THE PLANE IS NOT COMPACTLY GENERATED BY A FREE MAPPING

BY

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Abstract. Let X denote the plane, or the closed half-plane, and let $T: X \to X$ be a self homeomorphism which preserves orientation and has no fixed points. It is proved that, if A is any compact subset of X, then there exists an unbounded connected subset B of X which does not meet $T^n(A)$ for any integer n.

0. Introduction. The aim of this paper is to furnish a proof of the following result.

THEOREM. *Let X be the plane, or a closed half-plane.*

Let $T: X \to X$ be a homeomorphism which preserves orientation and has no fixed points. Then for any compact set $A \subset X$ there exists an unbounded connected set $B \subset X$ which avoids $\bigcup_{-\infty}^{+\infty} T^n A$.

This theorem was used by the author in [3] and [4], where flows and topological translations of the plane were studied.

For the duration of the paper we consider only the case $X = \mathbf{R}^2$. The extension of the theorem to half-planes will be deduced in the final section.

The first main step in proving the theorem is to replace the orientation condition by something more workable.

PROPOSITION 0. *Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be a homeomorphism without fixed points. Then the following conditions are equivalent:*

(a) *T preserves orientation.*

(b) *Let C be any arcwise connected subset of $\mathbf{R}^2$. If $TC \cap C = \varnothing$ then $T^n C \cap C = \varnothing$ for all $n \neq 0$.*

Proof. The implication (a) $\Rightarrow$ (b) is shown in [4, pp. 62–63] to be an elementary generalization of the following result of Brouwer:

LEMMA. *Let T be an orientation-preserving homeomorphism, without fixed points, of the plane $\mathbf{R}^2$ onto itself. Let $p \in \mathbf{R}^2$, and let $\alpha \subset \mathbf{R}^2$ be an arc with endpoints p and Tp. Put $\alpha_1 = \alpha - \{Tp\}$. Then $\alpha_1 \cap T\alpha_1 = \varnothing$ implies $\alpha_1 \cap T^n \alpha_1 = \varnothing$ for all $n \neq 0$.*

Brouwer's lemma and its proof appear in [1, pp. 38–40]. (The proof proceeds by a somewhat intricate computation of a certain winding number.)

The implication (b) $\Rightarrow$ (a) is not vital to our purposes; it can, however, be deduced quite easily from the material in §2.

In order to formalize the reduction of the problem which is achieved by Proposition 0, we adopt the following definition.

DEFINITION. A homeomorphism T of the plane R^2 onto itself is called a *free mapping* if

(1) T has no fixed points.

(2) If C is any arcwise connected subset of R^2 which satisfies $C \cap TC = \varnothing$, then $C \cap T^n C = \varnothing$ for all $n \neq 0$.

Throughout the paper we will be dealing with a single free mapping, which will always be denoted by T.

In the Appendix at the end of the paper we have listed, without proof, some of the basic facts about plane topology which we use.

1. Here, and in the succeeding section, we review some of the work of Brouwer in [1].

PROPOSITION 1.1. *Let $p \in R^2$. Then $T^n p \to \infty$ as $n \to \pm\infty$.*

Proof. Let $q \in R^2$ be arbitrary; we show that q is not an accumulation point for $\{T^n p\}$. Since $Tq \neq q$, there is an open disc D with $q \in D$ and $TD \cap D = \varnothing$. Since T is free, we conclude that $T^n D \cap D = \varnothing$ for all $n \neq 0$.

Hence D can contain no more than one point from the sequence $\{T^n p\}$. Therefore q is not an accumulation point for $\{T^n p\}$. Q.E.D.

REMARK. The proposition implies that for no bounded set $E \subset R^2$ is it true that $TE \subset E$ or $T^{-1}E \subset E$.

DEFINITION. A set $F \subset R^2$ is called a *flow line* for T if

(1) $TF = F$.

(2) F is the image of a one-to-one continuous function sending R into R^2.

PROPOSITION 1.2. *Flow lines exist.*

Proof. Let $D \subset R^2$ be an open disc which satisfies $D \cap TD = \varnothing$ and $\bar{D} \cap \text{Cl}(TD) \neq \varnothing$. Then there is a point p on the boundary of D whose image Tp is also on the boundary of D.

Let m be any arc whose endpoints are p and Tp and which satisfies $m - \{p, Tp\} \subset D$. Then set $F = \bigcup_{-\infty}^{+\infty} T^n m$.

F is, in an obvious way, a continuous image of R. One sees that $m_1 = m - \{Tp\}$ is disjoint from its first image under T; hence, by freeness, $F = \bigcup_{-\infty}^{+\infty} T^n m_1$ is a disjoint union. This shows that F has no self-intersections.

Finally, F obviously satisfies $TF = F$. Q.E.D.

REMARK. Let $G \subset R^2$ be the graph of $y = (1/x) \sin(1/x)$ for $x > 0$.

The set G actually occurs as a flow line for a certain free mapping; it shows, for example, that a flow line need not be a closed subset of R^2.

For the remainder of this section, F will denote a single given flow line.

PROPOSITION 1.3. *Let $C \subset R^2$ be arcwise connected and satisfy $C \cap TC = \varnothing$. Let $F \subset R^2$ be a flow line, p a point of F, and $m \subset F$ the closed arc from p to Tp along F.*
Suppose $C \cap m \neq \varnothing$, but that $C \cap T^{-1}m = \varnothing$ and $C \cap Tm = \varnothing$.
Then $C \cap F = C \cap m$.

Proof. Put $m_1 = m - \{Tp\}$. Then $m_1 \cap Tm_1 = \varnothing$, and $\bigcup_{-\infty}^{+\infty} T^n m_1 = F$.

Now put $K = m_1 \cup C$. Since $C \cap m_1 \neq \varnothing$, K is arcwise connected. Since the four sets $C \cap TC$, $m_1 \cap Tm_1$, $C \cap T^{-1}m_1$, and $C \cap Tm_1$ are all empty, we have $K \cap TK = \varnothing$.

Therefore $K \cap T^n K = \varnothing$ for all $n \neq 0$. In particular, $C \cap T^n m_1 = \varnothing$ for all $n \neq 0$. Since $F = \bigcup_n T^n m_1$, we have $C \cap F = C \cap m_1 = C \cap m$.   Q.E.D.

PROPOSITION 1.4. *Let w be any subarc of the flow line F, and let q be an interior point of w. Then there is an open set $D \subset R^2$ with $q \in D$ and $D \cap F = D \cap w$.*

Proof. Choose $p \in F$ so that $q \in m$ where $m \subset F$ is the subarc determined by p and Tp. The sets $T^{-1}m$ and Tm are compact, and do not contain p. If D is any disc centered at p which avoids $T^{-1}m$ and Tm then it follows from Proposition 1.3 that $D \cap F = D \cap m$.

Let D be chosen small enough so that $D \cap m = D \cap w$, as well as $D \cap Tm = D \cap T^{-1}m = \varnothing$. This D satisfies the requirements of Proposition 1.4.   Q.E.D.

PROPOSITION 1.5. *Let F be a flow line, and let C be a compact arcwise connected set which satisfies $C \cap \bar{F} = $ two points $= \{a, b\}$. Suppose in addition that $C \cap TC = \varnothing$. Then a and b are either both in F or both in $\bar{F} - F$.*

Proof. Suppose $a \in F$, and suppose if possible that $b \in \bar{F} - F$.

Then in particular $C \cap F = \{a\}$.

Choose $p \in F$ so that $a \in m$, where $m \subset F$ is the subarc determined by p and Tp. Then $C \cap m \neq \varnothing$ while $C \cap T^{-1}m = \varnothing$ and $C \cap Tm = \varnothing$.

Because the relevant sets are compact, there exists a connected open set $\mathcal{O}$, containing C, such that $\mathcal{O} \cap T\mathcal{O}$, $\mathcal{O} \cap T^{-1}m$, and $\mathcal{O} \cap Tm$ are all empty.

By Proposition 1.3, $\mathcal{O} \cap F = \mathcal{O} \cap m$. Hence $\mathcal{O} \cap \bar{F} = \mathcal{O} \cap \bar{m} = \mathcal{O} \cap m$. But this is impossible since $\mathcal{O}$ contains b, where $b \in \bar{F}$ but $b \notin F$.   Q.E.D.

