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HOMOLOGY OF DELETED PRODUCTS OF ONE-DIMENSIONAL SPACES(1)

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Abstract. The object of this paper is to investigate the homology of deleted products of finitely triangulated one-dimensional spaces. By direct calculation, we obtain upper bounds for the two-dimensional Betti numbers, and, using a rather small system of topological types of spaces appearing as subspaces of the space under consideration, we obtain lower bounds for these Betti numbers. We demonstrate that, in general, the two-dimensional Betti numbers are larger than they were originally thought to be.

1. **Introduction.** Our purpose in this paper is to investigate the homology and cohomology of deleted products of finitely triangulated one-dimensional spaces. It has long been known that these groups are finitely computable, but the computations are unreasonably tedious, and we have often wished for simpler ways of evaluating the groups. Since the homology and cohomology in dimensions greater than two are trivial, and since the connectivity and the Euler-Poincaré characteristic are readily computed, it suffices to find the first or the second homology group. We have chosen to concentrate on the second homology. Our original thesis, which we were not successful in demonstrating, was that the elements of H_2 arose from a rather small system of topological types of spaces appearing as subspaces of the space under consideration. However, this approach did lead us to discover lower bounds on the Betti numbers. A somewhat more direct procedure yields upper bounds.

2. **Notation.** If X is a topological space, then its *deleted product* is the topological product $X \times X$ with the diagonal removed. The *reduced deleted product* is the quotient space obtained from the deleted product by identifying each point (x_1, x_2) with (x_2, x_1). Both of these spaces play an important role in the study of embeddings of spaces. If X has a given triangulation, then

$$JX = \bigcup \{\sigma \times \tau \mid \sigma \text{ and } \tau \text{ are simplices of } X \text{ and } \sigma \cap \tau = \varnothing\}$$

is a deformation retract of the deleted product. The reduction of JX is called KX, and it is a deformation retract of the reduced deleted product. If A and B are subspaces of X, then we will write $A \square B = (A \times B) \cup (B \times A)$. ($A \square B$ is a subspace

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of $X \times X$). If v and w are the vertices of a 1-simplex σ, then we may also write $\sigma = \langle v, w \rangle$. Unless otherwise specified, homology and cohomology will be taken to be with coefficients in a principal ideal ring Λ. The Betti number $\beta_n(X) = \beta_n(X; \Lambda)$ is the rank of $H_n(X; \Lambda)$.

3. **Upper bounds.** Suppose X is a finitely triangulated space, A is a triangulated subspace (triangulated by a subcomplex of the triangulation on X), and σ is a simplex of X. Define $C(\sigma, A)$ to be the union of the simplices of A that do not meet σ.

Now suppose that X is one-dimensional, $X = A \cup \sigma$, where σ is a 1-simplex, and that $A \cap \sigma$ lies in the boundary, $\partial\sigma$, of σ. Let

$$\begin{aligned} B(X, A) &= \beta_1 C(\sigma, A) && \text{if the ends of } \sigma \text{ are joined by a path in } A, \\ &= 0 && \text{otherwise.} \end{aligned}$$

In this formula, β_1 is the Betti number formed relative to some homology theory; since the space involved is one-dimensional, it does not matter which homology theory is used. A *decomposition* of X (where X is connected) is a sequence $(X_n, X_{n-1}, \ldots, X_0)$ of subspaces of X such that $X_n = X$ and $X_i = X_{i-1} \cup \sigma_i$, where, for each $i = 1, 2, \ldots, n$, X_i is connected and σ_i is a 1-simplex having $X_{i-1} \cap \sigma_i \subset \partial\sigma_i$. The decomposition is called *complete* if X_0 is a single vertex. Note that in a complete decomposition, n is exactly the number of 1-simplices in X. When $(X_n, X_{n-1}, \ldots, X_0)$ is any decomposition of X, let

$$B(X_n, X_{n-1}, \ldots, X_0) = \sum_{i=1}^{n} B(X_i, X_{i-1}).$$

THEOREM 3.1. *For complete decompositions of X, the number $B(X_n, X_{n-1}, \ldots, X_0)$ is independent of the decomposition used.*

Proof. A decomposition determines an indexing of the 1-simplices σ_i of X, and, conversely, any indexing of the 1-simplices determines a decomposition. Thus it suffices to compare the B values for two decompositions related by the interchange of a successive pair of simplices. This is done in the following lemma.

LEMMA 3.2. *Suppose $X = A \cup \sigma_1 \cup \sigma_2$, where A is connected and σ_1 and σ_2 are distinct 1-simplices such that $\varnothing \neq A \cap \sigma_i \subset \partial\sigma_i$ for each $i = 1, 2$. If, for each $i = 1, 2$, $A_i = A \cup \sigma_i$, then*

$$B(X, A_1) + B(A_1, A) = B(X, A_2) + B(A_2, A).$$

Proof. Set $B_0 = B(A_1, A) + B(A_2, A)$. First suppose $\sigma_1 \cap \sigma_2 = \varnothing$. Then $C(\sigma_i, A_j) = C(\sigma_i, A) \cup \sigma_j$ $(i, j = 1, 2, i \neq j)$. For $i = 1, 2$, consider the two cases:

Case (a, i). $A \cap \sigma_i = \partial\sigma_i$.

