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ON THE SYMMETRIC CUBE OF A SPHERE

BY
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1. **Introduction.** Let X^m denote the cartesian product $X \times \cdots \times X$ (m factors) of the based space X. The full symmetric group $S(m)$ acts on X^m by permutation homeomorphisms and the quotient space $X^m/S(m)$ is defined to be the m-fold symmetric product $SP^m X$ of X.

Now let $X = S^n$, the n-sphere. A map $f\colon SP^m S^n \to S^n$ is *of type r* if the composite

$$S^n \xrightarrow{i} SP^m S^n \xrightarrow{f} S^n$$

has degree r. Here $i(x) = [x, e, \ldots, e]$ where e is the basepoint of S^n. For given n and m an elementary result of James asserts that the set of all "realizable types" is an ideal $(k^{m,n}) \subset Z$ in the ring of integers—whence the problem of determining the generator $k^{m,n}$. The main results of [1] and [6] determine $k^{2,n}$:

THEOREM 1.1. (i) $k^{2,2t} = 0$;
(ii) $k^{2,2t+1} = 2^{\varphi(2t)}$ *where* $\varphi(b)$ *is the number of integers* $0 < a \leqq b$ *with* $a \equiv 0, 1, 2$ *or* 4 mod 8.

In [7] we obtained a lower bound for $k^{m,n}$ for all m. Our main result in this paper asserts that this lower bound is best possible when $m = 3$, i.e. $k^{3,n}$ is determined as follows:

THEOREM 1.2. (i) $k^{3,2t} = 0$; (ii) $k^{3,2t+1} = 2^{\varphi(2t)} \cdot 3^t$.

1.2(i) is of course a simple consequence of 1.1(i). Moreover the main result of [7] implies that $2^{\varphi(2t)} \cdot 3^t | k^{3,2t+1}$. Thus 1.2(ii) will be proved by constructing a map $f\colon SP^3 S^n \to S^n$, $n = 2t+1$, of type $r = 2^{\varphi(2t)} \cdot 3^t$. For this Toda's notion of the suspension-order of a space X—the least positive integer s such that $s\iota_{EX} = 0$ in $[EX, EX]$—is fundamental (see §4).

Recall then Toda's result [6] which is an important step in the proof of 1.1(ii). Let $X^n_{2,1}$ be the quotient space $X/S(2)$ where X is the $S(2)$-invariant subspace $D^n \times S^{n-1} \cup S^{n-1} \times D^n$ of $(D^n)^2$. Then $X^n_{2,1}$ has the homotopy type of a suspension space (see Corollary 2.7 or [1, Lemma 2.1]) so that we may consider the "desuspension" $E^{-1}X^n_{2,1}$ of $X^n_{2,1}$. Toda's result asserts

THEOREM 1.3. (i) *The suspension-order of* $E^{-1}X^{2t}_{2,1}$ *is infinite.*
(ii) *The suspension-order of* $E^{-1}X^{2t-1}_{2,1}$ *is* $2^{\varphi(2t)}$.

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In a similar way 1.2(ii) depends on a suspension-order calculation. Let $X^n_{3,2}$ be the quotient space $Y/S(3)$ where Y is the $S(3)$-invariant subspace $D^n \times S^{n-1} \times S^{n-1} \cup S^{n-1} \times D^n \times S^{n-1} \cup S^{n-1} \times S^{n-1} \times D^n$ of $(D^n)^3$. The standard embedding $(D^n)^2 \to (D^n)^3$, $(x_1, x_2) \to (x_1, x_2, e)$, e the basepoint of D^n, induces an embedding $X^n_{2,1} \to X^n_{3,2}$. The quotient space $X^n_{3,2}/X^n_{2,1}$ has the homotopy type of a suspension space (see Proposition 2.6). Then Toda's methods can be applied to prove

THEOREM 1.4. (i) *The suspension-order of* $E^{-1}(X^{2t}_{3,2}/X^{2t}_{2,1})$ *is infinite.*
(ii) *The suspension-order of* $E^{-1}(X^{2t+1}_{3,2}/X^{2t+1}_{2,1})$ *is* 3^t.

Apart from some cohomological calculations our procedure is that of §4 of [1]—namely killing a certain attaching map by composition with a suitable map. In §1 we develop some geometry of SP^mS^n. In particular we study certain desuspension properties of related subcomplexes. §2 contains a lemma sharpening our focus on exactly which map should be killed. 1.4 and 1.3 are then proved in §3 and §4, respectively. The latter makes use of the mod 3 Hopf invariant.

Given a space Y many properties of its suspension EY, e.g. the group structure on $[EY, Z]$, the suspension-order of Y (which is a property defined in terms of EY), etc. admit definitions for any space X of the same homotopy type as EY. Thus we often speak of these properties for $E^{-1}X$ when we really mean the corresponding properties for Y (or EY), even though we don't know if X is actually desuspendable.

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2. **Geometry of SP^mS^n.** Let D^n be the unit n-ball in R^n and $S^{n-1} = \partial D^n$ its boundary $(n-1)$-sphere. Let $h_\tau: (D^n)^m \to (D^n)^m$ be the permutation homeomorphism corresponding to a given element $\tau \in S(m)$. Setting $A^n_{m,l} = (D^n)^{m-l} \times (S^{n-1})^l$, $0 \leq l \leq m$, we obtain an $S(m)$-invariant subspace of $A^n_{m,0} = (D^n)^m$ by

$$\tilde{X}^n_{m,l} = \bigcup_{\tau \in S(m)} h_\tau(A^n_{m,l}).$$

Hence we may take its orbit space under the $S(m)$-action and obtain, $X^n_{m,l} = \tilde{X}^n_{m,l}/S(m)$.

We give $X^n_{m,l}$ the quotient topology. The embedding $A^n_{m',l'} \to A^n_{m,l}$, $m' \leq m$ and $m'-l' \leq m-l$, given by $(x_1, \ldots, x_{m'}) \to (x_1, \ldots, x_{m'}, e, \ldots, e)$ induces an embedding $X^n_{m',l'} \to X^n_{m,l}$. $X^n_{m,0}$ and $X^n_{m,m}$ are the familiar symmetric products SP^mD^n and SP^mS^{n-1}, respectively. And $X^n_{2,1}$ is the "symmetric join" of S^{n-1} with itself defined in [1].

LEMMA 2.1. *$X^n_{m,0}$ is homeomorphic to the cone*

$$CX^n_{m,1} = X^n_{m,1} \times I / X^n_{m,1} \times \{0\}.$$

Proof. The map $X^n_{m,1}\times I\to X^n_{m,0}$ given by $([x],t)=([x_1,\ldots,x_m],t)\to[tx]=[tx_1,\ldots,tx_n]$ induces a topological map

$$CX^n_{m,1}\xrightarrow{\alpha}X^n_{m,0} \tag{1}$$

which sends $X^n_{m,1}\times\{1\}$ onto the subspace $X^n_{m,1}\subset X^n_{m,0}$.

REMARK. All cones CX in this paper will be "inverted", i.e. $CX=X\times I/X\times\{0\}$. As we define the suspension EX to to be the quotient $CX/X=CX/X\times\{1\}$, our suspensions will also be inverted.

Choose a relative homeomorphism $h\colon(D^n,S^{n-1})\to(S^n,e)$. Then the induced map

$$h_{S(m)}\colon(X^n_{m,0},X^n_{m,1})\to(X^{n+1}_{m,m},X^{n+1}_{m-1,m-1}) \tag{2}$$

also is a relative homeomorphism and so from Lemma 2.1 we have

LEMMA 2.2. *$X^{n+1}_{m,m}$ is homeomorphic to the adjunction space $X^{n+1}_{m-1,m-1}\cup CX^n_{m,1}$.*

Here the attaching map is given by the restriction of $h_{S(m)}$ to $X^n_{m,1}$.

We define the join $X\circ Y$ of X and Y as the space $X\times CY\cup CX\times Y$. The homeomorphism $c\colon C(X\circ Y)\to CX\times CY$ given by

$$\begin{aligned} c[(x,[y,t]),u] &= [[x,u],[y,tu]] \quad \text{if } (x,[y,t])\in X\times CY,\\ c[([x,t],y),u] &= [[x,tu],[y,u]] \quad \text{if } ([x,t],y)\in CX\times Y \end{aligned} \tag{3}$$

induces a homeomorphism

$$E(X\circ Y)=\frac{C(X\circ Y)}{X\circ Y}\xrightarrow{\hat c}\frac{CX\times CY}{X\circ Y}.$$

Consider the shuffle map $\beta\colon X^n_{k,l}\times X^n_{k',l'}\to X^n_{k+k',l+l'}$ defined by

$$\beta([x_1,\ldots,x_k],[y_1,\ldots,y_{k'}])=[x_1,\ldots,x_k,y_1,\ldots,y_{k'}].$$

Using the homeomorphism (1) we obtain a map

$$\psi\colon X^n_{k,1}\circ X^n_{k',1}\to X^n_{k+k',1} \tag{4}$$

by setting $\psi|X^n_{k,1}\times CX^n_{k',1}=\beta\circ(\mathrm{id}\times\alpha)$ and $\psi|CX^n_{k,1}\times X^n_{k',1}=\beta\circ(\alpha\times\mathrm{id})$. The maps α, β, ψ and c satisfy the commutative diagram

$$\begin{array}{ccc} C(X^n_{k,1}\circ X^n_{k',1}) & \xrightarrow{C(\psi)} & CX^n_{k+k',1}\\ \downarrow{\scriptstyle c} & & \\ CX^n_{k,1}\times CX^n_{k',1} & & \downarrow{\scriptstyle\alpha}\\ \downarrow{\scriptstyle\alpha\times\alpha} & & \\ X^n_{k,0}\times X^n_{k',0} & \xrightarrow{\beta} & X^n_{k+k',0} \end{array}$$

Here $C(\psi)$ denotes the conal extension of the map ψ. The verification of the commutativity is straightforward: for $(x, [y, t]) \in X^n_{k,1} \times CX^n_{k',1}$, $x=[x_1, \dots, x_k]$, $y=[y, \dots, y_{k'}]$ we have

$$\alpha \circ C(\psi)[(x, [y, t]), u] = \alpha[\psi(x, [y, t]), u] = \alpha[\beta(x, ty), u] = u\beta(x, ty) = [ux, tuy],$$
$$\beta \circ (\alpha \times \alpha) \circ c[(x, [y, t]), u] = \beta \circ (\alpha \times \alpha)[[x, u], [y, tu]] = \beta(ux, tuy) = [ux, tuy].$$

Similarly for $([x, t], y) \in CX^n_{k,1} \times X^n_{k',1}$. As an immediate consequence we have

Corollary 2.3. *The shuffle map β: $X^n_{m,0} \times X^n_{m',0} \to X^n_{m+m',0}$ induces the suspension map*

$$E(X^n_{m,1} \circ X^n_{m',1}) \xrightarrow{E\psi} X^n_{m+m',0}/X^n_{m+m',1} \cong EX^n_{m+m',1}.$$