NOTATION. $D_r(p)$ is the open disc with radius $r > 0$ and center $p \in R^2$.

DEFINITION. A point $p \in R^2 - \bar{F}$ is called an *edge point* of F if, for some $r > 0$, we have $D_r(p) \cap F \neq \varnothing$ and $D_r(p) \cap T(D_r(p)) = \varnothing$.

DEFINITION. Let p be an edge point of F, and let $D_r(p)$ satisfy $D_r(p) \cap F \neq \varnothing$ and $D_r(p) \cap T(D_r(p)) = \varnothing$. Then there exists a compact arcwise connected set C, $p \in C \subset D_r(p)$, with $C \cap \bar{F} = \{$one point$\} = a$. Then p is of *type F* or *type $\bar{F} - F$* according as $a \in F$ or $a \in \bar{F} - F$.

PROPOSITION 1.6. *Edge point type is well-defined.*

Proof. Suppose that C_1 and C_2 are compact arcwise connected sets with $p \in C_1 \cap C_2$ and $C_i \cap \bar{F} = \{a_i\}$ for $i = 1, 2$. If the C_i arise from the definition of edge

point type, then each C_i is contained in a disc $D_{r_i}(p)$ which satisfies $D_{r_i}(p) \cap T(D_{r_i}(p)) = \varnothing$. Put $K = C_1 \cup C_2$; we have $K \cap TK = \varnothing$ because K is contained in the larger of the two discs. K is connected since $C_1 \cap C_2 \neq \varnothing$; we also have $K \cap \bar{F} = \{a_1, a_2\}$.

By Proposition 1.5, a_1 and a_2 are both in F or both in $\bar{F} - F$.   Q.E.D.

REMARK. It is easy to see that the set of all edge points of F is open, and that the edge point type is constant on the connected components of this open set.

PROPOSITION 1.7. *Let α be an arc with endpoints p and q. Suppose $\alpha \cap \bar{F} = \{p, q\}$. Then p and q are either both in F or both in $\bar{F} - F$.*

Proof. We begin by assuming that α is a *polygonal* arc.

Suppose $p \in F$ and $q \in \bar{F} - F$. Choose $r > 0$ so that $D_{4r}(q) \cap \alpha$ is a straight line segment, and $D_{4r}(q) \cap T(D_{4r}(q)) = \varnothing$.

Choose $a \in F$ so that the arc $F_1 \subset F$ determined by p and a satisfies $F_1 \cap \mathrm{Cl}\,(D_r(q)) = \{a\}$. Let σ be the straight line segment with endpoints a and q; then $\Gamma = \alpha \cup \sigma \cup F_1$ is a Jordan curve.

Put $K = \sigma \cup \mathrm{Cl}\,(F \cap \mathrm{int}\,\Gamma)$; K is compact and connected. Moreover, we claim that $K \cap \Gamma = \sigma$. Indeed, $K \cap \alpha = \{q\}$ by the assumption that $\alpha \cap \bar{F} = \{p, q\}$, and $K \cap F_1 = \{a\}$ by an application of Proposition 1.4.
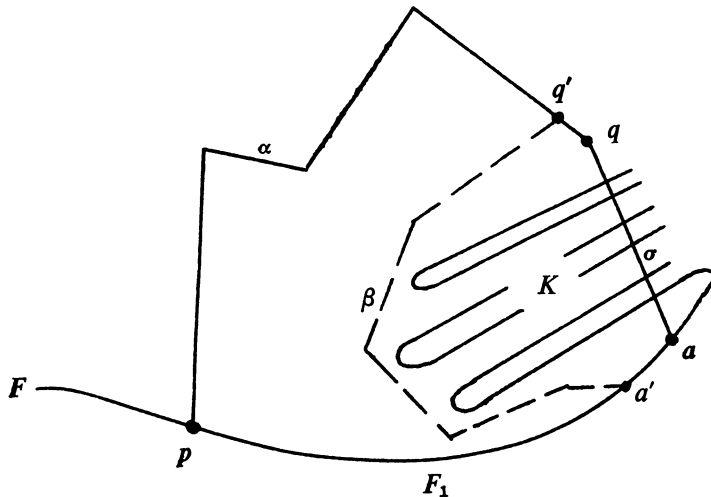
Now choose ε so that $0 < \varepsilon < r$, and $D_\varepsilon(x) \cap T(D_\varepsilon(x)) = \varnothing$ for all $x \in \mathrm{Cl}\,(\mathrm{int}\,\Gamma)$. By App. 5, there is an arc β, with endpoints q' and a', so that

(1) $\beta - \{a', q'\} \subset (\mathrm{int}\,\Gamma) - K$.

(2) $a' \in F_1 - \sigma$ and $q' \in \alpha - \sigma$.

(3) $D_\varepsilon(x)$ meets K for all $x \in \beta$.

It follows that $\beta \cap \bar{F} = \varnothing$. Furthermore, each point x of β is an edge point of F:

if $D_\varepsilon(x)$ meets F this claim is obvious; if $D_\varepsilon(x)$ meets σ then $D_{2r}(x)$ will meet $q \in \bar{F}$ and still satisfy $D_{2r}(x) \cap T(D_{2r}(x)) = \varnothing$.

Hence all the points of β are edge points. But $q' \in \beta$ is of type $\bar{F} - F$, while points of β near a' are obviously of type F. By the Remark following Proposition 1.6, we have reached a contradiction. This completes the proof for the case where α is polygonal.

Suppose now that α is a general arc. Because of the local constancy of edge point type, there would exist a *polygonal* arc α', with endpoints p' and q', such that $\alpha' \cap \bar{F} = \{p', q'\}$ with $p' \in F$ and $q' \in \bar{F} - F$. This is the case just treated, hence the proof is complete.   Q.E.D.

For the remainder of this section we consider a single flow line F, and a chosen subarc $m \subset F$ which satisfies $m \cap Tm \neq \varnothing$.

Let $m_0 = m - \{u, v\}$ where u and v are the endpoints of m.

DEFINITION. The *rays* of F are the two arcwise connected components of $F - m_0$.

PROPOSITION 1.8. *Let $f \subset F$ be a ray. Then $F \cap \bar{f} = f$.*

Proof. Let $q \in F - f$. Then q is in the interior of a subarc β of F which satisfies $\beta \cap f = \varnothing$. By Proposition 1.4, q is contained in a disc D which satisfies $D \cap F = D \cap \beta$; hence in particular $D \cap f = \varnothing$.

Thus $D \cap \bar{f} = \varnothing$, and we have shown that $q \notin \bar{f}$.   Q.E.D.

From now on we denote the two rays of F by i and f; for definiteness let the notation be chosen so that $i \subset Ti$ and $f \supset Tf$.

PROPOSITION 1.9. *Let C be an arcwise connected set which satisfies $C \cap \bar{F} = $ two points $= \{a, b\}$. Suppose in addition that $C \cap TC = \varnothing$. Then it is impossible that $a \in i$ and $b \in f$.*

Proof. Suppose $a \in i$ and $b \in f$. Then it is possible to choose $p \in F$ so that $\{p, a, Tp, T^2p, b\}$ is the order in which these five points occur along F.

Let $\alpha \subset F$ be the closed subarc determined by p and Tp. Then $C \cap \alpha \neq \varnothing$ while $C \cap T^{-1}\alpha$ and $C \cap T\alpha$ are both empty. By Proposition 1.3, $C \cap F = C \cap \alpha = \{a\}$. But this contradicts $b \in C \cap F$.   Q.E.D.

REMARK. Proposition 1.9 affords the following refinement of edge point type. Let p be an edge point of F, of type F. From the definition, there is an open disc D centered at p so that $D \cap F \neq \varnothing$ and $D \cap TD = \varnothing$.

But suppose D can be chosen so that it also satisfies $D \cap m = \varnothing$. Then p is of *type i* or *type f*, according as the set C in the definition of edge point type meets i or f.