Case (b, i). $A \cap \sigma_i \neq \partial\sigma_i$.

Note that

$$\begin{aligned} B(X, A_i) &= B(A_j, A) + 1 && \text{if } (a, 1) \text{ and } (a, 2), \\ &= B(A_j, A) && \text{otherwise.} \end{aligned}$$

Thus

$$\begin{aligned} B(X, A_1)+B(A_1, A) &= B_0+1 && \text{if } (a, 1) \text{ and } (a, 2),\\ &= B(A_1, A) = B_0 && \text{if } (a, 1) \text{ and } (b, 2),\\ &= B(A_2, A) = B_0 && \text{if } (b, 1) \text{ and } (a, 2),\\ &= 0 = B_0 && \text{if } (b, 1) \text{ and } (b, 2),\\ &= B(X, A_2)+B(A_2, A). \end{aligned}$$

Now suppose $\sigma_1 \cap \sigma_2 \neq \varnothing$. Note that $C(\sigma_i, A)=C(\sigma_i, A_j)$ $(i, j=1, 2; i\neq j)$. For $i=1, 2$, consider the three cases:

Case (a, i). $\sigma_i \cap A=\partial\sigma_i$.

Case (b, i). $\sigma_i \cap A\neq\partial\sigma_i$ but $\sigma_i \cap A_j=\partial\sigma_i$.

Case (c, i). $\sigma_i \cap A_j\neq\partial\sigma_i$.

Note that $(b, 1)$ and $(b, 2)$ are equivalent. Thus

$$\begin{aligned} B(X, A_1)+B(A_1, A) &= B_0 && \text{if } (a, 1) \text{ and } (a, 2),\\ &= B(A_1, A) = B_0 && \text{if } (a, 1) \text{ and } (c, 2),\\ &= B(A_2, A) = B_0 && \text{if } (b, 1) \text{ and } (b, 2),\\ &= B(A_2, A) = B_0 && \text{if } (c, 1) \text{ and } (a, 2),\\ &= 0 = B_0 && \text{if } (c, 1) \text{ and } (c, 2),\\ &= B(X, A_2)+B(A_2, A). \end{aligned}$$

The condition that A be connected is necessary in Lemma 3.2 as the following examples show.

EXAMPLE 3.3. Let

$$\begin{aligned} A = {}& \langle v_1, v_2\rangle \cup \langle v_1, v_3\rangle \cup \langle v_1, v_4\rangle \cup \langle v_2, v_3\rangle \cup \langle v_3, v_4\rangle \cup \langle v_5, v_6\rangle\\ & \cup \langle v_5, v_7\rangle \cup \langle v_5, v_8\rangle \cup \langle v_6, v_7\rangle \cup \langle v_7, v_8\rangle, \end{aligned}$$

let $\sigma_1=\langle v_1, v_5\rangle$, and let $\sigma_2=\langle v_4, v_8\rangle$. Then $B(X, A_1)+B(A_1, A)=2$, whereas $B(X, A_2)+B(A_2, A)=0$.

EXAMPLE 3.4. Let

$$\begin{aligned} A = {}& \langle v_1, v_2\rangle \cup \langle v_1, v_3\rangle \cup \langle v_1, v_4\rangle \cup \langle v_2, v_3\rangle \cup \langle v_3, v_4\rangle \cup \langle v_5, v_6\rangle\\ & \cup \langle v_5, v_7\rangle \cup \langle v_5, v_8\rangle \cup \langle v_6, v_7\rangle \cup \langle v_7, v_8\rangle, \end{aligned}$$

let $\sigma_1=\langle v_4, v_5\rangle$, and let $\sigma_2=\langle v_4, v_8\rangle$. Then $B(X; A_1)+B(A_1, A)=2$, whereas $B(X, A_2)+B(A_2, A)=1$.

THEOREM 3.5. *If (X, A) is a decomposition of X, then $\beta_2(JX)\leqq\beta_2(JA)+2B(X, A)$ and $\beta_2(KX)\leqq\beta_2(KA)+B(X, A)$.*

Proof. Let σ be the 1-simplex such that $X=A \cup \sigma$. First suppose $A \cap \sigma$ is a single vertex v of σ. Then $JX=JA \cup (\sigma \square C) \cup (b \square A)$, where $C=C(\sigma, A)$ and b is the other vertex of σ. The inclusion map $(\sigma \square C, v \square C) \to (JA \cup (\sigma \square C), JA)$

is an excision. Since $H_2(JA \cup (\sigma \square C), JA)=0$, $\beta_2(JA \cup (\sigma \square C))=\beta_2(JA)$. Similarly the inclusion map $(b \square A, b \square C) \to (JX, JA \cup (\sigma \square C))$ is an excision, and hence, since $H_2(JX, JA \cup (\sigma \square C))=0$, $\beta_2(JX)=\beta_2(JA)$. The corresponding argument for KX is similar.