Now consider the surjective map $X^n_{m-l,0} \times X^{n-1}_{l,0} \xrightarrow{\alpha} X^n_{m,l}$, $1 \leq l < m$, defined by the composite

$$X^n_{m-l,0} \times X^{n-1}_{l,0} \xrightarrow{\mathrm{id} \times h_{S(l)}} X^n_{m-l,0} \times X^n_{l,l} \xrightarrow{\beta} X^n_{m,l}$$

where $h_{S(l)}$ is given in (2) and β is the shuffle map. It is easy to check that the boundary of $X^n_{m-l,0} \times X^{n-1}_{l,0}$

$$\partial(X^n_{m-l,0} \times X^{n-1}_{l,0}) \cong \partial(CX^n_{m-l,1} \times CX^{n-1}_{l,1}) = X^n_{m-l,1} \circ X^{n-1}_{l,1}$$

is mapped by d onto the subspace $X^n_{m,l+1} \cup X^n_{m-1,l-1}$ and that d defines a relative homeomorphism of the pairs

$$(X^n_{m-l,0} \times X^{n-1}_{l,0}, X^n_{m-l,1} \circ X^{n-1}_{l,1}) \quad \text{and} \quad (X^n_{m,l}, X^n_{m,l+1} \cup X^n_{m-1,l-1}).$$

And so dc defines a relative homeomorphism of the pairs

$$(C(X^n_{m-l,1} \circ X^{n-1}_{l,1}), X^n_{m-l,1} \circ X^{n-1}_{l,1}) \quad \text{and} \quad (X^n_{m,l}, X^n_{m,l+1} \cup X^n_{m-1,l-1}).$$

Note $c|(X^n_{m-l,1} \circ X^{n-1}_{l,1})$ is the identity map.

Similarly β: $X^n_{m-l,0} \times X^n_{l,l} \to X^n_{m,l}$ itself is a surjective map which defines a relative homeomorphism of the pairs

$$(X^n_{m-l,0} \times X^n_{l,l}, X^n_{m-l,1} \times X^n_{l,l}) \quad \text{and} \quad (X^n_{m,l}, X^n_{m,l+1}).$$

Summarizing these observations we have

Lemma 2.4. *Let $1 \leq l < m$. Then*

(i) *$X^n_{m,l}$ is homeomorphic to the adjunction space $(X^n_{m,l+1} \cup X^n_{m-1,l-1}) \cup C(X^n_{m-l,1} \circ X^{n-1}_{l,1})$ with attaching map given by the restriction of dc (and hence of d) to $X^n_{m-l,1} \circ X^{n-1}_{l,1}$.*

(ii) *$X^n_{m,l}$ is homeomorphic to the adjunction space $X^n_{m,l+1} \cup CX^n_{m-l,1} \times X^n_{l,l}$ with attaching map given by the restriction of $\beta(\alpha \times \mathrm{id})$ (and hence of β) to $X^n_{m-l,1} \times X^n_{l,l}$.*

As a consequence of Lemma 2.4 we have

LEMMA 2.5. *Let* $1 \leqq l < m$. *Then*

(i) $X^n_{m,l}/(X^n_{m,l+1} \cup X^n_{m-1,l-1})$ *and* $EX^n_{m-l,1} \wedge EX^{n-1}_{l,1}$ *are homeomorphic.*

(ii) $X^n_{m,l}/X^n_{m,l+1}$ *and* $EX^n_{m-l,1} \vee E(X^n_{m-l,1} \wedge X^n_{l,l})$ *have the same homotopy type.*

(iii) $X^n_{m,l}/X^n_{m-1,l-1}$ *is homeomorphic to the adjunction space* $X^n_{m,l+1}/X^n_{m-1,l} \cup C(X^n_{m-l,1} \circ X^{n-1}_{l,1})$ *with attaching map given by the composition of the attaching map in* 2.4(i) *and the obvious collapsing map.*

Proof. (i) follows directly from Lemma 2.4(i) and the easy observation that $(C(X \circ Y), X \circ Y) \cong (CX \times CY, X \circ Y)$ and $(EX \times EY, EX \vee EY)$ are relatively homeomorphic. Similarly (ii) comes from Lemma 2.4(ii) and the facts (1) that $(CX \times Y, X \times Y)$ and $(EX \times Y$, point $\times Y)$ are relatively homeomorphic and (2) that the quotient $EX \times Y/\text{point} \times Y$ has the same homotopy type as $EX \vee E(X \wedge Y)$. Finally (iii) follows from Lemma 2.4(i) and the simple observation that $X^n_{m,l+1} \cap X^n_{m-1,l-1} = X^n_{m-1,l}$.

With these preliminaries over we can proceed to consider the desuspension properties of the above and related spaces. Our approach will be to "desuspend up to homotopy equivalence" certain of the above attaching maps.

First we recall some elementary homotopy theory. Suppose X and X' are 1-connected and A and A' are 0-connected finite CW complexes and

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A' & \xrightarrow[\ f'\ ]{} & X' \end{array}$$

is a commutative diagram of maps with p and q homotopy equivalences. Then the adjunction spaces $X \cup_f CA$ and $X' \cup_{f'} CA'$ have the same homotopy type. An explicit homotopy equivalence

$$F\colon X \cup_f CA \to X' \cup_{f'} CA'$$

is given by $F|X = q$ and $F|CA = C(p)$ the conal extension of p. We call F the conal homotopy equivalence determined by p and q. Our interest lies in the situation $A' = EA''$, $X' = EX''$ suspension spaces and $f' = Ef''$ a suspension map. Then $X \cup_f CA$ will have the homotopy type of the suspension space $EX'' \cup_{Ef''} C(EA'') \cong E(X'' \cup_{f''} CA'')$.

We now apply these remarks to the space $X^n_{m,m-1}/X^n_{m-1,m-2}$. According to Lemma 2.5(iii) the attaching map in the adjunction space

$$X^n_{m,m}/X^n_{m-1,m-1} \cup C(X^n_{1,1} \circ X^{n-1}_{m-1,1})$$

is the composite

$$X^n_{1,1} \circ X^{n-1}_{m-1,1} \to X^n_{m,m} \cup X^n_{m-1,m-2} \to \frac{X^n_{m,m} \cup X^n_{m-1,m-2}}{X^n_{m-1,m-2}} \cong \frac{X^n_{m,m}}{X^n_{m-1,m-1}}.$$

The first map on $X^n_{1,1}\times CX^{n-1}_{m-1,1}$ is $\beta\circ(\mathrm{id}\times(h_{S(m-1)}\circ\alpha))$ and on $CX^n_{1,1}\times X^{n-1}_{m-1,1}$ is $\beta\circ(\alpha\times h_{S(m-1)})$ where β: $X^n_{1,0}\times X^n_{m-2,m-2}\to X^n_{m-1,m-2}$ is the shuffle map. A more familiar description of this attaching map via the identifications $X^n_{1,1}=S^{n-1}$, $X^n_{m,m}/X^n_{m-1,m-1}=SP^mS^{n-1}/SP^{m-1}S^{n-1}$ is given by

$$S^{n-1}\times CX^{n-1}_{m-1,1}\to S^{n-1}\times SP^{m-1}S^{n-1}\to SP^mS^{n-1}\to SP^mS^{n-1}/SP^{m-1}S^{n-1},$$
$$CS^{n-1}\times X^{n-1}_{m-1,1}\to \text{basepoint}.$$

As point $\times CX^{n-1}_{m-1,1}$ is also mapped to the basepoint, this attaching map factors as

$$\begin{array}{ccc} S^{n-1}\circ X^{n-1}_{m-1,1} & \longrightarrow & SP^mS^{n-1} \\ \downarrow & & \downarrow \\ S^{n-1}\circ X^{n-1}_{m-1,1}/A & \longrightarrow & SP^mS^{n-1}/SP^{m-1}S^{n-1} \end{array} \tag{5}$$

where $A=\text{point}\times CX^{n-1}_{m-1,1}\cup CS^{n-1}\times X^{n-1}_{m-1,1}$ is contractible (in itself). Thus the collapsing map p is a homotopy equivalence. Moreover, setting $S=S^{n-1}$ and $X=X^{n-1}_{m-1,1}$ we can impose a suspension structure on $S\circ X/A$ by the following (obvious) homeomorphisms:

$$\begin{aligned} \frac{S\circ X}{A} &\cong \frac{S\times CX}{S\times X\cup \text{point}\times CX}\cong\frac{CS^{n-2}\times CX}{CS^{n-2}\times X\cup S^{n-2}\times CX}\\ &\cong\frac{C(S^{n-2}\circ X)}{S^{n-2}\circ X}=E(S^{n-2}\circ X).\end{aligned}\tag{6}$$

But $X^{n-1}_{1,0}\times X^{n-1}_{m-1,1}\cong CS^{n-2}\times CX$ and so the map inducing the lower horizontal map of (5) is obviously the shuffle map. Thus we have proven

Proposition 2.6. *$X^n_{m,m-1}/X^n_{m-1,m-2}$ has the homotopy type of the suspension space*

$$EX^{n-1}_{m,1}\cup_{E\psi}C(E(S^{n-2}\circ X^{n-1}_{m-1,1}))$$

where ψ is the map defined in (4), *i.e. ψ is given by*

$$S^{n-2}\times CX^{n-1}_{m-1,1}\xrightarrow{\mathrm{id}\times\alpha}X^{n-1}_{1,1}\times X^{n-1}_{m-1,0}\xrightarrow{\beta}X^{n-1}_{m,1},$$
$$CS^{n-2}\times X^{n-1}_{m-1,1}\xrightarrow{\alpha\times\mathrm{id}}X^{n-1}_{1,0}\times X^{n-1}_{m-1,1}\xrightarrow{\beta}X^{n-1}_{m,1}.$$

Moreover an explicit homotopy equivalence is given by the composition $F\circ G$, G the homeomorphism of Lemma 2.5(iii) and F the conal homotopy equivalence determined by the collapsing map p

$$p\colon S^{n-1}\circ X^{n-1}_{m-1,1}\to\frac{S^{n-1}\circ X^{n-1}_{m-1,1}}{A}$$

and the homeomorphism $q\colon X^{n-1}_{m,m}/X^{n-1}_{m-1,m-1}\cong EX^{n-1}_{m,1}$.

