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CONTINUA FOR WHICH THE SET FUNCTION T IS CONTINUOUS([1])

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Abstract. The set-valued set function T has been studied extensively as an aid to classifying metric and Hausdorff continua. It is a consequence of earlier work of the author with H. S. Davis that T, considered as a map from the hyperspace of closed subsets of a compact Hausdorff space to itself, is upper semicontinuous. We show that in a continuum for which T is actually continuous (in the exponential, or Vietoris finite, topology) semilocal connectedness implies local connectedness, and raise the question of whether any nonlocally connected continuum for which T is continuous must be indecomposable.

1. **Definitions and notation.** The letters S and Z will denote compact Hausdorff spaces. The definition of the set-function T and the notion of T-additivity, [1] and [2], are assumed. A continuum S is *T-symmetric* iff for each pair of closed sets $A, B \subseteq S$, $A \cap T(B) = \varnothing$ whenever $B \cap T(A) = \varnothing$. S is *point T-symmetric* iff this definition holds whenever A and B are singletons. (Compare this with Definition 1.1 of [4].) S is *almost connected im kleinen* [3] at $x \in S$ provided every open set containing x contains also a continuum with nonempty interior; S is *connected im kleinen* at x iff this W can always be chosen to be a continuum neighborhood of x. Observe that S is connected im kleinen at p if and only if: $p \in A$ iff $p \in T(A)$ for every closed set $A \subseteq S$ [2]. A closed set $A \subseteq S$ is a *closed domain* [5, p. 74] iff $A = \mathrm{Cl}\,\mathrm{Int}\,(A)$. If in addition A is connected, A is called a *continuum domain*. S will be called *semilocally connected* at p iff $T(p) = \{p\}$. (See [6, p. 19] and [2].)

$\mathscr{F}(S)$ denotes the space of nonempty closed subsets of S and $\mathscr{W}(S)$ the space of nonempty subcontinua of S with the usual exponential topology [5]. T is of course defined for all subsets of S. The phrase "T is continuous for S" will mean that

$$T|\mathscr{F}(S)\colon \mathscr{F}(S) \to \mathscr{F}(S)$$

is continuous. For $O \subseteq S$, define

$$\mathscr{F}(O) = \{A \in \mathscr{F}(S) : A \subseteq O\}, \qquad \mathscr{G}(O) = \{A \in \mathscr{F}(S) : A \cap O \neq \varnothing\}.$$

Finally, the set function aT is defined by: $p \in S - aT(X)$ iff there exists a finite collection of continua, $\{W_i\}_{i=1}^{n}$, such that $p \in \mathrm{Int} \bigcap_{i=1}^{n} W_i$ while $X \cap \bigcap_{i=1}^{n} W_i = \varnothing$.

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2. **Introduction.** The first result is an easy consequence of Theorem A of [1]. The proof is left to the reader.

LEMMA 1. *T: $\mathscr{F}(S) \to \mathscr{F}(S)$ is an upper semicontinuous mapping.*

This suggests that the continuity-related properties of, and points of discontinuity of T may be interesting. It is the purpose of this paper to examine the extreme case, when T is continuous for the continuum S. There are two trivial cases where this is true: (1) S is an indecomposable continuum. Here $T(A) = S$ for all $A \in \mathscr{F}(S)$, (2) S is connected im kleinen, or locally connected, in which case $T(A) = A$ for each $A \in \mathscr{F}(S)$. The question of whether these are the only possibilities appears to be difficult. It is shown here that if T is continuous and S is not connected im kleinen, then it also is neither almost connected im kleinen nor semilocally connected.

3. **Preliminary lemmas.**

LEMMA 2. *T is idempotent on S iff for every subcontinuum $W \subseteq S$ and $x \in \operatorname{Int} W$, there is a continuum M with $x \in \operatorname{Int} M \subseteq M \subseteq \operatorname{Int} W$.*

Indication of Proof. It is clear that this condition implies $T^2 = T$. To obtain the converse, apply the idempotency of T to $S - W$.