Now suppose $A \cap \sigma = \partial\sigma$. Then $JX = JA \cup (\sigma \square C)$, and $H_2(JX, JA)$ is isomorphic to the direct sum of two copies of $H_1(C)$. The exactness of the sequence $0 \to H_2(JA) \to H_2(JX) \xrightarrow{j_*} H_2(JX, JA)$ then shows $\beta_2(JX) \leqq \beta_2(JA) + 2\beta_1(C)$. The projection $JX \to KX$ identifies the two subspaces $C \times \sigma$ and $\sigma \times C$. Thus a similar argument shows $\beta_2(KX) \leqq \beta_2(KA) + \beta_1(C)$.

It is interesting to note that it is not necessary to assume that A is connected in Theorem 3.5. Suppose the endpoints a_1 and a_2 of σ lie in disjoint closed subsets A_1 and A_2 (respectively) of A, and that $A = A_1 \cup A_2$. For each $i=1, 2$, set $C_i = C \cap A_i$. Note that $JA = JA_1 \cup JA_2 \cup (A_1 \square A_2)$. Using this fact and the Künneth Theorem, we find a direct sum decomposition induced by inclusion maps:

$$H_1(JA_1)+H_1(JA_2)+H_1(a_1 \times A_2)+H_1(A_1 \times a_2)+H_1(a_2 \times A_1)+H_1(A_2 \times a_1) \to H_1(JA).$$

Any element of $H_2(JX, JA)$ has a representative relative 2-cycle $z = \sigma \times (z_1+z_2) + (z_1'+z_2') \times \sigma$ with z_i and z_i' 1-cycles on C_i. Then

$$\begin{aligned} \partial z = {} & (a_1 \times z_1 + z_1' \times a_1) + (-[a_2 \times z_2] - [z_2' \times a_2]) + (a_1 \times z_2) \\ & -(z_1' \times a_2) - (a_2 \times z_1) + (z_2' \times a_1). \end{aligned}$$

The terms in ∂z are parenthesized in agreement with the direct sum decomposition just established. Thus $\partial z = 0$ implies $z=0$. This shows image (j_*) = kernel $(\partial_*)=0$, whence

$$\beta_2(JX) = \beta_2(JA)+0 = \beta_2(JA)+2B(X, A).$$

The corresponding argument for KX is similar.

COROLLARY 3.6. *If $(X_n, X_{n-1}, \ldots, X_0)$ is a decomposition of X, then*

$$\beta_2(JX) \leqq \beta_2(JX_0)+2B(X_n, X_{n-1}, \ldots, X_0)$$

and

$$\beta_2(KX) \leqq \beta_2(KX_0)+B(X_n, X_{n-1}, \ldots, X_0).$$

COROLLARY 3.7. *If $(X_n, X_{n-1}, \ldots, X_0)$ is a complete decomposition of X, then*

$$\beta_2(JX) \leqq 2B(X_n, X_{n-1}, \ldots, X_0) \quad \textit{and} \quad \beta_2(KX) \leqq B(X_n, X_{n-1}, \ldots, X_0).$$

4. **The family $\mathscr{F}$.** The following three spaces play a crucial role in this paper.

(1) $\partial\Delta \times \partial I$, where $\partial\Delta$ is the (geometric) boundary of the standard 2-simplex Δ and ∂I is the boundary of the unit interval I. This space is the union of two disjoint simple closed curves.

(2) H & W, the houses-and-wells figure, is the join of two 3-point discrete spaces.

(3) CG5V, the complete graph on five vertices, is the 1-skeleton of the standard 4-simplex.

In each case, the above presentation suggests an obvious triangulation. Furthermore, this is the minimal triangulation for the space. Notice that these spaces are homogeneous in the following sense: for any two 1-simplexes of the space, there is an isomorphism of the space carrying the first simplex onto the second. Let $\mathscr{F}$ be the set of all PL-homeomorphs of these spaces. It is readily verified that if $F \in \mathscr{F}$ and σ is a 1-simplex in F, then there is a unique simple closed curve S in F that does not meet σ. We say that σ and S are *opposite* each other. Let $\mathscr{F}_0$ be the set of PL-homeomorphs of $\partial\Delta \times \partial I$.

The following table summarizes the homology and cohomology of the deleted products of spaces in $\mathscr{F}$.