The particular case $m=2$ yields a slightly stronger assertion:

COROLLARY 2.7. *$X^n_{2,1}$ has the homotopy type of the suspension space*

$$EX^{n-1}_{2,1} \cup_{E\psi} C(E(S^{n-2} \circ S^{n-2}))$$

and an explicit homotopy equivalence is given by $F \circ G \circ p$ where F, G, ψ are as in 2.6 *and p: $X^n_{2,1} \to X^n_{2,1}/X^n_{1,0}$ is the collapsing map (a homotopy equivalence).*

We now restrict our attention to the case $m=3$. We wish to prove

PROPOSITION 2.8. *$X^n_{3,2}/X^n_{2,1}$ has the homotopy type of a double suspension space.*

An important step for this is

PROPOSITION 2.9. *$X^{n-1}_{3,1}$ has the homotopy type of a suspension.*

Proof (of 2.9). By Lemma 2.4(i) $X^{n-1}_{3,1}$ is homeomorphic to the adjunction space

$$(X^{n-1}_{3,2} \cup X^{n-1}_{2,0}) \cup_{\phi} C(X^{n-1}_{2,0} \circ S^{n-3}). \tag{7}$$

Since $X^{n-1}_{2,0} \cong CX^{n-1}_{2,1}$ the collapsing map $X^{n-1}_{3,2} \cup X^{n-1}_{2,0} \to X^{n-1}_{3,2}/X^{n-1}_{2,1}$ is a homotopy equivalence and so the adjunction space (7) is homotopy equivalent to the adjunction space

$$X^{n-1}_{3,2}/X^{n-1}_{2,1} \cup_{\varphi} C(X^{n-1}_{2,1} \circ S^{n-3}).$$

Here the attaching map φ is given by

$$X^{n-1}_{2,1} \times CS^{n-3} \to X^n_{2,1} \times X^{n-1}_{1,1} \to X^{n-1}_{3,2} \to X^{n-1}_{3,2}/X^{n-1}_{2,1},$$
$$A = \text{point} \times CS^{n-3} \cup CX^{n-1}_{2,1} \times S^{n-3} \to \text{basepoint},$$

and so we again have a factorization

$$\begin{array}{ccc} X^{n-1}_{2,1} \circ S^{n-3} & \xrightarrow{\varphi} & X^{n-1}_{3,2}/X^{n-1}_{2,1} \\ & \searrow \quad \nearrow_{\varphi'} & \\ & X^{n-1}_{2,1} \circ S^{n-3}/A & \end{array}$$

Recall the homeomorphism $(X^{n-1}_{2,1} \circ S^{n-3})/A \cong (X^{n-1}_{2,1} \times CS^{n-3})/A'$, where $A' = X^{n-1}_{2,1} \times S^{n-3} \cup \text{point} \times CS^{n-3}$, and the homotopy equivalence

$$\frac{X^{n-1}_{2,1} \times CS^{n-3}}{A'} \to Z = \frac{\{SP^2S^{n-2}/S^{n-2} \cup_{\varphi} C(S^{n-2} \circ S^{n-2})\} \times CS^{n-3}}{A''}$$

where

$$A'' = \{SP^2S^{n-2}/S^{n-2} \cup_{\varphi} C(S^{n-2} \circ S^{n-3})\} \times S^{n-3} \cup \text{point} \times CS^{n-3}.$$

From this and the homeomorphisms

$$SP^2S^{n-2}/S^{n-2} \cong CX^{n-2}_{2,1}/X^{n-2}_{2,1} = EX^{n-2}_{2,1},$$

$$\frac{EX^{n-2}_{2,1} \times CS^{n-3}}{EX^{n-2}_{2,1} \times S^{n-3} \cup \text{point} \times CS^{n-3}} \cong \frac{CX^{n-2}_{2,1} \times CS^{n-3}}{X^{n-2}_{2,1} \circ S^{n-3}}$$

$$\cong \frac{C(X^{n-2}_{2,1} \circ S^{n-3})}{X^{n-2}_{2,1} \circ S^{n-3}} = E(X^{n-2}_{2,1} \circ S^{n-3})$$

we obtain a homeomorphism

$$Z \to E(X_{2,1}^{n-2} \circ S^{n-3}) \cup_\gamma C(S^{n-2} \circ S^{n-3}) \times CS^{n-3}$$

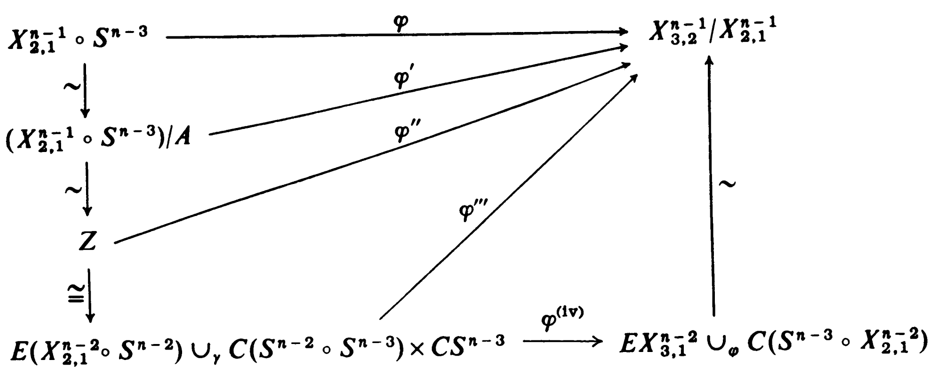
where the attaching map γ is given by

$$(S^{n-2} \circ S^{n-3}) \times CS^{n-3} \to \frac{SP^2 S^{n-3}}{S^{n-2}} \times CS^{n-3} \to \frac{(SP^2S^{n-2}/S^{n-2}) \times CS^{n-3}}{A}$$

$$\cong E(X_{2,1}^{n-2} \circ S^{n-3}),$$

$$C(S^{n-2} \circ S^{n-3}) \times S^{n-3} \to \text{basepoint}.$$

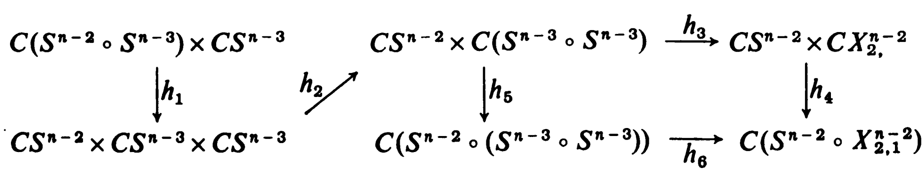
Since the map φ sends $D^{n-1} \times CS^{n-3}$ to the basepoint we obtain induced maps φ'', φ''', $\varphi^{(\mathrm{iv})}$ and the commutative diagram



The end is now in sight. To complete the proof of 2.9 we need only show that $\varphi^{(\mathrm{iv})}$ is homotopy equivalent to a suspension. First note that $\varphi^{(\mathrm{iv})}|E(X_{2,1}^{n-2} \circ S^{n-3})$ is already a suspension map since it is induced by the "shuffle" map

$$CX_{2,1}^{n-2} \times CS^{n-3} \to CX_{3,1}^{n-2}.$$

Furthermore $\varphi^{(\mathrm{iv})}$ is induced by the composite $h_4h_3h_2h_1$ of homeomorphisms



Now there exists a homeomorphism h_7

$$E(X_{2,1}^{n-2} \circ S^{n-3}) \cup_{\gamma'} C(S^{n-2} \circ (S^{n-3} \circ S^{n-3}))$$

$$\xrightarrow{h_7} E(X_{2,1}^{n-2} \circ S^{n-3}) \cup_\gamma C(S^{n-2} \circ S^{n-3}) \times CS^{n-3}$$

which is the identity on $E(X_{2,1}^{n-2} \circ S^{n-3})$ and $(h_5h_2h_1)^{-1}$ on

$$C(S^{n-2} \circ (S^{n-3} \circ S^{n-3})).$$

Here $\gamma' = \gamma(h_5h_2h_1)^{-1}|(S^{n-2} \circ (S^{n-3} \circ S^{n-3}))$. $\varphi^{(\mathrm{iv})}h_7$ is now induced by h_6. The attaching map φ factors as

$$\begin{array}{ccc} S^{n-2} \circ X_{2,1}^{n-2} & \xrightarrow{\varphi} & EX_{3,1}^{n-2} \\ & p_2 \searrow \quad \nearrow \varphi^* & \\ & (S^{n-2} \circ X_{2,1}^{n-2})/A & \end{array}$$

where $A = \text{point} \times CX_{2,1}^{n-2} \cup CS^{n-2} \times X_{2,1}^{n-2}$. p_2 is a homotopy equivalence and φ^* becomes $E\psi$ when $(S^{n-2} \circ X_{2,1}^{n-2})/A$ is identified as $E(S^{n-3} \circ X_{2,1}^{n-2})$.

The attaching map γ' also factors as

$$\begin{array}{ccc} S^{n-2} \circ (S^{n-3} \circ S^{n-3}) & \xrightarrow{\gamma'} & E(X_{2,1}^{n-2} \circ S^{n-3}) \\ & p_1 \searrow \quad \nearrow \gamma'' & \\ & (S^{n-2} \circ (S^{n-3} \circ S^{n-3}))/A' & \end{array}$$

where $A' = \text{point} \times C(S^{n-3} \circ S^{n-3}) \cup CS^{n-3} \times (S^{n-3} \circ S^{n-3})$ and p_1 is a homotopy equivalence. Thus we have a commutative diagram with $\varphi^{(\mathrm{v})}$ induced by $\varphi^{(\mathrm{iv})}$

$$\begin{array}{ccc} E(X_{2,1}^{n-2} \circ S^{n-3}) \cup_{\gamma'} C(S^{n-2} \circ (S^{n-3} \circ S^{n-3})) & \xrightarrow{\varphi^{(\mathrm{iv})}} & EX_{3,1}^{n-2} \cup_{\varphi} C(S^{n-2} \circ X_{2,1}^{n-2}) \\ \Big\downarrow p_1 & & \Big\downarrow p_2 \\ E(X_{2,1}^{n-2} \circ S^{n-3}) \cup_{\gamma''} C(E(S^{n-3} \circ (S^{n-3} \circ S^{n-3}))) & \xrightarrow{\varphi^{(\mathrm{v})}} & EX_{3,1}^{n-2} \cup_{E\psi} C(E(S^{n-3} \circ X_{2,1}^{n-2})). \end{array}$$

But $\varphi^{(\mathrm{v})}$ restricts to a suspension map $E(X_{2,1}^{n-2} \circ S^{n-3}) \to EX_{3,1}^{n-2}$ and moreover maps the suspension variable of $C(E(S^{n-3} \circ (S^{n-3} \circ S^{n-3})))$ linearly onto the suspension variable of $C(E(S^{n-3} \circ X_{2,1}^{n-2}))$. Thus $\varphi^{(\mathrm{v})}$ will be a suspension map (and $X_{3,1}^{n-1}$ homotopy equivalent to a suspension space) if the attaching map γ'' is also a suspension map. But this is a simple matter of direct verification (or is obvious by construction!).