This "refined" edge point type is well defined, by a straightforward application of Proposition 1.9. Moreover, it is locally constant wherever it is defined, i.e., on those points p of $R^2 - \bar{F}$ which admit open discs $D_r(p)$ satisfying $D_r(p) \cap F \neq \varnothing$, $D_r(p) \cap T(D_r(p)) = \varnothing$, and $D_r(p) \cap m = \varnothing$.

PROPOSITION 1.10. $i \cap \bar{f} = \varnothing$.

Proof. Suppose $p \in \bar{i} \cap \bar{f}$. Then in particular $p \notin m$. Choose $r > 0$ so that $D_{4r}(p)$ avoids m and satisfies $D_{4r}(p) \cap T(D_{4r}(p)) = \varnothing$.

Choose $x \in i$ and $y \in f$ so that the subarc $F_1 \subset F$ they determine satisfies $F_1 \cap \mathrm{Cl}\,(D_r(p)) = \{x, y\}$. Let σ be the straight line segment determined by x and y. Then $\Gamma = \sigma \cup F_1$ is a Jordan curve.
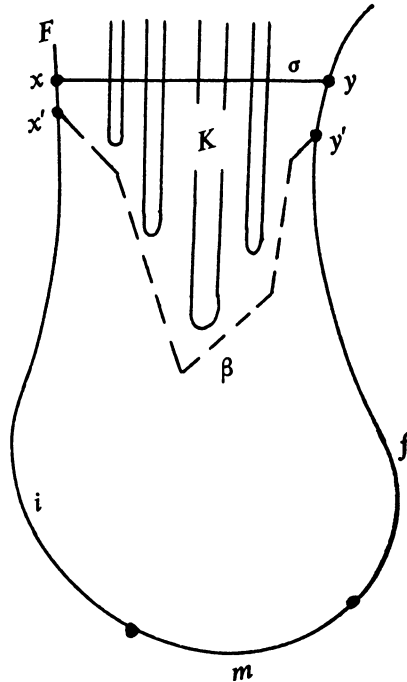
Put $K = \sigma \cup \mathrm{Cl}\,(F \cap \mathrm{int}\,\Gamma)$. Then K is compact, connected, and satisfies $K \cap \Gamma = \sigma$.

Let η denote the distance from K to m. Choose ε so that $0 < \varepsilon < r$, $\varepsilon < \frac{1}{2}\eta$, and $D_\varepsilon(q) \cap T(D_\varepsilon(q)) = \varnothing$ for all $q \in \mathrm{Cl}\,(\mathrm{int}\,\Gamma)$.

By App. 5, there is an arc β with endpoints x' and y', such that

(1) $\beta - \{x', y'\} \subset (\mathrm{int}\,\Gamma) - K$.

(2) $x' \in i - \sigma$ and $y' \in f - \sigma$.

(3) $D_\varepsilon(z) \cap K \neq \varnothing$ for all $z \in \beta$.

Now each point $z \in \beta$ is the center of some disc D which satisfies $D \cap F \neq \varnothing$, $D \cap m = \varnothing$, and $D \cap TD = \varnothing$. Indeed, if $D_\varepsilon(z)$ meets F, then $D_\varepsilon(z)$ itself will do. Otherwise, by (3), $D_\varepsilon(z)$ meets σ; in this case $D_{2r}(z)$ will do.

Consequently the *refined* edge point type is defined for all points of β. By inspection, β contains points of type i and of type f. Therefore we have reached a contradiction to the Remark following Proposition 1.9.   Q.E.D.

PROPOSITION 1.11. *Let α be an arc with endpoints p and q. Suppose $\alpha \cap \bar{F} = \{p, q\}$. If $p, q \in \bar{F} - F$, then p and q are either both in $\bar{i}$ or both in $\bar{f}$.*

Proof. Suppose $p \in i$ and $q \in \bar{f}$. Because $i \cap \bar{f} = \varnothing$, one can choose $r > 0$ such that

(1) $D_r(p) \cap \alpha$ and $D_r(q) \cap \alpha$ are straight line segments.

(2) $D_r(p)$ avoids $m \cup \bar{f}$, and $D_r(q)$ avoids $m \cup i$.

Let $x \in i$ and $y \in f$ be chosen so that the subarc $F_1 \subset F$ they determine satisfies $F_1 \cap \text{Cl}(D_r(p)) = \{x\}$ and $F_1 \cap \text{Cl}(D_r(q)) = \{y\}$. Let σ_p and σ_q be the straight line segments determined by p and x, q and y, resp. Finally put $\Gamma = \alpha \cup \sigma_p \cup \sigma_q \cup F_1$; it is a Jordan curve.

Put $K_i = \sigma_p \cup \text{Cl}(i \cap \text{int } \Gamma)$ and $K_f = \sigma_q \cup \text{Cl}(f \cap \text{int } \Gamma)$. The choice of r, together with the fact that $i \cap \bar{f} = \varnothing$, implies that $K_i \cap K_f = \varnothing$. Moreover $K_i \cap \Gamma = \sigma_i$ and $K_f \cap \Gamma = \sigma_f$; this follows from Proposition 1.4, the argument being the same as in the proof of Proposition 1.7.

By App. 4 there exists a polygonal arc β such that $\beta \cap (K_i \cup K_f) = \varnothing$, $\beta \cap \text{ext } \Gamma = \varnothing$, and β meets both components of $\Gamma - (K_i \cup K_f)$. But now β followed by α would give an arc γ which violated Proposition 1.7; except for its endpoints, γ would avoid $\bar{F}$, and one endpoint would be in F while the other would be in $\bar{F} - F$. This completes the proof for polygonal α.

The extension to general arcs α is deduced from the polygonal case in the same way as in the proof of Proposition 1.7; one uses here the fact that $i \cap \bar{f} = \varnothing$. Q.E.D.

The following is a summary of the results of §1.

SUMMARY. Let F be a flow line, $m \subset F$ a closed subarc satisfying $m \cap Tm \neq \varnothing$, and let $f, i \subset F$ be the associated rays.

Let $\alpha \subset R^2$ be an arc with endpoints p and q; suppose that $\alpha \cap \bar{F} = \{p, q\}$.

Then exactly one of the following three statements is true:

(1) p and q are both in F.

(2) p and q are both in $i - F$.

(3) p and q are both in $\bar{f} - F$.

In particular, $i \cap \bar{f} = \varnothing$.

2.   Let F be a flow line. To minimize technicalities, we assume that F arises in the following way.

Let D be an open disc which satisfies $D \cap TD = \varnothing$ and $\bar{D} \cap \mathrm{Cl}\,(TD) \neq \varnothing$. Then the boundary of D contains a pair of points of the form $\{p, Tp\}$. Let m be a polygonal arc, with endpoints p and Tp, which satisfies $m - \{p, Tp\} \subset D$. Then we shall consider the flow line $F = \bigcup_{-\infty}^{+\infty} T^n m$.

DEFINITION. A component Ω of $R^2 - F$ is called a *side domain* of F if there exists a line segment σ with endpoint q such that $q \in F$ and $\sigma - \{q\} \subset \Omega$.

PROPOSITION 2.1. *There exist at least two side domains.*

Proof. Let D_1 and D_2 be the two components of $D - m$. Since $D_n \cap \bar{F} = m$, D_1 and D_2 determine side domains Ω_1 and Ω_2. If Ω_1 and Ω_2 were the same, a polygonal Jordan curve Γ would exist which intersected F transversally in a single point of m. Then one of the two components of $F - \Gamma$ would be contained in int Γ. This, however, is impossible because int Γ is bounded while both components of $F - \Gamma$ are unbounded.   Q.E.D.

PROPOSITION 2.2. *The two side domains which are determined by the components of $D - m$ are invariant under T.*

Proof. Let D_1 be one of the components of $D - m$, and let $\Omega \supset D_1$ be the side domain it determines.

Let i and f be the components of $F - m$. There exists an arc α, with endpoints a and b, such that $\alpha \cap D_1 \neq \varnothing$ and $\alpha \cap \bar{F} = \{a, b\}$, with $a \in i$ and $b \in f$. Now $\alpha - \{a, b\} \subset \Omega$. By Proposition 1.8 we have $\alpha \cap T\alpha \neq \varnothing$. Thus, in particular, $\Omega \cap T\Omega \neq \varnothing$.