Table 4.1

$X=$	$\partial\Delta \times \partial I$	H & W	CG5V
$H_2(JX; Z_2)$	Z_2+Z_2	Z_2	Z_2
$H_2(KX; Z_2)$	Z_2	Z_2	Z_2
$H_2(JX; Z)$	$Z+Z$	Z	Z
$H_2(KX; Z)$	Z	0	0
$H^2(KX; Z)$	Z	Z_2	Z_2
$H^2(KX; Z_T)$	Z	Z	Z
$\chi(JX)$	0	-6	-10
$\chi(KX)$	0	-3	-5

If $X = \partial\Delta \times \partial I$, then JX has four components: two are simple closed curves, and two are tori. The projection $JX \to KX$ maps each simple closed curve two-fold upon a simple closed curve, and collapses the two tori onto one. If a triangulation other than the minimal one is used, the resulting JX and KX contain the minimal ones as deformation retracts. In any case, $H_2(JX)$ is a free Λ-module on two generators γ_1 and γ_2, and these generators may be taken to be orientation classes for the two tori. Let γ' be the projection of γ_1 into $H_2(KX)$, and note that γ_2 projects to $\pm\gamma'$. If σ is a 1-simplex in some triangulation of X, A is the complement of the interior of σ, and S is a simple closed curve opposite σ in X; then we clearly have

(4.2) $S \times \sigma$ and $\sigma \times S$ are relative 2-cycles on (JX, JA), and their homology classes are the images of γ_1 and γ_2 (up to signs) under the inclusion map $j: JX \to (JX, JA)$.

Let $X = \text{H \& W}$, with the minimal triangulation. If σ is a 1-simplex of X, then $C(\sigma, X)$ is a 4-sided simple closed curve. Thus every 1-cell $\sigma \times v$ (or $v \times \sigma$) lies in exactly two 2-cells of $\sigma \times C(\sigma, X)$ (respectively, $C(\sigma, X) \times \sigma$). It is easy to see that the link of each vertex of JX is a circle. In particular, eighteen of the links are octagons and twelve are dodecagons. Hence it is immediate that

$$JX = \bigcup_{\sigma} (\sigma \square C(\sigma, X))$$

is a connected 2-manifold. By counting its cells, we find the Euler-Poincaré characteristic of JX is -6. Since JX is a connected 2-fold covering of KX, it is an oriented manifold, a surface of genus 4. Its homology and cohomology are now easily found. In particular, $H_2(JX)$ is isomorphic to Λ, and an orientation class γ of JX may be taken to be the generator. If σ is a simplex in some (not necessarily minimal) triangulation of X, A is the complement of the interior of σ, and S is opposite σ; then we have

(4.3) $(S\times\sigma)-(\sigma\times S)$ is a relative 2-cycle on (JX, JA) and represents the image of $\pm\gamma$ under the inclusion map $j\colon JX \to (JX, JA)$.

The fundamental group $\pi_1(JX)$ has eight generators a_i, b_i $(i=1, 2, 3, 4)$ and the relation $\prod a_i b_i a_i^{-1} b_i^{-1} = 1$ $(i=1, 2, 3, 4)$. The space KX is a nonorientable surface of Euler characteristic $\chi(KX)=(1/2)\chi(JX)=-3$. Thus its integral homology groups are

$$H_0 \approx Z, \qquad H_1 \approx Z+Z+Z+Z+Z_2, \qquad H_2 = 0.$$

Since the orientation sheaf on KX is twisted, we have $H^i(KX; Z_T)=H_{2-i}(KX; Z)$ [2, p. 138]. Note that the van Kampen cocycle $m^2(X)$ generates $H^2(KX; Z_T)$ (see [1]). The remaining homology and cohomology of KX is obtained using universal coefficient theorems.

The discussion of $X=$CG5V is quite similar to the above. The Euler-Poincaré characteristics are $\chi(JX)=-10$ and $\chi(KX)=-5$. Thus JX is a surface of genus 6, while KX is nonorientable and has $H_1(KX; Z)\approx Z+Z+Z+Z+Z+Z+Z_2$. Again, $H^2(KX; Z_T)\approx Z$ is generated by the van Kampen cocycle. If γ is an orientation class in $H_2(JX)\approx\Lambda$ and σ, A, and S are chosen as above, then

(4.4) $(S\times\sigma)-(\sigma\times S)$ represents the image of $\pm\gamma$ under the inclusion map $j\colon JX\to(JX, JA)$.

5. **Lower bounds.** Suppose X is a finitely triangulated one-dimensional space, and (X, A) is a decomposition of length two for X. Let σ be the 1-simplex such that $X=\sigma\cup A$. Define $W(X, A)$ to be the submodule of $H_1(A)$ generated by simple closed curves S in A such that S is opposite σ in some $F\in\mathscr{F}$ lying in X. Let $W_0(X, A)$ be similarly generated by S opposite σ in $F\in\mathscr{F}_0$. Set

$$D(X, A) = \operatorname{rank} W(X, A) \quad\text{and}\quad E(X, A) = \operatorname{rank} W_0(X, A).$$