COROLLARY 1. *If T is idempotent on S, $W \subseteq S$ is a continuum, and K is a component of* Int W, *then K is open.*

Proof. Let $p \in K$. Let M be a continuum neighborhood of p with $M \subseteq \operatorname{Int} W$. Then $M \subseteq K$ so that $p \in \operatorname{Int} K$.

COROLLARY 2. *If T is idempotent on S, $x \in S$, and W is a continuum neighborhood of x, then x has a continuum neighborhood $M \subseteq W$ which is a continuum domain.*

Proof. Let M be the closure of that component of Int W containing x.

LEMMA 3. *If S is a continuum for which T is continuous, then T is idempotent on S also.*

Proof. Let $W \subseteq S$ be a subcontinuum and $x \in \operatorname{Int} W$. Now, $T^{-1}(\mathscr{F}(S - \operatorname{Int} W))$ is a closed set by continuity of T and

$$\mathscr{F}(S - W) \subseteq T^{-1}(\mathscr{F}(S - \operatorname{Int} W))$$

by definition of T. Since, for $W \neq S$, $S - \operatorname{Int} W$ is a limit point of $\mathscr{F}(S - W)$, it follows that $T(S - \operatorname{Int} W) \subseteq S - \operatorname{Int} W$. Then, x has a continuum neighborhood M missing $S - \operatorname{Int} W$. Thus $M \subseteq \operatorname{Int} W$. If $W = S$, it suffices to choose $M = S$, so that in either case the proof is complete by Lemma 2.

LEMMA 4. *If S is a continuum for which T is idempotent and in which $T(p, q)$ is a continuum for all $p, q \in S$, then S is indecomposable.*

Proof. Suppose not. Then by Corollary 2, there is a nonempty proper continuum domain $W \subseteq S$. Let p_0 and q_0 be any two points in $S - \operatorname{Int} W$. Since $T(p_0, q_0) \cap \operatorname{Int} W = \varnothing$ and $T(p_0, q_0)$ is connected, p_0 and q_0 lie in the same component of $S - \operatorname{Int} W$. Thus, $S - \operatorname{Int} W$ is a continuum. By Lemma 2, there is a continuum M, with nonempty interior, such that $M \subseteq \operatorname{Int}(S - \operatorname{Int} W) = S - W$. Then, let $S - (\operatorname{Int} W \cup \operatorname{Int} M) = L$ and let p_1, q_1 be any two points in L. $T(p_1, q_1)$ is a continuum contained in L, so that L is a continuum also. Now suppose $p \in \operatorname{Int} M$ and $q \in \operatorname{Int} W$. Then $T(p, q)$ is not a continuum since it misses $\operatorname{Int} L$, a contradiction.

The proofs of the next two lemmas are left to the reader. They involve standard compactness arguments.

Lemma 5. *If $A \subseteq S$ is closed, $aT(A) = \bigcup_{p \in A} T(p)$.*

Lemma 6. *S is T-additive iff $T(A) = aT(A)$ for every closed $A \subseteq S$.*

Lemma 7. *If T is continuous for S, so is aT.*

Proof. It is clear that aT is upper semicontinuous. (Mimic the proof of Theorem A of [1].) Thus suppose O is open in S, and $A \in aT^{-1}(\mathscr{G}(O))$, or $aT(A) \cap O \neq \varnothing$. Then by Lemma 5 there is a $p \in A$ with $T(p) \cap O \neq \varnothing$. By continuity of T, there is an open set $U \subseteq S$ containing p such that, for all $x \in U$, $T(x) \cap O \neq \varnothing$. Then if $B \in \mathscr{G}(U)$, $aT(B) \cap O \neq \varnothing$, so that $A \in \mathscr{G}(U) \subseteq aT^{-1}(\mathscr{G}(O))$. Thus $aT^{-1}(\mathscr{G}(O))$ is open.