Now Ω is a component of $R^2 - \bar{F}$; hence either $\Omega \cap T\Omega = \varnothing$ or $\Omega = T\Omega$. The first possibility is the one we have just ruled out.   Q.E.D.

PROPOSITION 2.3. *F has precisely two side domains, and each is invariant under T.*

Proof. Let Ω be an arbitrary side domain of F. Then there is an arc γ, with endpoint $q \in F$, such that $\gamma - \{q\} \subset \Omega$. We can assume without loss of generality that q is not an iterate of p; since $\lim_{n \to \infty} T^n p = \infty$, the original γ could be diverted slightly so that it meets F in $F - \bigcup T^n p$.

Thus, some point of $m - \{p, Tp\}$ is a boundary point of $T^n \Omega$ for some integer n. Hence $T^n \Omega$ meets D_1 or D_2 and hence $T^n \Omega = \Omega_1$ or Ω_2. By Proposition 2.2 we have $\Omega = \Omega_1$ or Ω_2.   Q.E.D.

PROPOSITION 2.4. *Let Ω be a side domain of F. Let σ be a curve with endpoint p, such that $\sigma - \{p\} \subset \Omega$ and $p \in \bar{F}$. Then actually $p \in F$.*

Proof. By the definition of side domain, one can extend σ to an arc α, with endpoints p and q, where $\alpha \cap \bar{F} = \{p, q\}$ and $q \in F$. The conclusion now follows by the result of §1.   Q.E.D.

THEOREM 2.5. *Let Γ be a polygonal Jordan curve with $m \subset$ int Γ. Let Ω be a side domain of F. Then Γ contains a subarc γ, with endpoints a and b, such that $\gamma \cap \bar{F} = \{a, b\}$, $\gamma - \{a, b\} \subset \Omega$, $a \in i$, and $b \in f$.*

Proof. Since $m \subset$ int Γ, we have $\Gamma \cap \bar{F} = (\Gamma \cap \bar{i}) \cup (\Gamma \cap \bar{f})$. Say a closed subarc I of Γ is *crossing* if I meets both $\bar{i}$ and $\bar{f}$, but no proper subarc of I meets $\bar{i}$ and $\bar{f}$. Because $\bar{i} \cap \bar{f} = \varnothing$, there is a nonzero distance from $\Gamma \cap \bar{i}$ to $\Gamma \cap \bar{f}$; hence there are only finitely many crossing subarcs. Let them be denoted by $I_1, I_2, \ldots, I_k$.

Let I_k be a typical crossing arc, with endpoints a and b. We have $I_k \cap \bar{F} = \{a, b\}$ with $a \in \bar{i}$ and $b \in \bar{f}$. But by Proposition 1.11, we actually have $a \in i$ and $b \in f$. Thus, the theorem will be proved once we show that among the $\{I_k\}$ there is one that meets Ω.

Put $I = \bigcup_{k=1}^{N} I_k$. The components of $\Gamma - I$ are open subarcs; let J be the closure of a typical such subarc. Then J cannot meet both $\bar{i}$ and $\bar{f}$, for otherwise a proper subarc of J would be among the $\{I_k\}$. On the other hand, J does meet $\bar{i} \cup \bar{f}$, because of the definition of crossing arc.

Thus the components of $\Gamma - I$ are classified according to whether their closures meet $\bar{i}$ or $\bar{f}$. Let $J^i \subset \Gamma$ be the union of those J's which meet $\bar{i}$; in like manner define $J^f \subset \Gamma$.

One has the relations $I \cup J^i \cup J^f = \Gamma$ and $(\bar{i} \cup J^i) \cap (\bar{f} \cup J^f) = \varnothing$.

Now choose closed subarcs $F^i \subset i$ and $F^f \subset f$; let them be large enough so that they include all endpoints of all crossing arcs. Put $C^i = F^i \cup J^i$, $C^f = F^f \cup J^f$, and $K = F^i \cup F^f \cup I$. Then C^i, C^f, and K are compact connected sets.

Now $C^i \cap C^f = \varnothing$, while $C^i \cap K$ and $C^f \cap K$ are both connected. Indeed, $C^i \cap K = F^i \cap K = F^i$, a subarc of the flow line F.

Let q be a point of m; we wish to show that $q \in$ int (K) where in general the notation $q \in$ int (C) means that q is in a bounded component of the complement of the closed set C.

Now, $q \in$ ext C^f, since q is on the unbounded set $i \cup m$ which avoids C^f. Likewise $q \in$ ext C^i.

Now suppose, if possible, that $q \in$ ext K. Since $K \cap C^f$ is connected, we could conclude by App. 2 that $q \in$ ext $K \cup C^f$. But $(K \cup C^f) \cap C^i$ is also connected; hence a second application of App. 2 would lead to the conclusion that

$$q \in \text{ext } (K \cup C^f \cup C^i).$$

This is impossible because $K \cup C^f \cup C^i$ contains Γ and we already know that $q \in$ int Γ.

Thus $q \in$ int K. It now follows that $I \cap \Omega \neq \varnothing$; for otherwise the unbounded connected set Ω would avoid K, and consequently points arbitrarily close to q would be in ext K.
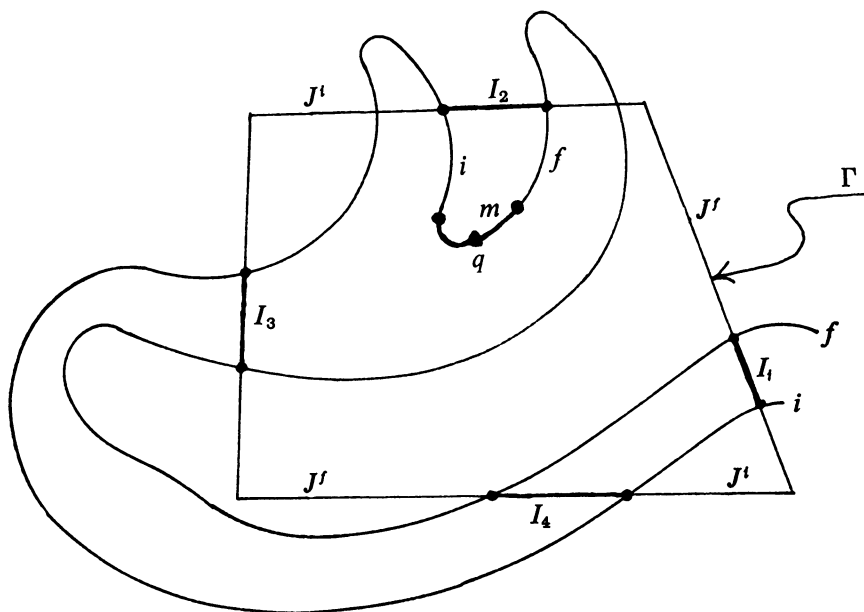
Hence $I_k \cap \Omega \neq \varnothing$ for some crossing arc I_k. This completes the proof of the theorem.   Q.E.D.

3.   Throughout the present section F_1 and F_2 will be two flow lines which satisfy $F_1 \cap F_2 = \varnothing$.

DEFINITION. An arc α with endpoints a_1 and a_2 is called a *crosscut* if $\alpha \cap F_n = \{a_n\}$ for $n = 1, 2$.

PROPOSITION 3.1. *Suppose F_1 and F_2 have a crosscut α. Then $\overline{F}_1 \cap F_2 = F_1 \cap \overline{F}_2 = \varnothing$, and $\alpha \cap \mathrm{Cl}\,(F_n) = \{a_n\}$, for $n = 1, 2$.*

Proof. By hypothesis $(\alpha \cup F_1) \cap F_2 = \{a_2\}$.

Since $\alpha \cup F_1$ is arcwise connected it meets $\mathrm{Cl}\,(F_2)$ only in the single point a_2, for otherwise $\alpha \cup F_1$ would contain an arc which violated Proposition 1.7.   Q.E.D.