Since S is opposite σ, it must be disjoint from σ, whence $S\subset C(\sigma, A)$. It is frequently convenient to regard $W(X, A)$ and $W_0(X, A)$ as submodules of $H_1(C(\sigma, A))$. Note that D and E are independent of the ring Λ. Also $D(X, A)=E(X, A)=0$ unless the ends of σ are joined by a path in A. If $(X_n, X_{n-1}, \ldots, X_0)$ is any decomposition of X, set

$$D(X_n, X_{n-1}, \ldots, X_0) = \sum_{i=1}^{n} D(X_i, X_{i-1})$$

and

$$E(X_n, X_{n-1}, \ldots, X_0) = \sum_{i=1}^{n} E(X_i, X_{i-1}).$$

THEOREM 5.1. *If X is a finitely triangulated one-dimensional space and A is a subspace such that (X, A) is a decomposition, then*

$$\beta_2(JX) \geqq \beta_2(JA)+D(X, A)+E(X, A),$$
$$\beta_2(KX; Z_2) \geqq \beta_2(KA; Z_2)+D(X, A),$$

and

$$\beta_2(KX; Z) \geqq \beta_2(KA; Z)+E(X, A).$$

Proof. Let σ be the 1-simplex of X such that $A \cup \sigma = X$. The result is trivial unless the ends of σ are joined by an arc in A, so we will immediately make this additional assumption. Then $\sigma \cap A = \partial\sigma$, whence the inclusion map

$$k: (C \square \sigma, C \square \partial\sigma) \to (JX, JA)$$

is an excision when $C = C(\sigma, A)$. The homomorphism

$$\theta': H_1(C)+H_1(C) \to H_2(C \square \sigma, C \square \partial\sigma),$$

given by assigning to $\theta'(z_1, z_2)$ the relative 2-cycle $[(z_1+z_2)\times\sigma]-[\sigma\times z_1]$ when z_1, z_2 are 1-cycles on C, is easily seen to be an isomorphism. Let

$$\theta = k_* \circ (\theta' \mid W(X, A)+W_0(X, A))$$

and note that θ is injective. On the other hand, we have the exact sequence $0 \to H_2(JA) \xrightarrow{i_*} H_2(JX) \xrightarrow{j_*} H_2(JX, JA)$. The portion of the theorem that concerns JX will be proved as soon as we show that image $\theta \subset$ image j_*. Let S be a simple closed curve lying opposite σ in some $F \in \mathscr{F}$. Then it follows from (4.2), (4.3), and (4.4) that $(S\times\sigma)-(\sigma\times S)$ is in the image of j_*. If, in addition, $F \in \mathscr{F}_0$, then (4.2) shows that $S\times\sigma \in$ image j_*. Thus θ maps the generators of $W(X, A)+W_0(X, A)$ into image j_*, and the result follows.

Next, let $\Lambda = Z_2$. Define φ: $W(X, A) \to H_2(KX, KA)$ to be the homomorphism sending the 1-cycle z into the relative equivariant 2-cycle $(z\times\sigma)+(\sigma\times z)$. As before, one easily verifies that φ is injective and that image $\varphi \subset$ image j_* when j_*: $H_2(KX) \to H_2(KX, KA)$ is induced by inclusion.

Finally, let $\Lambda = Z$ and define ψ: $W_0(X, A) \to H_2(KX, KA)$ to be the homomorphism sending the 1-cycle z into the relative equivariant 2-cycle $(z\times\sigma)-(\sigma\times z)$. As before, one easily verifies that ψ is injective and image $\psi \subset$ image j_*.

COROLLARY 5.2. *If* $(X_n, X_{n-1}, \ldots, X_0)$ *is a decomposition, then*

$$\beta_2(JX_n) \geqq \beta_2(JX_0)+D(X_n, X_{n-1}, \ldots, X_0)+E(X_n, X_{n-1}, \ldots, X_0)$$

for any coefficient ring Λ,

$$\beta_2(KX_n; Z_2) \geqq \beta_2(KX_0; Z_2)+D(X_n, X_{n-1}, \ldots, X_0),$$

and

$$\beta_2(KX_n; Z) \geqq \beta_2(KX_0; Z)+E(X_n, X_{n-1}, \ldots, X_0).$$

COROLLARY 5.3. *If* $(X_n, X_{n-1}, \ldots, X_0)$ *is a complete decomposition, then*

$$\beta_2(JX_n) \geqq D(X_n, X_{n-1}, \ldots, X_0)+E(X_n, X_{n-1}, \ldots, X_0),$$
$$\beta_2(KX_n; Z_2) \geqq D(X_n, X_{n-1}, \ldots, X_0),$$

and

$$\beta_2(KX_n; Z) \geqq E(X_n, X_{n-1}, \ldots, X_0).$$

The lower bounds on $\beta_2(KX; Z_2)$ given in Corollary 5.3 have turned out to be the exact values in all of the examples we tested, and we conjecture that $\beta_2(KX; Z_2) = D(X_n, X_{n-1}, \ldots, X_0)$ always holds. On the other hand, it may happen that $\beta_2(JX) > (D+E)(X_n, X_{n-1}, \ldots, X_0)$. For example, let $X = X_{13}$ be the space with the eight vertices $v_1, v_2, \ldots, v_8$ and the thirteen 1-simplexes $\langle v_i, v_j\rangle$ ($i=1, 2$; $j=3, 4, 5, 6$), $\langle v_3, v_7\rangle$, $\langle v_4, v_7\rangle$, $\langle v_5, v_8\rangle$, $\langle v_6, v_8\rangle$, and $\langle v_7, v_8\rangle$ (see Figure 5.4). The subspace X_{12} is obtained by omitting the simplex $\langle v_7, v_8\rangle$ from X. Let $(X_{12}, X_{11}, \ldots, X_0)$ be any decomposition of X_{12}. Then

$$\beta_2(JX_{12}) = (D+E)(X_{12}, X_{11}, \ldots, X_0) = 4,$$
$$\beta_2(JX_{13}) = 7, \quad \text{and} \quad (D+E)(X_{13}, X_{12}, \ldots, X_0) = 6.$$