Lemma 8. *If S is a point T-symmetric continuum for which T is continuous, then $T(p, q) = T(p) \cup T(q)$ for every $p, q \in S$.*

Proof. For each $p \in S$, define

$$A(p) = \{q : T(p, q) \in \mathscr{W}(S)\}, \qquad B(p) = \{q : T(p, q) = T(p) \cup T(q)\}.$$

It follows from the continuity of T and aT and the fact that $\mathscr{W}(S)$ is closed in $\mathscr{F}(S)$ that both $A(p)$ and $B(p)$ are closed in S. If $x \in T(p)$, $T(x) = T(p) = T(x, p)$ by point T-symmetry and idempotency. Since $T(p)$ is a continuum (by Corollary 1 of [1]), $x \in A(p) \cap B(p)$. Also, if $x \in A(p) \cap B(p)$, $T(p, x) = T(p) \cup T(x)$, and this set is a continuum. Hence $T(p) \cap T(x) \neq \varnothing$. Let $q \in T(p) \cap T(x)$. Then $x \in T(q) \subseteq T^2(p) = T(p)$. Thus, $A(p) \cap B(p) = T(p)$.

Now, suppose there is a $p \in S$ such that $B(p) \neq T(p)$. Let $y \in A(p)$ and $x \in B(p) - T(p)$ be arbitrary points. Then $T(x, p) = T(x) \cup T(p)$ and $T(x) \cap T(p) = \varnothing$. Hence, $T(x) \cap A(p) = \varnothing$, since otherwise $(T(x) \cap A(p)) \cup (T(x) \cap B(p))$ is a separation of $T(x)$. Let U be an open set with $\bar{U} \cap A(p) = \varnothing$ while $T(x) \subseteq U$. Now suppose $q \in \operatorname{Bd}(U)$. Since $q \notin T^2(x, p)$, there is a continuum W with $q \in \operatorname{Int} W$ and $W \cap T(x, p) = \varnothing$. Then $W \subseteq B(p)$, since otherwise $(A(p) \cap W) \cup (B(p) \cap W)$ is a separation of W. Therefore, $y \notin W$, and $q \notin T(x, y)$. Then

$$T(x, y) = (T(x, y) \cap U) \cup (T(x, y) \cap (S - \bar{U})) \text{ sep}$$

and by Corollary 2 of [1], $T(x, y)=T(x) \cup T(y)$, so that $x \in B(y)$. Thus, $B(p) - T(p) \subseteq B(y)$, and since $B(y)$ is closed and $p \in \mathrm{Cl}\,(B(p)-T(p))$, (If not, $p \in \mathrm{Int}\, A(p)$, and since $A(p)$ is a continuum, there is a continuum M with $p \in \mathrm{Int}\, M \subseteq M \subseteq \mathrm{Int}\, A(p)$. Hence M misses some $q \in T(p)$, and $p \notin T(q)$, contradicting the point T-symmetry of S.) it follows that $p \in B(y)$, or that $y \in B(p)$. But $y \in A(p)$, so that $y \in T(p)$, and $A(p)=T(p)$. By contraposition, if $A(p) \neq T(p)$, $B(p)=T(p)$, so that for each $p \in S$ either $A(p)=S$ or $B(p)=S$. Suppose that there is a $p \in S$ such that $A(p)=S$. Let $q \in S$ be arbitrary. Either $q \in T(p)$, in which case $A(q)=A(p)=S$; or $q \notin T(p)$, in which case $q \in A(p)$ so that $p \in A(q)$. Since $p \notin T(q)$, $A(q) \neq T(q)$, so that $A(q)=S$. Thus, either $A(p)=S$ for every $p \in S$ or $B(p)=S$ for every $p \in S$. If $B(p)=S$ for all p, the lemma is proved, so suppose $A(p)=S$ for every p. By definition of $A(p)$ and Lemma 4, S is indecomposable, so that $B(p)=S$ for all p in this case also.

