n) \in C; a_i = c_i\}$. A is a connected sublattice of C with Br $(A) = n - 1$. It is easily seen that $A \cap L$ is a closed, locally convex, connected sublattice of L such that Br $(A \cap L) \leqq n - 1$. By our induction hypothesis $A \cap L$ is separable. $A \cap L$ is then a closed separable subset of L which separates a and b. Thus the hypothesis of Theorem 1.2 is satisfied. Hence we may conclude that L is separable.

Case (ii). Suppose that L has separating points. $S(L)$ possesses those properties listed in Theorem 4.1. Let K be a countable dense subset of $S(L)$. From Theorem 3.1 we can embed L into a compact lattice. Let a and b be the universal bounds of L in that lattice. The number of gaps in $S(L)$ is at most countable so it follows that the number of gaps in $S(L) \cup \{a, b\}$ is at most countable. Let $\{\langle a_i, b_i \rangle; i \in N\}$ be the set of gaps of $S(L) \cup \{a, b\}$. For each i, $[a_i, b_i]$ satisfies the conditions of Case (i). Thus $[a_i, b_i]$ has a countable dense subset, say K_i. The set $(\bigcup_{i=1}^{\infty} K_i) \cup K$ is a countable dense subset of L. Thus L is separable.

In the previous theorem distributivity was only used in Case (i). There it was used to show that if L is a distributive lattice with breadth n and a and b are distinct

points of L then there is a sublattice L_0 of L such that L_0 separates a and b in L and $\mathrm{Br}(L_0) < n$. Whether a wider class of lattices than distributive lattices share this property is unknown to the author. For such lattices Theorem 5.2 will hold.

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