A slight modification of this example reveals an essential feature of such failures. Let $X_{14} = X_{13} \cup \langle v_4, v_5\rangle$. Then $(D+E)(X_{14}, X_{13}) = 4$, whence

$$(D+E)(X_{14}, X_{13}, \ldots, X_0) = 10.$$

On the other hand, if $Y_i = X_i$ for $i=0, 1, \ldots, 12, 14$ and $Y_{13} = Y_{12} \cup \langle v_4, v_5\rangle$, then $(D+E)(Y_{14}, Y_{13}, \ldots, Y_0) = 11$. The thrust of this second example is that $D+E$ depends on the decomposition used and is not an invariant of the space. Notice that E is not invariant, either. Similar examples may be constructed using the spaces sketched in Figure 5.5.

6. **Complete graphs.** Let A_k be the complete graph with vertices $v_1, v_2, \ldots, v_k$. That is, A_k is the 1-skeleton of the $(k-1)$-simplex spanned by the independent points $v_1, v_2, \ldots, v_k$. In this section, we study the functions B, D, and E on suitable decompositions of A_k in order to obtain estimates of the accuracy with which they reflect the values of β_2.

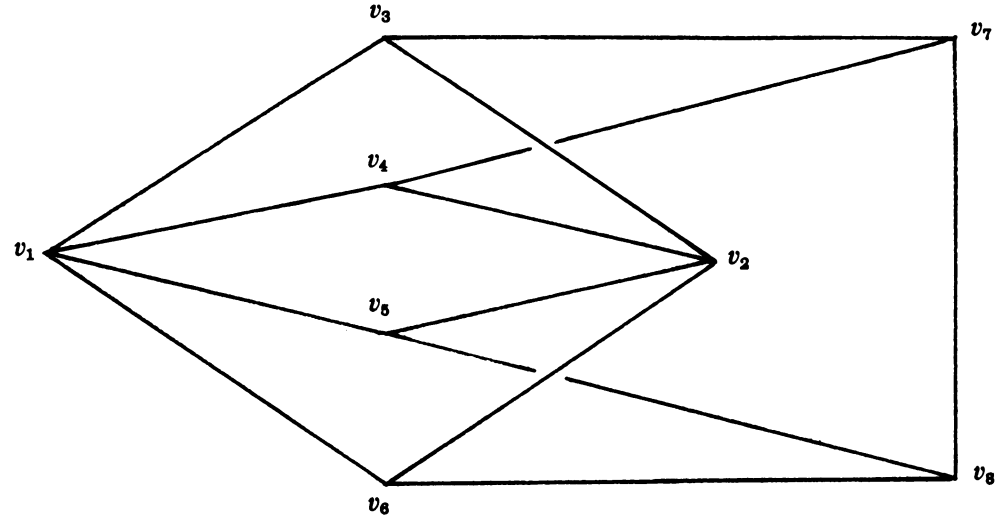

FIGURE 5.4

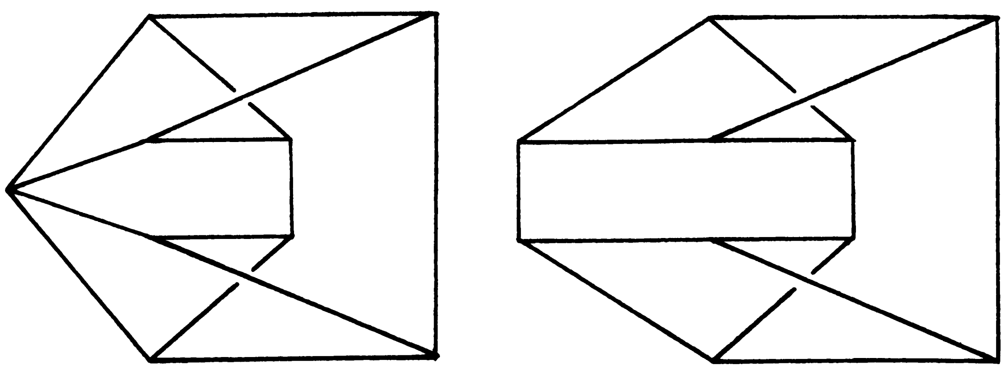

FIGURE 5.5

LEMMA 6.1. *The Euler-Poincaré characteristics of A_k, JA_k, and KA_k are*

$$\chi A_k = 1-(k-1)(k-2)/2, \chi JA_k = k(k-1)(k-3)(k-6)/4,$$

and

$$\chi KA_k = k(k-1)(k-3)(k-6)/8.$$

Proof. Now A_k has k vertices and $(k-1)k/2$ 1-simplices. The difference is χA_k. The 0-cells of JA_k are of the form $\langle v_i \rangle \times \langle v_j \rangle$ with $i \neq j$; there are $k(k-1)$ of these. Each vertex v_i has order $k-1$, and hence there are $k(k(k-1)/2-k+1)$ 1-cells of the form $\langle v_i \rangle \times \langle v_j, v_k \rangle$, and the same number of the form $\langle v_j, v_k \rangle \times \langle v_i \rangle$. Each 1-simplex $\langle v_i, v_j \rangle$ meets $2k-3$ of the 1-simplices of A_k (including itself). Thus JA_k has $(k(k-1)/2)(k(k-1)/2-2k+3)$ cells of dimension 2. A little arithmetic gives the value of χJA_k. Since KA_k has exactly half as many cells in each dimension as JA_k, $\chi KA_k = \chi JA_k/2$.

THEOREM 6.2. *If $k \geqq 2$, there is a complete decomposition $(X_n, X_{n-1}, \ldots, X_0)$ of A_k with*

$$B(X_n, X_{n-1}, \ldots, X_0) = (k-1)(k-2)(k-3)(k-4)/8.$$

For $k \geqq 5$, we have the following:

$$D(X_n, X_{n-1}, \ldots, X_0) = (k^2-5k+2)(k-1)(k-4)/8.$$

$H_2(JA_k; Z)$ is free abelian with rank=

$$(D+E)(X_n, X_{n-1}, \ldots, X_0) = 2D(X_n, X_{n-1}, \ldots, X_0)-1.$$