We need examine only how γ' behaves on $S^{n-2} \times C(S^{n-3} \circ S^{n-3})$ since γ' sends $\text{point} \times C(S^{n-3} \circ S^{n-3}) \cup CS^{n-2} \times (S^{n-3} \circ S^{n-3})$ to the basepoint (south pole). Recall that the suspension structure of $E(S^{n-3} \circ (S^{n-3} \circ S^{n-3}))$ comes from $(S^{n-2} \times C(S^{n-3} \circ S^{n-3}))/A''$,

$$A'' = \text{point} \times C(S^{n-3} \circ S^{n-3}) \cup CS^{n-2} \times (S^{n-3} \circ S^{n-3})$$

via

$$\frac{CS^{n-3} \times C(S^{n-3} \circ S^{n-3})}{S^{n-3} \circ (S^{n-3} \circ S^{n-3})} \cong \frac{C(S^{n-3} \circ (S^{n-3} \circ S^{n-3}))}{S^{n-3} \circ (S^{n-3} \circ S^{n-3})}.$$

Similarly that of $E(X_{2,1}^{n-2} \circ S^{n-3})$ comes from

$$\frac{CX_{2,1}^{n-2} \times CS^{n-3}}{X_{2,1}^{n-2} \circ S^{n-3}} \cong \frac{C(X_{2,1}^{n-2} \circ S^{n-3})}{X_{2,1}^{n-2} \circ S^{n-3}}.$$

Let $(x_2, [x_3, t_3]) \in S^{n-3} \times CS^{n-3} \subset S^{n-3} \circ S^{n-3}$ so that

$$a = ([x_1, t_1], [(x_2, [x_3, t_3]), u] \in CS^{n-3} \times C(S^{n-3} \circ S^{n-3}).$$

Then a is mapped to $b = ([[x_1, t_1], [x_2, u]], [x_3, tu]) \in X^{n-2}_{2,0} \times CS^{n-3}$. As an element of $E(S^{n-3} \circ (S^{n-3} \circ S^{n-3}))$ a has t_1 or u as suspension variable according as $u \leqq t_1$ or $t_1 \leqq u$. As an element of $CX^{n-2}_{2,1} \times CS^{n-3}$ b has the description

$$\begin{aligned} b &= ([(x_1, [x_2, u/t_1]), t_1], [x_1, t_3 u]) \quad \text{if } u \leqq t_1, \\ &= ([([x_1, t_1/u], x_2), u], [x_3, t_3 u]) \quad \text{if } t_1 > u. \end{aligned}$$

Hence the suspension variable of b is t_1 or u according as $t_3 u \leqq t_1$ or $t_1 \leqq u$. So the condition $u \leqq t_1$ implies $t_3 u \leqq t_1$ in which case t_1 is the suspension variable for both a and b. The other alternative $t_1 \leqq u$ makes u the common suspension variable.

On the other hand let $([x_2, t_2], x_3) \in CS^{n-3} \times S^{n-3}$ so that

$$a = ([x_1, t_1], [([x_2, t_2], x_3), u]) \in CS^{n-3} \times C(S^{n-3} \circ S^{n-3}).$$

Then a is mapped to $b = ([[x_1, t_1], [x_2, t_2 u]], [x_3, u]) \in X^{n-2}_{2,0} \times CS^{n-3}$. As an element of $CX^{n-2}_{2,1} \times CS^{n-3}$ b looks like

$$\begin{aligned} b &= ([(x_1, [x_2, t_2 u/t_1]), t_1], [x_3, u]) \quad \text{if } t_2 u \leqq t_1, \\ &= ([([x_1, t_1/t_2 u], x_2), t_2 u], [x_3, u]) \quad \text{if } t_1 \leqq t_2 u. \end{aligned} \tag{8}$$

Now if $u \leqq t_1$ then a has suspension variable t_1. But then so does b. If $t_1 \leqq u$ then a has suspension variable u. So does b but this requires a little checking: if $t_2 u \leqq t_1$ then b has suspension variable u from the top half of (8), and if $t_1 \leqq t_2 u$ then b has suspension variable u from the bottom half of (8).

Hence γ'' is a suspension map and the proof of Proposition 2.9 is complete.

The proof of 2.8 is based on the following elementary observation, the proof of which is left to the reader: suppose $f\colon X \circ Y \to Z$ is a map which factors as $f'p = f$

$$\begin{array}{ccc} X \circ Y & \xrightarrow{f} & Z \\ & {}_{p}\searrow \quad \nearrow_{f'} & \\ & W & \end{array}$$

where p collapses $X \times \text{vertex}_{CY}$ to a point and collapses $\text{vertex}_{CX} \times Y$ to a (different) point and W is the resulting quotient space (easy to see that $W \simeq E(X \times Y)$ with these collapsed points corresponding to the north and south poles). Then f is homotopic to the map g given by

$$\begin{aligned} g(x, [y, t]) &= f(x, [y, 2t]), \qquad 0 \leqq t \leqq \tfrac{1}{2}, \\ &= f(x, [y, -2t+2]), \quad \tfrac{1}{2} \leqq t \leqq 1; \\ g(CX \times Y) &= \text{basepoint}. \end{aligned}$$

By Proposition 2.6 there is a homotopy equivalence

$$X^n_{3,2}/X^n_{2,1} \to EX^{n-1}_{3,1} \cup_{E\psi} C(E(S^{n-2} \circ X^{n-1}_{2,1})).$$

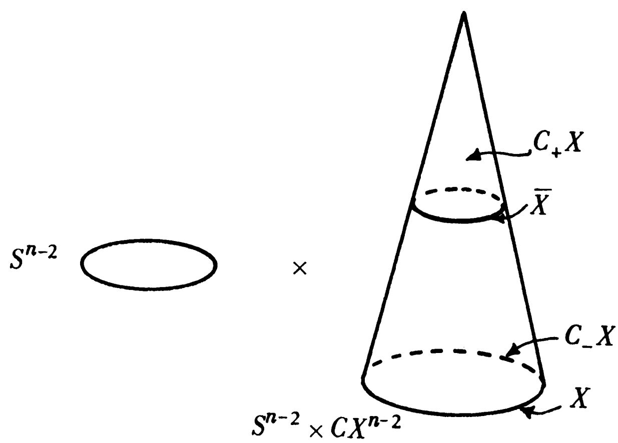
From the above observation the map $\psi: S^{n-2} \circ X^{n-1}_{2,1} \to X^{n-1}_{3,1}$ is homotopic to the map $\psi': S^{n-2} \circ X^{n-1}_{2,1} \to X^{n-1}_{3,1}$ defined above. Thus we have a commutative diagram

$$\begin{array}{ccccc} S^{n-2} \circ X^{n-1}_{2,1} & \xrightarrow{\psi'} & X^{n-1}_{3,1} & \longrightarrow & X^{n-1}_{3,2}/X^{n-1}_{2,1} \cup C(X^{n-1}_{2,1} \circ S^{n-3}) \\ & {\scriptstyle p}\searrow & & \nearrow{\scriptstyle f'} & \\ & & (S^{n-2} \circ X^{n-1}_{2,1})/A & & \end{array}$$

where $A = \text{point} \times CX^{n-1}_{2,1} \cup CS^{n-2} \times X^{n-1}_{2,1}$. Set

$$\bar{X} = \{[x, \tfrac{1}{2}] \in CX^{n-1}_{2,1} \mid x \in X^{n-1}_{2,1}\},$$
$$C_+X = \{[x, t] \in CX^{n-1}_{2,1} \mid x \in X^{n-1}_{2,1},\ 0 \leqq t \leqq \tfrac{1}{2}\},$$
$$C_-X = \{[x, t] \in CX^{n-1}_{2,1} \mid x \in X^{n-1}_{2,1},\ \tfrac{1}{2} \leqq t \leqq 1\}.$$

From the definition of ψ' we have that the f'-pre-image of $X^{n-1}_{3,2}/X^{n-1}_{2,1}$ is the p-image of $S^{n-2} \times \bar{X} \cup CS \times X^{n-1}_{2,1} \cup \text{pt} \times CX^{n-1}_{2,1}$.



We define maps

(i) $CS^{n-3} \times C_+X \to S^{n-2} \times C_+X$,

(ii) $CS^{n-3} \times C_-X \to S^{n-2} \times C_-X$,

via the maps $CS^{n-3} \cong D^{n-2} \xrightarrow{h} S^{n-2}$ and $C_\pm X \xrightarrow{\text{id}} C_\pm X$. In (i) the boundary $S^{n-3} \times C_+X \cup CS^{n-3} \times X$ is mapped onto $\text{point} \times C_+X \cup S^{n-2} \times \bar{X}$ and in (ii) the boundary

$$S^{n-2} \times C_-X \cup CS^{n-3} \times X \text{ onto } \text{pt} \times C_-X \cup S^{n-2} \times (\bar{X} \cup X).$$

Consequently there exists a homeomorphism

$$Y \cong (S^{n-2} \times \bar{X} \cup \text{point} \times CX \cup CS^{n-2} \times X) \cup CS^{n-3} \times C_+X \cup CS^{n-3} \times C_-X$$

and so a homeomorphism

$$\frac{Y}{A} \cong Z_1 = \frac{S^{n-2}\times \bar{X}}{\text{point}\times \bar{X}} \cup_{\alpha_1} CS^{n-3}\times C_+X \cup_{\alpha_2} CS^{n-3}\times C_-X$$

with attaching maps α_1 and α_2 derived from (i) and (ii) above together with the collapsing map $Y \to Y/A$.

Now the collapsing map

$$\frac{S^{n-2}\times \bar{X}}{\text{point}\times \bar{X}} \to \frac{S^{n-2}\times \bar{X}}{S^{n-2}\vee \bar{X}} \tag{10}$$

induces a collapsing map

$$Z_1 \to Z_2 = \left(\frac{S^{n-2}\times \bar{X}}{S^{n-2}\vee \bar{X}}\right) \cup_{\alpha_1'} CS^{n-3}\times C_+X \cup_{\alpha_2'} CS^{n-3}\times C_-X. \tag{11}$$

But

$$\frac{S^{n-2}\times \bar{X}}{\text{point}\times \bar{X}} \sim S^{n-2} \vee (S^{n-2}\wedge \bar{X})$$

and so the map (10) is obviously homotopy equivalent to a suspension map. And it follows easily that the map (11) is also homotopy equivalent to a suspension map. As the map f' of (9) factors as

$$\begin{array}{ccc} Z_1 & \xrightarrow{f'} & X_{3,2}^{n-1}/X_{2,1}^{n-1} \cup C(X_{2,1}^{n-2}\circ S^{n-3}) \\ & \searrow \quad \nearrow_{f''} & \\ & Z_2 & \end{array}$$

we see that to prove f' is homotopy equivalent to a suspension, it suffices to prove that f'' is. Note that

$$(f'')^{-1}\left(\frac{X_{3,2}^{n-1}}{X_{2,1}^{n-1}}\right) = \frac{S^{n-2}\times \bar{X}}{S^{n-2}\vee \bar{X}}.$$

The rest of the proof consists of two steps.