DEFINITION. Let α and β be crosscuts. We define $K(\alpha, \beta)$ to be the set $\alpha \cup \beta \cup S_1 \cup S_2$, where $S_n \subset F_n$ is the subarc determined by the endpoints of α and β.

REMARK. $K(\alpha, \beta)$ is not in general a Jordan curve, since $\alpha \cap \beta \neq \varnothing$ is still possible. But the notation int $K(\alpha, \beta)$ will still be employed to designate any bounded component of $\mathbf{R}^2 - K(\alpha, \beta)$.

The following is the main result of this section.

THEOREM 3. *Let α be a polygonal crosscut which satisfies $\alpha \cap T\alpha = \varnothing$. Let p be a point of α.*

Let β be any other crosscut. Then for certain integers r and s we have

$$p \in \mathrm{int}\, K(T^r\beta, T^s\beta).$$

The proof of Theorem 3 begins with the following facts. (We use $F_n(a)$ to designate the rays of a flow line F which are determined by a point $a \in F$.)

PROPOSITION 3.2. *Let $F_1(a_1) \subset F_1$ and $F_2(a_2) \subset F_2$ be any choice of rays. Then $T^{-n}p$ and $T^n p$ are separated by $\mathrm{Cl}\,(F_1(a_1)) \cup \alpha \cup \mathrm{Cl}\,(F_2(a_2))$ for every $n > 0$.*

Proof. Since $T\alpha \cap \alpha = \varnothing$, we have that $K(T^{-n}\alpha, \alpha)$ and $K(\alpha, T^n\alpha)$ are both Jordan curves; moreover their intersection is just α.

By App. 6, their interior domains are either disjoint or nested. But T^n sends int $K(T^{-n}\alpha, \alpha)$ onto int $K(\alpha, T^n\alpha)$; hence the interiors cannot be nested or else we would have a bounded set taken into itself by T^n.

Therefore int $K(T^{-n}\alpha, \alpha) \cap$ int $K(\alpha, T^n\alpha) = \varnothing$. Consequently, if σ is a short line segment which meets α transversally in a single point, then σ meets both int $K(T^{-n}\alpha, \alpha)$ and int $K(\alpha, T^n\alpha)$.

It follows that int $K(T^{-n}\alpha, \alpha)$ and int $K(\alpha, T^n\alpha)$ are separated by $\mathrm{Cl}\,(F_1(a_1)) \cup \alpha \cup \mathrm{Cl}\,(F_1(a_2))$; for otherwise there would exist a polygonal Jordan curve Γ whose intersection with $\mathrm{Cl}\,(F_1(a_1)) \cup \alpha \cup \mathrm{Cl}\,(F_2(a_2))$ would consist precisely of the transversal intersection $\sigma \cap \alpha$. But this is absurd because one of the two unbounded sets $F_n(a_n)$ would then lie in int Γ.

Now the points $T^{-n}p$ and $T^n p$ are, respectively, in the same components of $\mathbf{R}^2 - (\mathrm{Cl}\,(F_1(a_1)) \cup \alpha \cup \mathrm{Cl}\,(F_2(a_2)))$ as int $K(T^{-n}\alpha, \alpha)$ and int $K(\alpha, T^n\alpha)$. Hence they too are separated by $\mathrm{Cl}\,(F_1(a_1)) \cup \alpha \cup \mathrm{Cl}\,(F_2(a_2))$.   Q.E.D.

PROPOSITION 3.3. *Let α and β be any crosscuts.*
Put $C = \alpha \cup \mathrm{Cl}\,(F_1(a_1)) \cup \mathrm{Cl}\,(F_2(a_2))$, where the $F_n(a_n) \subset F_n$ are any choice of rays emanating from the endpoints of α.
Then $C \cap K(\alpha, \beta)$ is connected.

Proof. First $K(\alpha, \beta) = \alpha \cup \beta \cup S_1 \cup S_2$, where $S_n \subset F_n$ is the subarc determined by the endpoints a_n and b_n of α, β, respectively.

By Proposition 3.1, $\mathrm{Cl}\,(F_1)$ meets $\alpha \cup \beta \cup F_2$ only in the points a_1 and $b_1 \in F_1$. Therefore $\mathrm{Cl}\,(F_1) \cap K(\alpha, \beta) = S_1$, and hence $\mathrm{Cl}\,(F_1(a_1)) \cap K(\alpha, \beta) = S_1 \cap \mathrm{Cl}\,(F_1(a_1))$. But we know from Proposition 1.8 that $\mathrm{Cl}\,(F_1(a_1)) \cap F_1 = F_1(a_1)$; therefore

$$\mathrm{Cl}\,(F_1(a_1)) \cap K(\alpha, \beta) = S_1 \cap F_1(a_1).$$

Thus $C \cap K(\alpha, \beta) = \alpha \cup (S_1 \cap F_1(a_1)) \cup (S_2 \cap F_2(a_2))$. Being, therefore, the union of α with two arcs that meet α, $C \cap K(\alpha, \beta)$ is arcwise connected.   Q.E.D.

PROPOSITION 3.4. *There exists $N_0 > 0$ such that, for all $n > N_0$, the points $T^{-n}p$ and $T^n p$ are separated by the set $C = \beta \cup \mathrm{Cl}\,(F_1(b_1)) \cup \mathrm{Cl}\,(F_2(b_2))$, where the $F_i(b_i) \subset F_i$ are any choice of rays.*

Proof. Set $K = K(\alpha, \beta)$. Choose $N_0 > 0$ so that $n \geqq N_0$ implies that $T^{-n}p$ and $T^n p$ are not separated by K. This is possible because K is bounded and $T^n p \to \infty$ as $n \to \pm\infty$.

We prove the lemma for all $n \geq N_0$. Suppose, if possible, that $T^n p$ and $T^{-n} p$ were not separated by C. Now K is compact, C is closed, and $C \cap K$ is connected by Proposition 3.3. Therefore, by App. 2, $T^{-n} p$ and $T^n p$ would not be separated by $C \cup K$.

But $C \cup K$ contains a set of the form $\alpha \cup Cl\,(F_1(a_1)) \cup Cl\,(F_2(a_2))$. By Proposition 3.2, $T^{-n} p$ and $T^n p$ are separated by any such set; therefore they must also be separated by the larger set $C \cup K$.

This completes the proof.   Q.E.D.

PROPOSITION 3.5. *There exist integers u and v, and a choice of rays $F_1(b_1) \subset F_1$ and $F_2(b_2) \subset F_2$, such that the points $T^u p$ and $T^v p$ are not separated by*

$$\beta \cup Cl\,(F_1(b_1)) \cup Cl\,(F_2(b_2)).$$

Furthermore, the choice of integers and rays can be made so that $u - v = 2k$, where k is an integer larger than any preassigned $N > 0$.

Proof. Let D be an open disc which contains β. Let k be any integer which is large enough so that $n > k$ implies $T^n p \notin \bar{D}$.

Our integers u and v will be chosen from the set $\Delta = \{2k, 4k, 6k, 8k, 10k\}$. (The condition that $u - v$ be a large even integer will obviously be satisfied.)

Consider a given $n \in \Delta$. Since $T^n p \notin \bar{D}$, there exists a polygonal arc γ, with endpoints $T^n p$ and q, such that $\gamma \cap \beta = \varnothing$ and $\gamma \cap (\bar{F}_1 \cup \bar{F}_2) = \{q\}$.

By Proposition 3.1, $p \in \Omega_1 \cap \Omega_2$, where Ω_k is that side domain of F_k which contains α. Hence, by invariance of side domains, the arc γ lies, except for its endpoint q, in $\Omega_1 \cap \Omega_2$. By Proposition 2.3, q actually lies in $F_1 \cup F_2$, and thus lies on one of the four rays determined by the points $b_k \in F_k$ for $k = 1, 2$.

Now Δ has five elements while there are only four possible rays. Hence let u and $v \in \Delta$ be chosen so that $T^u p$ and $T^v p$ have connecting arcs γ_u and γ_v which run to the same ray.

Now choose rays $F_1(b_1) \subset F_1$ and $F_2(b_2) \subset F_2$ so that $F_1(b_1) \cup F_2(b_2)$ does not meet $\gamma_u \cup \gamma_v$. This is possible because only one of the four possible rays meets $\gamma_u \cup \gamma_v$.