$H_2(KA_k; Z)$ is free abelian with rank=

$$E(X_n, X_{n-1}, \ldots, X_0) = D(X_n, X_{n-1}, \ldots, X_0)-1.$$

$H_2(KA_k; Z_T)$ is free abelian, and

$$\text{rank } H_2(KA_k; Z_T) = \text{rank } H_2(KA_k; Z_2) = D(X_n, X_{n-1}, \ldots, X_0).$$

Proof. A complete decomposition of A_2 is (A_2, X_0), where X_0 is the 1-point space. Suppose a complete decomposition $(X_m, X_{m-1}, \ldots, X_0)$ of A_{k-1}, $k \geqq 3$, has been defined; note that $m=(k-1)(k-2)/2$. Let σ_i be the 1-simplex $\sigma_i=\langle v_i, v_k\rangle$ for $i=1, 2, \ldots, k-1$, and set $X_{m+i}=X_{m+i-1} \cup \sigma_i$. In the following statements, assume $k>5$, except when the statement concerns B. The homology groups with unspecified coefficients are to be interpreted as follows: $H_*(JX)$ has coefficients in Z or Z_2; $H_*(KX)$ has coefficients in Z. Let $C_i=C(\sigma_i, X_{m+i-1})$. Note that C_i is an isomorph of A_{k-2} and hence has $(k-3)(k-4)/2$ independent 1-cycles. Since σ_1 is attached to X_m at only one of its ends, v_1, we have

$$H_2(JX_{m+1}, JX_m) = H_2(C_1 \square \sigma_1, C_1 \square v_1) = 0.$$

Thus $H_2(JX_m)$ is isomorphic to $H_2(JX_{m+1})$, and $H_2(KX_m)$ to $H_2(KX_{m+1})$, both under inclusion, whence $B(X_{m+1}, X_m)=D(X_{m+1}, X_m)=0$.

For $i>1$, σ_i is attached to X_{m+i-1} at both ends. Thus $B(X_{m+i}, X_{m+i-1})=(k-3)(k-4)/2$ and $B(X_n, X_{n-1}, \ldots, X_m)=(k-2)(k-3)(k-4)/2$. Every arc in X_{m+1} from v_2 to v_k contains v_1. Thus a simple closed curve lying opposite σ_2 in a figure F of type $\partial\Delta \times \partial I$ in X_{m+2} must lie in $C'=C(v_1, C_2)$. But C' is a complete graph on $k-3$ vertices. Conversely, every simple closed curve in C' is opposite σ_2 in some $F \in \mathscr{F}_0$. This shows $D(X_{m+2}, X_{m+1}) \geqq (k-4)(k-5)/2$. A basis for $H_1(C_2)$ may be obtained by taking a basis for $H_1(C')$ and adding to it the $k-4$ cycles

$$z_r = \langle v_1, v_r\rangle+\langle v_r, v_{r+1}\rangle-\langle v_1, v_{r+1}\rangle, \qquad 3 \leqq r \leqq k-2.$$

The image of

$$j_*: H_2(JX_{m+2}) \to H_2(JX_{m+2}, JX_{m+1})$$

contains all the relative 2-cycles $z \times \sigma_2$ and $\sigma_2 \times z$ with z a cycle on C'. On the other hand,

$$\partial_*: H_2(JX_{m+2}, JX_{m+1}) \to H_1(JX_{m+1})$$

maps each cycle

$$z = \sum_r [\alpha_r(z_r \times \sigma_2) + \beta_r(\sigma_2 \times z_r)]$$

onto

$$\partial z = \sum_r [\alpha_r(z_r \times (v_k - v_2)) + \beta_r((v_k - v_2) \times z_r)].$$