Step 1. $f''|(S^{n-2}\times \bar{X})/(S^{n-2}\vee \bar{X})$ is homotopy equivalent to a suspension.

Recall the homotopy equivalence

$$\bar{X}\,(=X_{2,1}^{n-1}) \to EX_{2,1}^{n-2}\cup C(S^{n-2}\circ S^{n-3})$$

from which we obtain a homotopy equivalence

$$\frac{S^{n-2}\times \bar{X}}{S^{n-2}\vee \bar{X}} \to Z_3 = \frac{S^{n-2}\times (EX_{2,1}^{n-2}\cup C(S^{n-2}\circ S^{n-3}))}{S^{n-2}\vee (EX_{2,1}^{n-2}\cup C(S^{n-2}\circ S^{n-3}))}$$

and a factoring

$$\begin{array}{ccc} \dfrac{S^{n-2}\times \bar{X}}{S^{n-2}\vee \bar{X}} & \xrightarrow{\ f''\ } & X^{n-1}_{3,2}/X^{n-1}_{2,1} \\ & \searrow \qquad \nearrow {\scriptstyle f'''} & \\ & Z_3 & \end{array}$$

In an obvious way Z_3 is homeomorphic to the space

$$Z_4 = \left(\frac{S^{n-2}\times EX^{n-2}_{2,1}}{S^{n-2}\vee EX^{n-2}_{2,1}}\right) \cup_\delta CS^{n-3}\times C(S^{n-2}\circ S^{n-3})$$

where δ is given by

$$\begin{aligned} &S^{n-3}\times C(S^{n-2}\circ S^{n-3}) \to \text{basepoint},\\ &CS^{n-3}\times S^{n-2}\times CS^{n-3} \to S^{n-2}\times EX^{n-2}_{2,1},\\ &CS^{n-3}\times CS^{n-2}\times S^{n-3} \to \text{basepoint}. \end{aligned}$$

With the inverse of the above homeomorphism $Z_3\to Z_4$ the map f''' defines a map $f^{(\text{iv})}\colon Z_4\to X^{n-1}_{3,2}/X^{n-1}_{2,1}$. Checking the above definitions we see that $f^{(\text{iv})}$ is induced by the composite map

$$\begin{aligned} CS^{n-3}\times C(S^{n-2}\circ S^{n-3}) &\to CS^{n-3}\times CS^{n-2}\times CS^{n-3} \to CS^{n-2}\times C(S^{n-3}\circ S^{n-3})\\ &\to CS^{n-2}\times CX^{n-2}_{2,1} \to C(S^{n-2}\circ X^{n-2}_{2,1}). \end{aligned}$$

Therefore if we replace Z_4 by the homeomorphic (to Z_4)

$$Z_5 = \frac{CS^{n-3}\times CX^{n-2}_{2,1}}{S^{n-3}\circ X^{n-2}_{2,1}} \cup CS^{n-2}\times C(S^{n-3}\circ S^{n-3}),$$

the resulting map $Z_5\to EX^{n-2}_{3,1}\cup C(S^{n-2}\circ X^{n-2}_{2,1})$ is induced by the composite

$$CS^{n-2}\times C(S^{n-3}\circ S^{n-3}) \to CS^{n-2}\times CX^{n-2}_{2,1} \to C(S^{n-2}\circ X^{n-2}_{2,1})$$

and so restricts to a suspension map $E(S^{n-3}\circ X^{n-2}_{2,1})\to EX^{n-2}_{3,1}$, and on

$$C(S^{n-2}\circ (S^{n-3}\circ S^{n-3}))$$

is the conal extension of the map

$$S^{n-2}\circ (S^{n-3}\circ S^{n-3}) \xrightarrow{\ \text{id}\circ\psi\ } S^{n-2}\circ X^{n-2}_{2,1}.$$

Hence via the usual homotopy equivalences we obtain a commutative diagram

$$\begin{array}{ccc} E(S^{n-3}\circ X^{n-2}_{2,1})\cup C(S^{n-2}\circ (S^{n-3}\circ S^{n-3})) & \longrightarrow & EX^{n-2}_{3,1}\cup C(S^{n-2}\circ X^{n-2}_{2,1}) \\ \downarrow & & \downarrow \\ E(S^{n-3}\circ X^{n-2}_{2,1})\cup C(E(S^{n-3}\circ (S^{n-3}\circ S^{n-3}))) & \to & EX^{n-2}_{3,1}\cup C(E(S^{n-3}\circ X^{n-2}_{2,1})) \end{array}$$

where the bottom horizontal map is a suspension map and the vertical maps homotopy equivalences. Thus $f''|(S^{n-2}\times \bar{X})/(S^{n-2}\vee \bar{X})$ is homotopy equivalent to a suspension.

Step 2. We have constructed a commutative diagram

$$\begin{array}{ccc}
Z_1 \simeq' Y/A & \longrightarrow & X^{n-1}_{3,2}/X^{n-1}_{2,1}\cup_{\varphi} C(X^{n-1}_{2,1}\circ S^{n-3}) \\
\downarrow & & \downarrow q_1 \\
Z_2 & \longrightarrow & \{EX^{n-2}_{3,1}\cup C(S^{n-2}\circ X^{n-2}_{2,1})\}\cup C(X^{n-1}_{2,1}\circ S^{n-3}) \\
\downarrow & \nearrow & \\
\vdots & & \\
\downarrow & & \downarrow q_2 \\
Z_5 & & \\
\downarrow & & \downarrow \\
EA\cup_{\alpha_1} C(S^{n-3}\circ X^{n-1}_{2,1})\cup_{\alpha_2} C(S^{n-3}\circ X^{n-1}_{2,1}) & \rightarrow & EB\cup_{\gamma} C(X^{n-2}_{2,1}\circ S^{n-3})
\end{array}$$

where the bottom horizontal map restricted to EA is a suspension map $\tilde{h}\colon EA\to EB$,

$$EA = E(S^{n-3}\circ X^{n-2}_{2,1})\cup C(E(S^{n-3}\circ (S^{n-3}\circ S^{n-3})))$$

and

$$EB = EX^{n-2}_{3,1}\cup C(E(S^{n-3}\circ X^{n-2}_{2,1})).$$

The attaching maps satisfy the relation $\tilde{h}\circ \alpha_i=\gamma\circ T$ where $T\colon S^{n-3}\circ X^{n-1}_{2,1}\to X^{n-1}_{2,1}\circ S^{n-3}$ is the switch homeomorphism. As

$$\alpha_i(S^{n-3}\times CX^{n-1}_{2,1}\cup CS^{n-3}\times \text{point}) = \text{basepoint} \qquad (i = 1 \text{ and } 2)$$

and

$$\gamma(CX^{n-1}_{2,1}\times S^{n-3}\cup \text{point}\times CS^{n-3}) = \text{basepoint},$$

we will be done if $X^{n-1}_{2,1}$ is a suspension space $X^{n-1}_{2,1}=EY$, for then we have a commutative diagram

$$\begin{array}{ccc}
EA\cup_{\alpha_1} C(S^{n-3}\circ X^{n-1}_{2,1})\cup_{\alpha_2} C(S^{n-3}\circ X^{n-1}_{2,1}) & \rightarrow & EB\cup_{\gamma} C(X^{n-1}_{2,1}\circ S^{n-3}) \\
\downarrow & & \downarrow \\
EA\cup C(E(S^{n-3}\circ Y))\cup C(E(S^{n-3}\circ Y)) & \rightarrow & EB\cup C(E(Y\circ S^{n-3}))
\end{array} \tag{12}$$

with the bottom horizontal map a suspension. And in fact by Lemma 2.1 of [1] $X^{n-1}_{2,1}$ is a suspension. However we can also give a self-contained proof using the homotopy equivalence $X^{n-1}_{2,1}\to EY'$ of Corollary 2.7. Since the attaching maps α_1, α_2 (and γ) send $(S^{n-2}\circ X^{n}_{1,0})(X^{n}_{1,0}\circ S^{n-3})$ to the basepoint, we obtain a new diagram similar to (12) with EY' and Y' replacing $X^{n-1}_{2,1}$ and Y respectively. The conclusion again is that the bottom horizontal map is a suspension. Whence $X^{n}_{3,2}/X^{n}_{2,1}$ has the homotopy type of a double suspension.

REMARKS. 1. For $m=2$ the preceding argument would suffice to prove that $X^n_{2,1}/X^n_{1,0}$ (and hence $X^n_{2,1}$ itself) has the homotopy type of a double suspension. However Lemma 2.1 of [1] asserts much more: $X^n_{2,1}$ is homeomorphic to the join $S^{n-1}\circ P^{n-1}$ of S^{n-1} with real projective n-space P^{n-1}. The subsets of $X^n_{2,1}$ corresponding to the "ends" of this join are $S^{n-1}\cong\{[x,x]\in X^n_{2,1}\mid x\in S^{n-1}\}$ and $P^{n-1}\cong\{[x,-x]\in X^n_{2,1}\mid x\in S^{n-1}\}$. It seems reasonable to conjecture that either $X^n_{m,m-1}$ or $X^n_{m,m-1}/X^n_{m-1,m-2}$ is homeomorphic to a join of the form $S^{n-1}\circ Z$ for some subcomplex Z (the problem is to determine Z).

2. Of immediate concern for constructing nontrivial maps $SP^mS^n\to S^n$ would be the result that $X^n_{m,m-1}/X^n_{m-1,m-2}$ has the homotopy type of a double suspension for all m (not just $m=2$ or 3). The techniques of this paragraph seem suitable for this. What is needed is (a) $X^{n-1}_{m,1}$ has the homotopy type of a suspension, and (b) the "shuffle" map $S^{n-2}\circ X^{n-1}_{m-1,1}\to X^{n-1}_{m,1}$ is homotopy equivalent to a suspension map. For (a) an induction could be based on the following

$$X^n_{m,1}\sim X^n_{m,2}/X^n_{m-1,1}\cup C(X^n_{m-1,1}\circ S^{n-2}),$$
$$X^n_{m,j}/X^n_{m-1,j-1}\cong X^n_{m,j+1}/X^n_{m-1,j}\cup C(X^n_{m-j,1}\circ X^{n-1}_{j,1}).$$

First $X^n_{m,m}/X^n_{m-1,m-1}\cong EX^{n-1}_{m,1}$ and next $X^n_{m,m-1}/X^n_{m-1,m-2}\sim EK$ by Proposition 2.6 and so on until one reaches $X^n_{m,2}/X^n_{m-1,1}$.