Thus $T^u p$ is connected to $T^v p$ by the arc $\gamma_u \cup S \cup \gamma_v$, where S is a subarc of one of the flow lines; by construction $F_1(b_1) \cup \beta \cup F_2(b_2)$ is disjoint from

$$\gamma_u \cup S \cup \gamma_v.$$

To complete the proof we must check that $Cl\,(F_1(b_1)) \cup Cl\,(F_2(b_2)) \cup \beta$ avoids $\gamma_u \cup \gamma_v \cup S$; but this is easily deduced from Propositions 3.1 and 1.8.   Q.E.D.

Proof of Theorem 3. Let $N > 0$ be chosen so that $n > N$ implies that $T^n p$ and $T^{-n} p$ are separated by $\beta \cup Cl\,(F_1(b_1)) \cup Cl\,(F_2(b_2))$, where $F_n(b_n)$, $n = 1, 2$, are any choice of rays. This is possible by Proposition 3.4.

Let integers u and v, and particular rays $F_1(b_1)$ and $F_2(b_2)$ be chosen, so that

(1) $T^u p$ and $T^v p$ are not separated by $\beta \cup Cl\,(F_1(b_1)) \cup Cl\,(F_2(b_2))$, and

(2) $u - v = 2k$, with $k > N$.

This is possible by Proposition 3.5.

Now apply T^l, where $l = k - u$. We conclude that the points $T^{-k}p$ and $T^k p$ *are separated* by $\beta \cup \mathrm{Cl}\,(F_1(b_1)) \cup \mathrm{Cl}\,(F_2(b_2))$, but are *not* separated by

$$T^l \beta \cup T^l(\mathrm{Cl}\,(F_1(b_1)) \cup \mathrm{Cl}\,(F_2(b_2))).$$

We now claim that $T^{\pm k}p$ are separated by $K(T^l\beta, \beta)$. For suppose not. The set $C = T^l\beta \cup T^l(\mathrm{Cl}\,(F_1(b_1)) \cup \mathrm{Cl}\,(F_2(b_2)))$ is of the type treated in Proposition 3.3; thus its intersection with $K(T^l\beta, \beta)$ is connected.

Hence, from App. 2, we would have $T^{\pm k}p$ not separated by $C \cup K(T^l\beta, \beta)$. But this set contains $\beta \cup \mathrm{Cl}\,(F_1(b_1)) \cup \mathrm{Cl}\,(F_2(b_2))$, and we know that $T^{\pm k}p$ are separated by this set.

Hence $T^{-k}p$ and $T^k p$ lie in different components of $\mathbf{R}^2 - K(T^l\beta, \beta)$. Since $K(T^l\beta, \beta)$ is compact, one of these components is bounded. After an application of T^k or T^{-k}, one sees that p itself is in a bounded component of $K(T^r p, T^s p)$ for certain r, s.   Q.E.D.

4. DEFINITION. An open set $\Omega \subset \mathbf{R}^2$ is *simply connected* if, given any Jordan curve $\Gamma \subset \Omega$, it is true that int $\Gamma \subset \Omega$.

PROPOSITION 4.1. *Let $C \subset \mathbf{R}^2$ be compact, and suppose that $p \in \mathbf{R}^2$ is in a bounded component of $\mathbf{R}^2 - C$. Let $\Omega \subset \mathbf{R}^2$ be a simply connected open set.*

If $C \subset \Omega$ then $p \in \Omega$.

Proof. Let $\mathcal{O}$ be the component of $\mathbf{R}^2 - C$ which contains p. Then $\bar{\mathcal{O}}$ is a connected compact set whose boundary lies in Ω.

Let $\varepsilon > 0$ be arbitrary. By App. 3, there exists a Jordan curve Γ such that $\bar{\mathcal{O}} \subset \mathrm{int}\,\Gamma$, and each point of Γ is within ε of a point of $\bar{\mathcal{O}}$. Since the boundary of $\bar{\mathcal{O}}$ is in the open set Ω, we conclude that $\Gamma \subset \Omega$ if ε is small enough.

By assumption Ω is simply connected; therefore int $\Gamma \subset \Omega$. Since $p \in \mathcal{O} \subset \mathrm{int}\,\Gamma$, we are done.   Q.E.D.

DEFINITION. Let $\Omega \subset \mathbf{R}^2$ be an open set and $T: \mathbf{R}^2 \to \mathbf{R}^2$ a free mapping. Then Ω is a *regular domain* for T if Ω is connected, simply connected, and satisfies $T\Omega = \Omega$.

THEOREM 4. *Let F_1 and F_2 be flow lines with $F_1 \cap F_2 = \varnothing$. Let α be a polygonal arc with endpoints a_1 and a_2; suppose that $\alpha \cap T\alpha = \varnothing$, and $\alpha \cap F_n = \{a_n\}$ for $n = 1, 2$.*

Now let $\Omega \subset \mathbf{R}^2$ be a regular domain for T. If $F_1 \cup F_2 \subset \Omega$ then $\alpha \subset \Omega$.

Proof. Because Ω is connected, there exists an arc $\gamma \subset \Omega$ such that γ meets both F_1 and F_2.

By Proposition 3.1, the set $\alpha \cup F_1$ meets $\mathrm{Cl}\,(F_2)$ in the single point $a_2 \in F_2$. Hence a subarc $\gamma' \subset \gamma$ can be chosen so that $\gamma' \cap F_1 \neq \varnothing$, and $\alpha \cup F_1 \cup \gamma'$ meets $\mathrm{Cl}\,(F_2)$ in exactly two points $\{a_2, b_2\}$, where b_2 is an endpoint of γ'.

Because $\alpha \cup F_1 \cup \gamma'$ is arcwise connected, we conclude from the results of §1 that $b_2 \in F_2$.

Now the set $\alpha \cup F_2$ meets $\text{Cl}\,(F_1)$ in the single point $a_1 \in F_1$. Hence a subarc $\gamma'' \subset \gamma'$ can be chosen so that $b_2 \in \gamma''$, and $\alpha \cup F_2 \cup \gamma''$ meets $\bar{F}_1$ in exactly two points $\{a_1, b_1\}$ where $b_1 \in \gamma''$.

Again it follows from the results of §1 that b_1 is actually in F_1.

Put $\beta = \gamma''$. Then β has endpoints b_1 and b_2, and satisfies $\beta \subset \Omega$ and $\beta \cap F_n = \{b_n\}$ for $n = 1, 2$. For a given $p \in \alpha$, Theorem 3 implies that $p \in \text{int}\,K(T^r\beta, T^s\beta)$ for certain r, s. But $K(T^r\beta, T^s\beta)$ is a compact subset of R^2, completely contained in Ω.

Hence by Proposition 4.1 we have $p \in \Omega$.   Q.E.D.

5.   We begin by defining the natural way to proceed from an open set to the simply connected open set it determines.

DEFINITION. Let $\mathcal{O}$ be an arbitrary open subset of R^2.

Then $SC(\mathcal{O}) = \bigcup_{\Gamma \subset \mathcal{O}} \text{int}\,\Gamma$, where Γ runs through all Jordan curves contained in $\mathcal{O}$.

Note that $SC(\mathcal{O})$ is open and contains the original set $\mathcal{O}$.

PROPOSITION 5.1. *Let $\mathcal{O}$ be open, and let K be a compact connected subset of $SC(\mathcal{O})$. Then there is a Jordan curve $\Gamma \subset \mathcal{O}$ with $K \subset \text{int}\,\Gamma$.*

Proof. By the definition of $SC(\mathcal{O})$ and by compactness of K, there exist finitely many Jordan curves $\{\Gamma_k\}$, with $K \subset \bigcup (\text{int}\,\Gamma_k)$ and $\Gamma_k \subset \mathcal{O}$ for all k.

Put $K' = \bigcup \text{Cl}\,(\text{int}\,\Gamma_k)$. Then K' is compact, connected, and its boundary $\partial K'$ satisfies $\partial K' \subset \mathcal{O}$.