Suppose there is a 2-chain c on JX_{m+1} having $\partial c = \partial z$. The coefficient of $\langle v_1\, v_r\rangle \times \langle v_k\rangle$ in ∂z is α_3, if $r=3$, and $\alpha_r - \alpha_{r-1}$, if $3 < r \leqq k-2$. The only 1-simplex in X_{m+1} containing v_k is $\langle v_1, v_k\rangle$. Thus no 2-cell in JX_{m+1} has $\langle v_1, v_r\rangle \times v_k$ on its boundary. This shows $\alpha_r = 0$ for all r. Similarly, each $\beta_r = 0$, whence $z=0$. From this we conclude that rank (image $\partial_*) \geqq 2k-8$, whence $(k-3)(k-4)-(2k-8) = (k-4)(k-5) \geqq \operatorname{rank} j_*$. But $\operatorname{rank} j_* \geqq 2D(X_{m+2}, X_{m+1}) \geqq (k-4)(k-5)$, so equality holds. This shows

$$\text{rank } H_2(JX_{m+2}) = \text{rank } H_2\,(JX_{m+1}) + 2D(X_{m+2}, X_{m+1}).$$

Similarly,

$$\text{rank } H_2(KX_{m+2}) = \text{rank } H_2(KX_{m+1}) + D(X_{m+2}, X_{m+1}).$$

Any simple closed curve in C_3 that does not contain $\langle v_1, v_2\rangle$ lies opposite σ_3 in some F of type $\partial\Delta \times \partial I$. Conversely, if z is a cycle in C_3 containing $\langle v_1, v_2\rangle$ with nonzero coefficient, then $\partial(z \times \sigma_3)$ contains $\langle v_1, v_2\rangle \times \langle v_k\rangle$ with nonzero coefficient. Since this 1-cell does not lie on the face of any 2-cell in JX_{m+2}, we can argue as above and conclude that

$$\begin{aligned} D(X_{m+3}, X_{m+2}) &= (k-3)(k-4)/2 - 1, \\ \text{rank } H_2(JX_{m+3}) &= \text{rank } H_2(JX_{m+2}) + 2D(X_{m+3}, X_{m+2}), \\ \text{rank } H_2(KX_{m+3}) &= \text{rank } H_2(KX_{m+2}) + D(X_{m+3}, X_{m+2}). \end{aligned}$$

For $r > 3$, the matter is still simpler: $W(X_{m+r}, X_{m+r-1}) = H_1(C_r)$, whence

$$\begin{aligned} D(X_{m+r}, X_{m+r-1}) &= (k-3)(k-4)/2, \\ \text{rank } H_2(JX_{m+r}) &= \text{rank } H_2(JX_{m+r-1}) + 2D(X_{m+r}, X_{m+r-1}), \\ \text{rank } H_2(KX_{m+r}) &= \text{rank } H_2(KX_{m+r-1}) + D(X_{m+r}, X_{m+r-1}). \end{aligned}$$

Combining these results, we get

$$\begin{aligned} \text{rank } H_2(KA_k) - \text{rank } H_2(KA_{k-1}) &= (\text{rank } H_2(JA_k) - \text{rank } H_2(JA_{k-1}))/2 \\ &= \sum_{r=1}^{k-1} (\text{rank } H_2(KX_{m+r}) - \text{rank } H_2(KX_{m+r-1})) \\ &= \sum_{r=1}^{k-1} D(X_{m+r}, X_{m+r-1}) \\ &= (k-4)(k-4)(k-1)/2 - 1. \end{aligned}$$

Summing again, we obtain

$$\begin{aligned}\operatorname{rank} H_2(KA_k)-\operatorname{rank} H_2(KA_5) &= (\operatorname{rank} H_2(JA_k)-\operatorname{rank} H_2(JA_5))/2\\ &= \sum_{j=6}^{k} (\operatorname{rank} H_2(KA_j)-\operatorname{rank} H_2(KA_{j-1}))\\ &= \sum_{i=11}^{n} D(X_i, X_{i-1})\\ &= (k^2-5k+2)(k-4)(k-1)/8-1.\end{aligned}$$

The proof is now completed by recalling, from §4, that

$$\begin{aligned}\operatorname{rank} H_2(JA_5; Z) &= \operatorname{rank} H_2(JA_5; Z_2) = \operatorname{rank} H_2(KA_5; Z_2)\\ &= \operatorname{rank} H_2(KA_5; Z_T) = 1\\ &= D(X_{10}, X_9, \ldots, X_0)\\ &= E(X_{10}, X_9, \ldots, X_0)+1,\end{aligned}$$

and

$$\operatorname{rank} H_2(KA_5, Z) = 0 = D(X_{10}, X_9, \ldots, X_0)-1.$$

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