3. **An important lemma.** Recall the attaching map $\varphi\colon X^n_{m,1}\to X^{n+1}_{m-1,m-1}=SP^{m-1}S^n$ defined in Lemma 2.2. If $f\colon SP^{m-1}S^n\to S^n$ is any map of type q such that $f\varphi$ is nullhomotopic, say by $N_t\colon\varepsilon\sim f\varphi$, then a map $g\colon SP^mS^n\to S^n$ of type q is defined by $g|SP^{m-1}S^n=f$ and $g|CX^n_{m,1}=N_t$. As $\varphi^{-1}(SP^iS^n)=X^n_{m,m-i}$ we define maps

$$\varphi_i\colon X^n_{m,m-i}\to SP^iS^n,\qquad \varphi'_i\colon X^n_{m,m-i}\cup X^n_{m-1,m-i-2}\to SP^{i+1}S^n$$

by the restrictions $\varphi_i=\varphi|X^n_{m,m-i}$ and $\varphi'_i=\varphi|(X^n_{m,m-i}\cup X^n_{m-1,m-i-2})$.

LEMMA 3.1. *If $f\colon S^n\to S^n$ is a map of degree q such that $f\varphi_1\colon X^n_{m,m-1}\to S^n$ is nullhomotopic, then there exists a map $F\colon SP^{m-1}S^n\to S^n$ of type q such that $F\varphi_{m-1}=F\varphi$ is nullhomotopic. Thus the above construction provides the existence of a map $SP^mS^n\to S^n$ of type q.*

Proof. We have a filtration of $X^n_{m,1}$ given by $C_1\subset D_1\subset C_2\subset D_2\subset\cdots\subset C_{m-2}\subset D_{m-2}\subset C_{m-1}$ where $C_i=X^n_{m,m-i}$ and $D_i=X^n_{m,m-i}\cup X^n_{m-1,m-i-2}$, and so we have maps $\varphi_i\colon C_i\to SP^iS^n$ and $\varphi'_i\colon D_i\to SP^{i+1}S^n$ determined by φ. Suppose inductively $f_i\colon SP^iS^n\to S^n$ is a map of type q with nullhomotopy $N^i_t\colon\varepsilon\sim f_i\varphi_i$. We will then construct a map $f_{i+1}\colon SP^{i+1}S^n\to S^n$ of type q and nullhomotopies $N'^i_t\colon\varepsilon\sim f_{i+1}\varphi'_i$ and $N^{i+1}_t\colon\varepsilon\sim f_{i+1}\varphi_{i+1}$. Clearly this will suffice to prove the lemma. The construction of f_{i+1}, N'_t and N_t is based on the geometry of Lemma 2.4:

(i) $C_{i+1}\cong D_i\cup C(X^n_{i+1,1}\circ X^{n-1}_{m-i-1,1})$,
(ii) $D_i\cong C_i\cup CX^n_{i+1,1}\times X^n_{m-i-2,m-i-2}$.

(ii) is an easy consequence of the observation that

$$C_i \cap X^n_{m-1,m-i-2} = X^n_{m-1,m-i-1}.$$

Construction of f_{i+1}: we have the standard inclusion $j: X^n_{i+1,1} \subset X^n_{m,m-i}$ and so let $M^i_t = N^i_t j$ (more precisely $M^i_t = N^i_t(j \times \mathrm{id})$) be the restriction of the nullhomotopy N^i_t to $X^n_{i+1,1}$. Thus M^i_t is also a nullhomotopy. But recall $SP^{i+1}S^n \cong SP^iS^n \cup CX^n_{i+1,1}$ and so we may define f_{i+1} by $f_{i+1}|SP^iS^n = f_i$ and $f_{i+1}|CX^n_{i+1,1} = M^i_t$. Because $\varphi_i j$ is the attaching map in $SP^iS^n \cup CX^n_{i+1,1}$, f_{i+1} is well defined.

Construction of the nullhomotopy $N'^i_u : \varepsilon \sim f_{i+1}\varphi'_i$: this is simply given by $N'^i_u | C_i = N^i_u$ and

$$N'^i(([x, t], y), u) = f_{i+1}\varphi'_i([x, ut], y)$$

on $CX^n_{i+1,1} \times X^n_{m-i-2,m-i-2}$. Here $([x, ut], y)$ also denotes a point of $CX^n_{i+1,1} \times X^n_{m-i-2,m-i-2}$. As $\varphi'_i|\{([x, 0], y)\}$ is the constant map, so is N'^i_0. And for $u=1$, N'^i_1 is just $f_{i+1}\varphi'_i$ as desired.

Construction of the nullhomotopy $N^{i+1}_t : \varepsilon \sim f_{i+1}\varphi_{i+1}$: define N^{i+1}_t by $N^{i+1}_t | D_i = N'^i_t$ and on $C(X^n_{i+1,1} \circ X^{n-1}_{m-i-1,1})$ by

(iii) $N^{i+1}([(x, [y, t_2]), s], u) = M^i[x, su]$,

(iv) $N^{i+1}([([x, t_1], y), s], u) = M^i[x, st_1u]$.

The common domain of (iii) and (iv) occurs when $t_1 = t_2 = 1$ in which case (i) and (ii) reduce to $M^i[x, su]$. When $s=1$ (i) and (ii) are consistent with $N^{i+1}_t | D^i = N'^i_t$. This is clear for $X^n_{i+1,1} \times CX^{n-1}_{m-i-1,1}$ since the attaching map of (i) sends $X^n_{i+1,1} \times CX^{n-1}_{m-i-1,1}$ to C_i and $N'^i_t | C_i = N^i_t$. On the other hand $\varphi'_i | CX^n_{i+1,1} \times X^{n-1}_{m-i-1,1}$, viewed as a map into $SP^iS^n \cup CX^n_{i+1,1}$, is the identity on $CX^n_{i+1,1}$; furthermore $f_{i+1}|CX^n_{i+1,1}$ is given by M^i. Thus $N'^i | CX^n_{i+1,1} \times X^{n-1}_{m-i-1,1}$ agrees with (iv) above. Finally when $u=0$ we get the constant map and when $u=1$ we get $f_{i+1}\varphi_{i+1}$. This completes the proof of 3.1.

We conclude this paragraph with a simple observation about the integer $k^{m,n}$ defined in the introduction. Note that $[X^n_{m,m-1}, S^n]$ is a *distinguished set* and the function

$$\psi_q : [X^n_{m,m-1}, S^n] \to [X^n_{m,m-1}, S^n]$$

defined by composition with a map $f_q : S^n \to S^n$ of degree q, respects distinguished elements. Set $\mathrm{Ker}\, \psi_q = \psi_q^{-1}(0)$. Then

PROPOSITION 3.2. *Let* $n = 2t+1$ *and* $\varphi, \varphi_1, \ldots, \varphi_{m-1}$ *be as in* 3.1. *Then* $k^{m,n}$ *is the least positive integer* q *such that* $[\varphi_1] \in \mathrm{Ker}\, \psi_q$.

Proof. For any $q > 0$ with $[\varphi_1] \in \mathrm{Ker}\, \psi_q$ the preceding lemma implies the existence of a map $f: SP^mS^n \to S^n$ of type $q > 0$, whence $k^{m,n} | q$. But by definition of $k^{m,n}$ there exists a map $g: SP^mS^n \to S^n$ of type $k^{m,n}$. Set $g_i = g | SP^iS^m$. Then $g\varphi: X^n_{m,1} \to S^n$ extends over the cone $CX^n_{m,1}$ and so is nullhomotopic. Thus so is $g_1\varphi_1$. But $g_1: S^n \to S^n$ has degree $k^{m,n}$ and so $[\varphi_1] \in \mathrm{Ker}\, \psi_r$ for $r = k^{m,n}$. Therefore $q = k^{m,n}$.

4. **Suspension-order.** The suspension-order of a space X is the order of the class $\iota_{EX} \in [EX, EX]$ of the identity map of EX. In [6] Toda has computed the suspension-order of $E^r X^n_{2,1}$, $r \geqq -(n-1)$. Our main concern is the determination of the suspension-order of $E^r(X^n_{3,2}/X^n_{2,1})$, $r \geqq -1$. However we begin with an easy result about $E^r X^n_{m,m-1}$, $r \geqq -1$.

PROPOSITION 4.1. (i) *For $n=2t$ the suspension-order of $E^r X^n_{m,m-1}$, $r \geqq -1$, is infinite.*

(ii) *For $n=2t+1$ the suspension-order of $E^r X^n_{m,m-1}$, $r \geqq -1$, is a divisor of $2^{r_1}3^{r_2}\cdots p^{r_1}$ for some positive integers $r_1, \ldots, r_1$ where p is the largest prime not exceeding m.*

Proof. (i) By Toda [6] we need only prove that $H^i(X^n_{m,m-1}; Z)$ contains an element of infinite order for some $i>0$. We assert that this is the case for $H^{2n-1}(X^n_{m,m-1}; Z)$. For $m=2$, $EX^n_{2,1}$ and SP^2S^n/S^n are homeomorphic but

$$H^{2n}(SP^2S^n; Z) \cong H^{2n}(SP^2S^n/S^n; Z) \approx Z$$

is well known.

So assume inductively on m that $H^{2n-1}(X^n_{m,m-1}; Z)$ contains an element of infinite order. From Lemma 2.4 $X^n_{m+1,m}$ is homeomorphic to the adjunction space

$$(X^n_{m+1,m+1} \cup X^n_{m,m-1}) \cup C(S^{n-1} \circ X^{n-1}_{m,1})$$

(recall $X^n_{m+1,m+1}=SP^{m+1}S^{n-1}$). As $X^n_{m+1,m+1} \cap X^n_{m,m-1}=X^n_{m,m}=SP^mS^{n-1}$ and $H^i(SP^mS^{n-1}; Z)$ is finite for $i=2n-1$, $2n-2$ (Nakaoka [3]), a straightforward application of the cohomology Mayer-Vietoris exact sequence implies that $H^{2n-1}(X^n_{m+1,m+1} \cup X^n_{m,m-1}; Z)$ contains an element of infinite order. The pair $(SP^mS^{n-1}, SP^{m-1}S^{n-1}) \cong (CX^{n-1}_{m,1}, X^{n-1}_{m,1})$ is n-connected and so the pair (using Lemma 2.4) $(X^n_{m+1,m}, X^n_{m+1,m+1} \cup X^n_{m,m-1})$ has trivial integral cohomology in dimensions $0<i<2n$. And since $n-1$ is odd H^{2n} is finite. So by the cohomology exact sequence of the pair $(X^n_{m+1,m}, X^n_{m+1,m+1} \cup X^n_{m,m-1})$ we obtain that

$$H^{2n-1}(X^n_{m+1,m}; Z)$$

contains an element of infinite order.