Therefore, by App. 3, there exists a Jordan curve Γ with $K' \subset \text{int}\,\Gamma$ and $\Gamma \subset \mathcal{O}$. Q.E.D.

PROPOSITION 5.2. *For any open set $\mathcal{O}$, the set $SC(\mathcal{O})$ is simply connected.*

Proof. Let $\Gamma \subset SC(\mathcal{O})$ be any Jordan curve.

Since Γ is compact, we can by Proposition 5.1 find a Jordan curve $\Gamma' \subset \mathcal{O}$ with $\Gamma \subset \text{int}\,\Gamma'$.

Hence int $\Gamma \subset \text{int}\,\Gamma'$, where int $\Gamma' \subset SC(\mathcal{O})$ by the definition of $SC(\mathcal{O})$.   Q.E.D.

THEOREM 5. *Let Ω be a regular domain with $\Omega \neq R^2$. Let p be a boundary point of Ω. Let D_0 be a disc centered at p which satisfies $\text{Cl}\,(D_0) \cap \text{Cl}\,(TD_0) = \varnothing$.*

Then there exists a regular domain Ω' which satisfies $\Omega \subset \Omega'$, $D_0 \subset \Omega'$, and $\Omega' \neq R^2$.

Proof. Let D be the largest open disc centered at p which satisfies $D \cap TD = \varnothing$. Let a point q be chosen so that q and Tq are both on the boundary of D.

Choose simple polygonal arcs m_1 and m_2 with the following properties: m_1 and m_2 both have endpoints q, Tq; $m_1 \cup m_2$ is a Jordan curve with $\text{Cl}\,(D_0)$ in its interior; and $m_1 \cup m_2 - \{q, Tq\} \subset D$.

Put $\mathcal{O} = \text{int}\,(m_1 \cup m_2)$; we have $D_0 \subset \mathcal{O}$. Set $\Omega' = SC(\bigcup_{-\infty}^{+\infty} T^n\mathcal{O} \cup \Omega)$. We claim that Ω' satisfies all the conditions of Theorem 5.

It is trivial to verify that Ω' is a regular domain; connectivity follows from $\mathcal{O} \cap \Omega \neq \varnothing$, and simple connectivity follows from Proposition 5.2.

The only difficult thing to verify is that $\Omega' \neq R^2$, and this indeed is the heart of the matter.
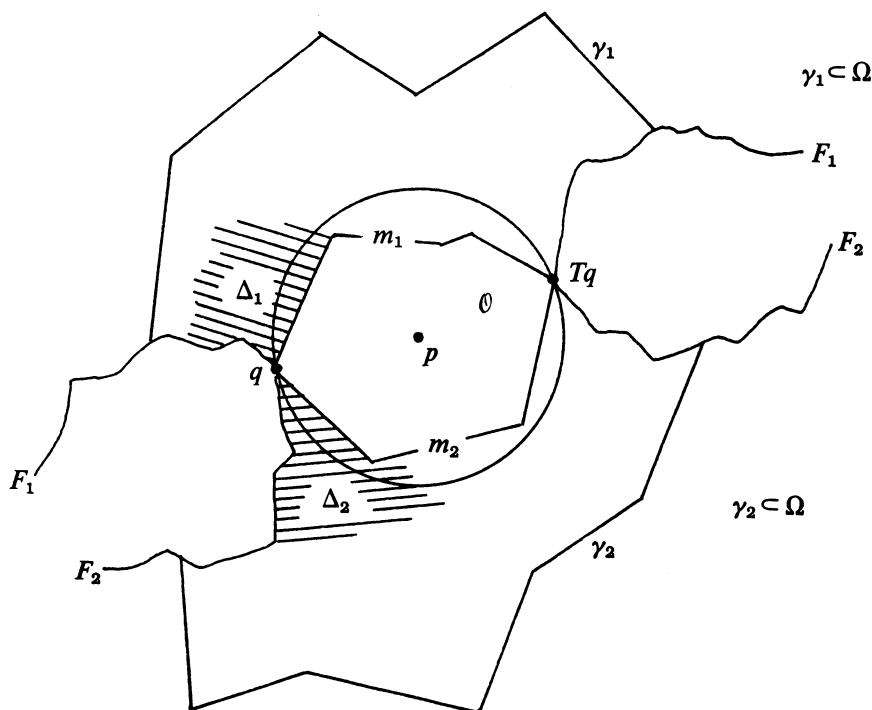
Put $F_k = \bigcup_n T^n m_k$ for $k = 1, 2$; then the F_k are flow lines by virtue of the construction of m_k. Let Δ_1 be the side domain of F_1 which avoids m_2; likewise let Δ_2 be that side domain of F_2 which avoids m_1.

Then the sets Δ_1, Δ_2, and $\bigcup_n T^n \mathcal{O}$ are mutually disjoint.

Now suppose, if possible, that $\Omega' = SC(\bigcup_n T^n \mathcal{O} \cup \Omega)$ is all of R^n. By Proposition 5.1 there exists a polygonal Jordan curve $\Gamma \subset \bigcup_n T^n \mathcal{O} \cup \Omega$ which satisfies $m_1 \cup m_2 \subset \text{int } \Gamma$.

We apply Theorem 2.5 to the flow lines F_k and the side domains Δ_k. The conclusion is that Γ contains closed subarcs γ_1 and γ_2 with the following properties:

(1) γ_k is, except for its endpoints, contained in Δ_k, and

(2) γ_k meets both components of $F_k - m_k$.

Now $\gamma_k \subset \bigcup_n T^n \mathcal{O} \cup \Omega$; since γ_k is contained in Cl (Δ_k) which avoids $\bigcup_n T^n \mathcal{O}$, we must conclude that $\gamma_k \subset \Omega$ for $k = 1, 2$.

Because γ_k meets both components of $F_k - m_k$ while it avoids m_k, we have by Proposition 1.9 that $\gamma_k \cap T\gamma_k \neq \varnothing$ for $k = 1, 2$.

We now claim that for each side domain Δ_k there exists a flow line F_k' and a polygonal arc β_k which have the following properties:

(1) $\beta_k \cap F_k' = \{b_k'\}$ and $\beta_k \cap F_k = \{b_k\}$ where b_k, b_k' are the endpoints of β_k;

(2) $\beta_k \cap T\beta_k = \varnothing$; and

(3) $F'_k \subset \Omega \cap \Delta_k$.

The construction of F'_k and β_k is done as follows. Let b_k be one endpoint of γ_k. Let $\gamma'_k \subset \gamma_k$ be a subarc with endpoints b_k and c_k; let it be the smallest such subarc which satisfies $\gamma'_k \cap T(\gamma'_k) \neq \varnothing$. Then γ'_k contains Tc_k or $T^{-1}c_k$; let b'_k denote this point.

Now γ'_k is the union of two arcs; β_k from b_k to b'_k, and δ_k from b'_k to c_k. Put $F'_k = \bigcup T^n\delta_k$; then the flow line F'_k and the arc β_k satisfy all the stipulations (1)–(3).

Replacing β_k by one of its iterates $T^n\beta_k$, we can assume that $b_k \in m_k$.

Now choose a polygonal arc τ, with endpoints b_1 and b_2, which contains p and satisfies $\tau - \{b_1, b_2\} \subset \mathrm{int}\,(m_1 \cup m_2)$. Finally consider $\sigma = \beta_1 \cup \tau \cup \beta_2$.

The three constituents of σ are respectively contained in the three disjoint invariant sets Δ_1, $\bigcup T^n\mathscr{O}$, and Δ_2. Moreover, the three constituents are separately disjoint from their images; therefore σ itself satisfies $\sigma \cap T\sigma = \varnothing$.

To summarize: The original point $p \in \partial\Omega$ is now contained in a polygonal arc σ, which meets the two flow lines F'_1 and F'_2 only in its endpoints. Moreover F'_1 and $F'_2 \subset \Omega$.

Hence, we conclude by Theorem 4 that $p \in \Omega$. But this contradicts the fact that p was chosen to be in the boundary of Ω.   Q.E.D.