(ii) That $E^r X^n_{m,m-1}$, $r \geqq -1$, is simply connected is an easy consequence of (1) SP^mS^{n-1} is $(n-2)$-connected, (2) Lemma 2.4, (3) induction on m, and (4) the van Kampen Theorem. Hence to prove (ii) it suffices by Theorem 1.5 of [6] to prove that the reduced homology $\tilde{H}^*(X^n_{m,m-1}; Z)$ is finite and has q torsion (q a prime) exactly when $q \leqq m$. This we do by induction on m. For $m=2$ it is well known that $\tilde{H}^*(X^n_{2,1}; Z)$ consists of only 2-torsion. By the cohomology exact sequence of the pair $(X^n_{m+1,m}, X^n_{m,m-1})$ and induction it suffices to prove that

$$\tilde{H}^*(X^n_{m+1,m}, X^n_{m,m-1}; Z)$$

is finite and has q-torsion exactly when $q \leqq m+1$. By Lemma 2.5 we have

$$X^n_{m+1,m}/X^n_{m,m-1} \simeq X^n_{m+1,m+1}/X^n_{m,m} \cup C(S^{n-1} \circ X^{n-1}_{m,1})$$
$$\simeq EX^{n-1}_{m+1,1} \cup C(S^{n-1} \circ X^{n-1}_{m,1})$$

with the attaching map sending $CS^{n-1} \times X^{n-1}_{m,1}$ to the basepoint and on $S^{n-1} \times CX^{n-1}_{m,1}$ given by the composition

$$S^{n-1} \times CX^{n-1}_{m,1} \to S^{n-1} \times SP^m S^{n-1}$$
$$\to SP^{m+1}S^{n-1} \to SP^{m+1}S^{n-1}/SP^m S^{n-1} \simeq EX^{n-1}_{m+1,1}.$$

Now rank $\tilde{H}^*(EX^{n-1}_{m+1,1}, Z) = 1$, a generator of infinite order occurring in dimension $(m+1)(n-1)$. From the above description of the attaching map this generator is mapped by the coboundary homomorphism onto $(m+1)$-times a generator of infinite order (plus possibly something of finite order) in $H^{i+1}(E^{n+1}X^{n-1}_{m,1}; Z)$, $i=(m+1)(n-1)$. Thus $\tilde{H}^*(X^n_{m+1,m}/X^n_{m,m-1}; Z)$ is finite with q-torsion, $q \leqq m+1$.

REMARK. The assertion of 4.1(ii) is also true for $E^r(X^n_{m,m-1}/X^n_{m-1,m-2})$, $r \geqq -1$, in place of $E^r X^n_{m,m-1}$. The proof is essentially the same, but of course the first step is unnecessary.

LEMMA 4.2. *Let $n=2t+1$ and p be an odd prime. Then $(KU^1)^\sim(X^n_{p,p-1})$ contains an element of order p^t.*

Proof. Consider the cofibration

$$X^n_{p-1,p-2} \to X^n_{p,p-1} \to X^n_{p,p-1}/X^n_{p-1,p-2}.$$

As n is odd each of these spaces has finite reduced integral cohomology groups. However Nakaoka's results and Proposition 4.1 imply that $\tilde{H}^*(X^n_{p-1,p-2}; Z)$ has q-torsion exactly for those primes $q<p$. Hence (via the Atiyah-Hirzebruch spectral sequence) $(KU^*)^\sim(X^n_{p-1,p-2})$ contains no elements of order p^r, $r \geqq 1$. Thus we need only prove that $(KU^1)^\sim(X^n_{p,p-1}/X^n_{p-1,p-2})$ contains an element of order p^t.

Similar application of Nakaoka's results shows that the reduced integral cohomology of $E^{n+1}X^{n-1}_{p-1,1} \simeq E^n(SP^{p-1}S^{n-1}/SP^{p-2}S^{n-1})$ has rank 1 and has no p-torsion. Again from Lemma 2.5 we have

$$X^n_{p,p-1}/X^n_{p-1,p-2} \simeq EX^{n-1}_{p,1} \cup C(S^{n-1} \circ X^{n-1}_{p-1,1}).$$

In addition to q-torsion, $q<p$, $\tilde{H}^*(EX^{n-1}_{p,1}; Z)$ has one copy of Z (and that in dimension $(n-1)p$) and has cyclic p-torsion only in dimensions $(n-1)+2k(p-1)+1$, $k=1, 2, \ldots, t-1$. From the previous description of the attaching map it follows that $\tilde{H}^*(X^n_{p,p-1}/X^n_{p-1,p-2}; Z)$ is finite, has q-torsion for $q<p$, and has cyclic p-torsion only in dimension $(n-1)+2k(p-1)+1$, $k=1, 2, \ldots, t$.

Now the remaining steps in the proof are a repeat of those given in §2 of [7]. Since all the p-torsion occurs in odd dimensions, it all survives to E_∞. Clearly since there are t E_∞-terms (all contributing to $(KU^1)^\sim$) containing cyclic p-torsion

$(KU^1)^{\sim}(X^n_{p,p-1}/X^n_{p-1,p-2})$ will contain an element of order p^t if its p-primary part is cyclic. Coefficient K-theory and the Universal Coefficient Theorem will suffice for this. $\tilde{H}^*(X^n_{p,p-1}/X^n_{p-1,p-2}; Z_p)$ consists of $2t$ copies of Z_p and the initial differential d_{2p-1} (of the Atiyah-Hirzebruch spectral sequence converging to $(KU^*)^{\sim}(\ ; Z_p)$) kills all but two copies of Z_p, one in an odd dimension and the other in an even dimension. The Universal Coefficient Theorem then implies that $(KU^1)^{\sim}(X^n_{p,p-1}/X^n_{p-1,p-2})$ is cyclic. This completes the proof of 3.2.

For the suspension-order determination of $X^n_{3,2}/X^n_{2,1}$ and $X^n_{3,2}$ we require more precise information on $\tilde{H}^*(X^n_{3,2}/X^n_{2,1}; Z)$ and $\tilde{H}^*(X^n_{3,2}; Z)$.

LEMMA 4.3. *For* $n=2t+1$,

$$\begin{aligned}\tilde{H}^i(X^n_{3,2}/X^n_{2,1}; Z) &\cong Z_3 \quad \textit{if } i=(n-1)+4k+1,\ k=1,2,\ldots,t,\\ &\cong 0 \quad \textit{otherwise.}\end{aligned}$$

Proof. We start with the familiar

$$X^n_{3,2}/X^n_{2,1} \cong SP^3S^{n-1}/SP^2S^{n-1} \cup C(S^{n-1} \circ X^{n-1}_{2,1}).$$

Nakaoka [2] has shown that $H^*(SP^3S^{n-1}, SP^2S^{n-1}; Z) \cong H^*(E^nX^{n-1}_{2,1}; Z) \oplus K$ where $K=K_1 \oplus \cdots \oplus K_t$, $K_i \cong Z_3$ and dimension $K_i=(n-1)+4i+1$.

Thus in the commutative diagram

$$\begin{array}{ccc} H^*(SP^3S^{n-1}/SP^2S^{n-1}; Z) & \xrightarrow{\ \delta\ } & H^*(E^{n+1}X^{n-1}_{2,1}; Z) \\ & {\scriptstyle\varphi^*}\searrow \quad\quad \nearrow{\scriptstyle\cong} & \\ & H^*(E^nX^{n-1}_{2,1}; Z) & \end{array}$$

we need only prove that φ^* is an isomorphism when restricted to the direct summand $H^*(E^nX^{n-1}_{2,1}; Z)$.

Now φ factors as

$$S^{n-1} \circ X^{n-1}_{2,1} \longrightarrow \frac{S^{n-1} \circ X^{n-1}_{2,1}}{A} \cong \frac{S^{n-1} \times CX^{n-1}_{2,1}}{\text{point} \times CX^{n-1}_{2,1} \cup S^{n-1} \times X^{n-1}_{2,1}}$$
$$\cong \frac{S^{n-1} \times SP^2S^{n-1}}{S^{n-1} \vee SP^2S^{n-1}} \xrightarrow{\ \bar{\varphi}\ } \frac{SP^3S^{n-1}}{SP^2S^{n-1}}$$

where $\bar{\varphi}$ is induced by the composite

$$S^{n-1} \times SP^2S^{n-1} \xrightarrow{p_1} SP^3S^{n-1} \xrightarrow{p_2} SP^3S^{n-1}/SP^2S^{n-1}.$$

But $p_1^*u=u_1+u_2$ where u, u_1, u_2 are $(n-1)$-dimensional cohomology generators (of infinite order). By Nakaoka [2] the direct summand $\tilde{H}^*(E^nX^{n-1}_{2,1}; Z)$ of $H^*(SP^3S^{n-1}/SP^2S^{n-1}; Z)$ is represented by the classes $u\cdot(\delta_2Sq^{2i}u)$ and so $\varphi^*(u\cdot\delta_2Sq^{2i}u)=u_1 \otimes \delta_2Sq^{2i}u$. But $H^*(E^nX^{n-1}_{2,1}; Z)$ lies in the image of p_2^* and so the result follows.

LEMMA 4.4. *Let $n=2t+1$. The exact cohomology sequence of the pair $(X^n_{3,2}, X^n_{2,1})$ breaks up into short exact sequences, and so*

$$\tilde{H}^*(X^n_{3,2}; Z) \cong \tilde{H}^*(X^n_{2,1}; Z) \oplus \tilde{H}^*(X^n_{3,2}/X^n_{2,1}; Z).$$

Proof. We have a commutative diagram

$$\begin{array}{ccc} X^n_{2,1} & \xrightarrow{\quad i \quad} & X^n_{3,2} \\ & {}_{i_1}\searrow \quad \nearrow_{i_2} & \\ & SP^3S^{n-1} \cup X^n_{2,1} & \end{array}$$

Since $X^n_{3,2}/(SP^3S^{n-1} \cup X^n_{2,1}) \simeq E^{n+1}X^n_{2,1}$ is $(2n+1)$-connected, i_2^* is an isomorphism in dimensions $\leqq 2n-1$. But a straightforward application of the Mayer-Vietoris cohomology exact sequence (note $SP^3S^{n-1} \cap X^n_{2,1} = SP^2S^{n-1}$) shows that i_1^* is epimorphic. And so i^* is epimorphic and the result follows.

LEMMA 4.5. *Let $n=2t+1$. Then $(KO^n)^\sim(X^n_{3,2})$ contains an element of order $2^{\varphi(2t)}$.*

Proof. Consider the exact sequence

$$(KO^n)^\sim(X^n_{3,2}) \to (KO^n)^\sim(X^n_{2,1}) \to (KO^{n+1})^\sim(X^n_{3,2}, X^n_{2,1}).$$

From Lemma 4.3 and the Atiyah-Hirzebruch spectral sequence

$$(KO^{n+1})^\sim(X^n_{3,2}, X^n_{2,1})$$

has no 2-torsion. But $(KO^n)^\sim(X^n_{2,1}) \simeq Z_{2^{\varphi(2t)}}$ since $X^n_{2,1}$ and E^nP^{n-1} are homeomorphic (actually all one needs for this particular situation is that $H^*(X^n_{2,1}; Z) \simeq H^*(E^nP^{n-1}; Z)$ as modules over the Steenrod algebra) and so the result follows by exactness.

THEOREM 4.6. *Let $n=2t+1$. The suspension-order of $E^r(X^n_{3,2}/X^n_{2,1})$, $r \geqq -1$, is 3^t.*

Proof. It follows from Lemma 4.2 (for $p=3$) and Toda [6] that the suspension-order of $E^r(X^n_{3,2}/X^n_{2,1})$, $r \geqq -1$, is a multiple of 3^t. So we need only show that it is also a divisor of 3^t. By Lemma 1.3 of [6] and our Lemma 4.3 there is a sequence of subcomplexes $*=L_1 \subset L_2 \subset \cdots \subset L_{r-1} \subset L_r \subset \cdots \subset L = E^r(X^n_{3,2}/X^n_{2,1})$ with $L_i = L_{i-1} \cup e^i \cup e^{i+1}$ or $L_i = L_{i-1}$ according as $i-r=n-1+4k$ $(k=1, 2, \ldots, t)$ or not. Set $K_j = L_i$ for $i-r=n-1+4j$ so we have $* \subset K_1 \subset K_2 \subset \cdots \subset K_t$ and $K_j = K_{j-1} \cup e^i \cup e^{i+1}$.

By Lemma 1.4 of [6] and the second paragraph of the proof of Theorem 4.4 of [6] the class $3\iota_{EZ}$, $Z=E^r(X^n_{3,2}/X^n_{2,1})$, has a representative f satisfying $f(EK_j) \subset EK_{j-1}$. So the class $3^t\iota_{EZ}$ is represented by the map $\hat{f} = f \circ f \circ \cdots \circ f$ (t times) which satisfies $\hat{f}(EK_t) = *$ i.e. $\hat{f}$ is nullhomotopic. Whence the suspension-order of Z is a divisor of 3^t.

For completeness we include

THEOREM 4.7. *Let $n=2t+1$. The suspension-order of $E^rX^n_{3,2}$, $r\geqq -1$, is $2^{\varphi(2t)}\cdot 3^t$.*

Proof. First by Lemmas 4.2 and 4.5 and Toda's Theorem 1.1 [6] the suspension-order of $E^rX^n_{3,2}$ is a multiple of $2^{\varphi(2t)}\cdot 3^t$. To prove it is also a divisor of this number consider the fibration

$$X^n_{2,1}\to X^n_{3,2}\to X^n_{3,2}/X^n_{2,1}.$$

By the corollary to Theorem 2.6 of [6] the suspension-order of $E^rX^n_{2,1}$, $r\geqq -1$, is a divisor of $2^{\varphi(2t)}$. And from Lemma 4.6 above the suspension-order of $E^r(X^n_{3,2}/X^n_{2,1})$ is a divisor of 3^t. So by Theorem 1.2 of [6] we have that the suspension-order of $E^rX^n_{3,2}$ is a divisor of $2^{\varphi(2t)}\cdot 3^t$.

5. **The** mod 3 **Hopf invariant.** The generalized Hopf invariant is a homomorphism

$$[EK, S^n]\xrightarrow{H}[EK, S^{2n-1}]$$

for which the sequence

$$[K, S^{n-1}]\xrightarrow{E}[EK, S^n]\xrightarrow{H}[EK, S^{2n-1}]$$

is exact when dimension $EK\leqq 3(n-1)$. H also satisfies the property $\psi_{q^2}H=H\psi_q$. In §4 of [1] H is used to prove $\psi_{4r}[h]=4r[h]$ for any $[h]\in[X^n_{2,1}, S^n]$, $n=2t+1\geqq 3$.

Our situation is analogous but with one important exception: the dimension restriction is not applicable. So we consider the mod p Hopf invariant H_p, p an odd prime (we assume the reader is familiar with the relevant sections of [5]). H_p is defined by the commutative diagram

$$\begin{array}{ccccc}
[E^2K, S^{2m+1}; p] & & \xrightarrow{\quad H_p \quad} & & [E^2K, S^{2pm+1}; p] \\
\cong\downarrow & \searrow^{j} & & \nearrow & \downarrow h_{m'} \\
[K, \Omega^2S^{2m+1}; p] & \xrightarrow[j']{} & [K, (\Omega^2S^{2m+1}, S^{2m+1}); p] & \xrightarrow[k]{} & [K, (\Omega^2S^{2m+1}, \Omega S^{2m}_{p-1}); p]
\end{array}$$

where [; p] denotes the p-primary component. It is known that $h_{m'}$ is an isomorphism and that Ker j is the p-primary component of image E^2 where E^2: $[K, S^{2m-1}]\to[E^2K, S^{2m+1}]$. So whenever k is a monomorphism, we will have Ker $H_p=$ Ker $j=p$-primary component of image E^2. k occurs in the homotopy exact sequence of the triple $(\Omega^2S^{2m+1}, \Omega S^{2m}_{p-1}, S^{2m-1})$ as

$$[K, (\Omega S^{2m}_{p-1}, S^{2m-1}); p]\longrightarrow[K, (\Omega^2S^{2m+1}, S^{2m-1}); p]$$
$$\xrightarrow{k}[K, (\Omega^2S^{2m+1}, \Omega S^{2m}_{p-1}); p].$$

The group on the left is isomorphic to $[EK, S^{2pm-1}; p]$ so whenever the latter is trivial, k is a monomorphism. We use this observation in 5.2, but first we have

LEMMA 5.1. *Suppose $H^i(E^2K; Z)=0$ for all $i>pn-(p-1)$, $n=2m+1$, and $qH^{pn-(p-1)}(E^2K; Z)=0$ for some integer q.*

Then Ker H_p *contains that the subgroups* $q[E^2K, S^{2m+1}; p]$ *and* $\psi_q[E^2K, S^{2m+1}; p]$.

Proof. By the Hopf Theorem we have $[E^2K, S^{pn-(n-1)}] \simeq H^{pn-(n-1)}(E^2K; Z)$. So

$$H_p q[E^2K, S^{2m+1}; p] = qH_p[E^2K, S^{2m+1}; p] \subset q[E^2K, S^{2pm+1}] = q[E^2K, S^{pn-(n-1)}] = 0.$$

And since $H_p\psi_q = \psi_{q^p} H_p$ we have

$$H_p\psi_q[E^2K, S^{2m+1}; p] = \psi_{q^p} H_p[E^2K, S^{2m+1}; p] \subset \psi_{q^p}[E^2K, S^{2pm+1}] = 0$$

since ψ_{q^p} acts on $[E^2K, S^{2pm+1}] \simeq H^{2pm+1}(E^2K; Z)$ as multiplication by q^p.

LEMMA 5.2. *Let* $r = kq^2$ *and assume* $[EK, S^{2pm-1}; p] = 0$ *in addition to the hypotheses of* 5.1. *Then* ψ_r *acts on* $[E^2K, S^{2m+1}; p]$ *as multiplication by r.*

Proof. For $\alpha \in [E^2K, S^{2m+1}; p]$ we have $\psi_r\alpha = \psi_{kq}(\psi_q\alpha) = kq(\psi_q\alpha) = k\psi_q(q\alpha) = kq(q\alpha) = r\alpha$ using 5.1 and the fact that Ker $H_p =$ Image E^2 when $[EK, S^{2pm-1}; p] = 0$.

Proof of Theorem 1.2(ii). For $n = 1$ the result is given by the group structure on $S^1 \cong U(1)$. For $n = 2t+1 \geqq 3$ we consider the cofibration

$$X^n_{2,1} \longrightarrow X^n_{3,2} \xrightarrow{p} X^n_{3,2}/X^n_{2,1}$$

and the associated exact sequence

$$[X^n_{3,2}/X^n_{2,1}, S^n] \xrightarrow{p^*} [X^n_{3,2}, S^n] \xrightarrow{i^*} [X^n_{2,1}, S^n].$$

The main results of [1] and [6] imply that $i^*\psi_r\alpha = 0$ for $r = 2^{\varphi(2t)}$ and so by exactness we obtain an element $\beta \in [X^n_{3,2}/X^n_{2,1}, S^n]$ with $p^*\beta = \psi_r\alpha$. So it is enough to show that $\psi_s\beta = 0$ for $s = 3^t$.

For $n = 3$ we have from Lemma 2.5 that $X^3_{3,2}/X^n_{2,1}$ is homeomorphic to the complex $S^6 \cup_f e^7$, degree $f = 3$—note $CS^2 \times CX^2_{2,1}$ is just a 7-cell since $X^2_{2,1}$ is a 3-cell. Hence the only obstructions to nullhomotoping a given map $S^6 \cup_f e^7 \xrightarrow{g} S^3$ lie in $H^6(S^6 \cup_f e^7; \pi_6(S^3)) \cong Z_3$ and $H^7(S^6 \cup_f e^7; \pi_7(S^3)) = 0$. But $\psi_3\colon \pi_6(S^3) \to \pi_6(S^3)$ is multiplication by 3 (since $E\colon \pi_6(S^3) \to \pi_7(S^4)$ is a monomorphism, or alternatively since $\pi_6(S^3) = \pi_6(S^3; 3)$ and we can invoke (13.13) of [5]). Thus $\psi_3\beta = 0$ as required.

For $n = 2t+1 \geqq 5$ we can apply 5.2 whenever the condition

$$[E^{-1}(X^n_{3,2}/X^n_{2,1}), S^{6t-1}; 3] = 0$$

obtains—in which case we again obtain the desired result $\psi_s\beta = s\beta = 0$ for $s = 3^t$ (using both Theorem 1.4(ii) and Lemma 5.2). Now $H^i(E^{-1}(X^n_{3,2}/X^n_{2,1}); Z) = 0$ for $i = 3n-4$ or $i > 3n-3$, and is $\cong Z_3$ for $i = 3n-3$. As dimension $E^{-1}(X^n_{3,2}/X^n_{2,1}) = 6t$ the Steenrod Classification Theorem [4, p. 460] immediately implies that

$$[E^{-1}(X^n_{3,2}/X^n_{2,1}), S^{6t-1}] = 0.$$

REMARKS. 1. It is possible to show that the suspension-order of $E^r X^n_{3,1}$, $r \geqq -1$ and $n = 2t+1$, is also $2^{\varphi(2t)}3^t$. Only minor modification of the hypotheses of Toda's Lemma 1.4 is needed for this. Indeed this was our first approach. However Lemma 3.1 showed that the suspension-order of $E^r(X^n_{3,2}/X^n_{2,1})$ would be more useful.

2. We conjecture that the lower bound for $k^{m,n}$ given in [7] is not best possible for any value $m > 3$, in particular for $m = 4$. It is really just a first-order result. Although we feel that K-theory will provide the best possible lower bound for all m, higher-order considerations (in the use of coefficients K-theory) must enter for $m > 3$. Finally we conjecture that for each prime p, the largest power p^v of p dividing $k^{m,n}$ is an unbounded, nondecreasing function of m (for fixed $n = 2t+1 \geqq 3$). The lower bound of [7] does *not* reflect this kind of behavior.

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