6.   The announced theorems in the Introduction are deduced in the following fashion.

THEOREM 6.1. *Let $A \subset \mathbf{R}^2$ be a compact set, and let $\Omega_0 \subset \mathbf{R}^2$ be a regular domain for T, such that $\Omega_0 \neq \mathbf{R}^2$.*

Then there exists a regular domain Ω, such that $A \cup \Omega_0 \subset \Omega$ and $\Omega \neq \mathbf{R}^2$.

Proof. Replacing A by a larger compact set if necessary, we can assume that $A \cap \Omega_0 \neq \varnothing$.

Choose $\varepsilon > 0$ so that $\mathrm{Cl}\,(D_\varepsilon(a)) \cap \mathrm{Cl}\,(T(D_\varepsilon(a))) = \varnothing$ for all $a \in A$. ($D_\varepsilon(a)$ is the open disc of radius ε centered at a.)

Choose $N > 0$ so that for any set of points $a_1, \ldots, a_N$ of A we have $d(a_i, a_j) < \varepsilon$ for some $i \neq j$.

If $\Omega_0 \not\supset A$, choose $a_1 \in A \cap \partial\Omega_0$ and apply Theorem 5 to $D_\varepsilon(a_1)$. The result is a regular domain $\Omega_1 \neq \mathbf{R}^2$ with $\Omega_1 \supset D_\varepsilon(a_1) \cup \Omega_0$.

If $\Omega_1 \not\supset A$, choose $a_2 \in A \cap \partial\Omega_1$ and repeat the procedure, obtaining a regular domain $\Omega_2 \neq \mathbf{R}^2$ with $\Omega_2 \supset D_\varepsilon(a_2) \cup \Omega_1 \supset D_\varepsilon(a_2) \cup D_\varepsilon(a_1) \cup \Omega_0$.

Since $a_k \in \partial\Omega_{k-1}$ and $\Omega_{k-1} \supset D_\varepsilon(a_{k-1}) \cup \cdots \cup D_\varepsilon(a_1)$ we have that the distance from a_k to a_s is $\geq \varepsilon$ for all $k > s$.

Hence this iterative procedure must exhaust A in at most N steps. The result is a regular domain Ω_m with $\Omega_m \neq \mathbf{R}^2$ and $\Omega_m \supset \Omega_0 \cup A$. We then take $\Omega = \Omega_m$ in the statement of the theorem.   Q.E.D.

THEOREM 6.2. *Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be a free mapping.*

Then for any compact set $A \subset R^2$ there is an unbounded connected set $B \subset R^2$ which avoids $\bigcup T^n A$.

Proof. Let Ω_0 be any regular domain with $\Omega_0 \neq R^2$.

For example, one could take Ω_0 to be a side domain of a flow line.

By Theorem 6.1 there is a regular domain Ω_1 with $A \subset \Omega_1$ and $\Omega_1 \neq R^2$. By App. 7, $R^2 - \Omega_1$ contains an unbounded connected set.   Q.E.D.

THEOREM 6.3. *Let $E \subset R^2$ be the closed upper half-plane.*

Let T be a homeomorphism of E onto itself which has no fixed points and preserves orientation.

Then for any compact set $A \subset E$ there is an unbounded connected set $B \subset E$ which avoids $\bigcup T^n A$.

Proof. Let $J: R^2 \to R^2$ be the reflection $(x, y) \leftrightarrow (x, -y)$. Extend T to be a mapping T' of the whole plane by setting $T' = JTJ$ on the lower half-plane. The T' is easily seen to preserve orientation and have no fixed points; by Proposition 0 it is a free mapping of R^2 onto itself.

Let Ω_0 be the open lower half-plane; it is a regular domain for T'. By Theorem 6.1 there is an open set $\Omega_1 \subset R^2$ which is a regular domain for T' and satisfies $A \cup \Omega_0 \subset \Omega_1$ and $\Omega_1 \neq R^2$. Thus, we conclude from App. 7 that $R^2 - \Omega_1$ contains an unbounded connected set C; by construction $C \subset E$.   Q.E.D.

Appendix.   App. 1 (JORDAN CURVE THEOREM). Let $\Gamma \subset R^2$ be a Jordan curve. Then $R^2 - \Gamma$ has two components D_1 and D_2; moreover $\Gamma \subset \mathrm{Cl}\,(D_n)$ for $n = 1, 2$.

App. 2 (ALEXANDER'S LEMMA). Let $C \subset R^2$ be closed and $K \subset R^2$ be compact, and let p, q be two points of $R^2 - (C \cup K)$.

Suppose that p and q are not separated by C and not separated by K. If $C \cap K$ is connected then p and q are not separated by $C \cup K$.

For proofs of Apps. 1 and 2 see Newman [2, pp. 101–116].

Recall that $D_\varepsilon(p)$ is the open disc of radius $\varepsilon > 0$ centered at the point $p \in R^2$.

App. 3. Let K be a compact connected subset of R^2, and let $\varepsilon > 0$ be arbitrary. Then there exists a (polygonal) Jordan curve Γ with $\Gamma \cap K = \varnothing$, $K \subset \mathrm{int}\,\Gamma$, and such that $D_\varepsilon(p) \cap K \neq \varnothing$ for all $p \in \Gamma$.

The remaining facts in this Appendix can be deduced from Apps. 1, 2, and 3.

App. 4. Let $\Gamma \subset R^2$ be a Jordan curve, and let $K_1, K_2 \subset R^2$ be disjoint compact connected sets which satisfy $K_n \subset \mathrm{Cl}\,(\mathrm{int}\,\Gamma)$ and $K_n \cap \Gamma = \gamma_n$ for $n = 1, 2$, where γ_1 and γ_2 are disjoint closed subarcs of Γ.

Then there exists an arc α, with endpoints a_1 and a_2, such that

(1) $\alpha \subset \mathrm{Cl}\,(\mathrm{int}\,\Gamma)$ and $\alpha \cap \Gamma = \{a_1, a_2\}$,

(2) $\alpha \cap (K_1 \cup K_2) = \varnothing$, and

(3) a_1 and a_2 are in the two components of $\Gamma - (\gamma_1 \cup \gamma_2)$.

App. 5. Let $\Gamma \subset R^2$ be a Jordan curve, and let $K \subset R^2$ be a compact connected set which satisfies $K \subset \mathrm{Cl}\,(\mathrm{int}\,\Gamma)$ and $K \cap \Gamma = \gamma$, when $\gamma \subset \Gamma$ is a subarc with endpoints b_1 and b_2. Let $\varepsilon > 0$ be arbitrary.

Then there exists an arc α, with endpoints a_1 and a_2, such that

(1) $\alpha \subset \mathrm{Cl}\,(\mathrm{int}\,\Gamma)$ and $\alpha \cap \Gamma = \{a_1, a_2\}$,

(2) $\alpha \cap K = \varnothing$ but $D_\varepsilon(p) \cap K \neq \varnothing$ for all $p \in \alpha$,

(3) $D_\varepsilon(a_n)$ contains b_n for $n = 1, 2$.

App. 6. Let Γ_1 and Γ_2 be Jordan curves, and suppose that $\Gamma_1 \cap \Gamma_2$ is a common subarc. If $\mathrm{int}\,\Gamma_1 \cap \mathrm{int}\,\Gamma_2 \neq \varnothing$ then either $\mathrm{int}\,\Gamma_1 \subset \mathrm{int}\,\Gamma_2$ or $\mathrm{int}\,\Gamma_2 \subset \mathrm{int}\,\Gamma_1$.

App. 7. Let Ω be a simply connected open subset of R^2. If $\Omega \neq R^2$ then there exists an unbounded closed connected set C which avoids Ω.

A proof of App. 7 is outlined as follows. Choose a fixed $p_0 \notin \Omega$, and say that a closed set E is "good" if $p_0 \in E$, $E \cap \Omega = \varnothing$, and $E \cap \Gamma \neq \varnothing$ for every Jordan curve Γ which contains p_0 in its interior domain. For example, $R^2 - \Omega$ is good.

The collection of good sets is invariant under arbitrary nested intersections. By Zorn's lemma, take C to be a minimal good set. This C satisfies all the conditions of App. 7; for example, the connectivity of C follows from App. 3.

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