LEMMA 9. *If S is a point T-symmetric continuum for which T is continuous, then S is T-additive.*

Proof. By Lemma 6, it suffices to prove that $T(A)=aT(A)$ for every $A \in \mathscr{F}(S)$. Since both T and aT are continuous, and the set $\{A \in \mathscr{F}(S) : A \text{ is finite}\}$ is dense in $\mathscr{F}(S)$, it suffices to prove that $aT(A)=T(A)$ for finite sets A. Thus, suppose M is a finite set of smallest cardinal number such that $T(M) \neq \bigcup_{p \in M} T(p)$.

As a consequence of Lemma 8, M contains at least three points. $T(M)$ is a continuum, since if $A \cup B$ is a separation of $T(M)$ by Lemma 2 of [1] and the minimality of M,

$$\begin{aligned} T(M) &= T(M \cap A) \cup T(M \cap B) \\ &= \bigcup_{p \in M \cap A} T(p) \cup \bigcup_{p \in M \cap B} T(p) = \bigcup_{p \in M} T(p) \end{aligned}$$

contrary to the choice of M. Further, if $p, q \in M$ are distinct points, then $T(p) \cap T(q)=\varnothing$, since if not, then the point T-symmetry and idempotency yield $T(p)=T(q)$ and then

$$\begin{aligned} T(M) &\subseteq T^2(M-\{p\}) \subseteq T(M-\{p\}) \\ &\subseteq aT(M-\{p\}) \subseteq aT(M) \end{aligned}$$

and since always $aT(M) \subseteq T(M)$, this contradicts the choice of M.

Now, let $p \in M$ be arbitrary and set $N=M-\{p\}$. Then N has at least two points, and since for distinct points $a, b \in N$, $T(a) \cap T(b)=\varnothing$, and $aT(N)=T(N)$, it follows that $T(N)$ is not a continuum. Set

$$L = \{x \in S : T(N \cup \{x\}) = aT(N \cup \{x\})\}, \qquad K = \{x \in S : T(N \cup \{x\}) \in \mathscr{W}(S)\}.$$

$L \neq \varnothing$ since $N \subseteq L$. $K \neq \varnothing$ since $p \in K$. L is closed since T, aT, and $\cup$ are continuous, and K is closed since $\mathscr{W}(S)$ is closed in $\mathscr{F}(S)$, and T and $\cup$ are continuous. If $y \in K \cap L$, then $T(y) \cap T(q) \neq \varnothing$ for every $q \in N$. By point T-symmetry and idempotency, $T(y)=T(q)$ for every $q \in N$, a contradiction to the fact that for

$a, b \in N$, if $a \neq b$, then $T(a) \neq T(b)$. Thus $K \cap L = \varnothing$. But if $x \notin L$, $T(N \cup \{x\})$ is a continuum by the argument applied to M, above, and $x \in K$. Hence, $K \cup L$ is a separation of the continuum S, and this contradiction completes the proof.

LEMMA 10. *If S is a continuum for which T is continuous, $W \subseteq S$ is a continuum with nonvoid interior, and O is open in S with $W \subseteq O$, then there is a point p such that $T(p) \subseteq O$.*

Proof. Either $S-W$ is connected or it is not. If $S-W$ is connected let $p \in \operatorname{Int} W$. Then $\operatorname{Cl}(S-W)$ is a continuum neighborhood of every point outside W missing p, and $T(p) \subseteq W \subseteq O$. Thus, suppose $M \cup N$ is a separation of $S-W$. Then, if $x \in M$, $T(x) \subseteq \overline{M}$, since $N \cup W$ is a continuum neighborhood of every point outside of $\overline{M}$ which misses x. Similarly, if $x \in N$, $T(x) \subseteq \overline{N}$. Now let

$$A = \{x : T(x) \cap M \cap (S-O) \neq \varnothing\}, \qquad B = \{x : T(x) \cap M \neq \varnothing\}.$$

$A \subseteq B$, $B \neq \varnothing$ since $M \subseteq B$; and B is open while A is closed by continuity of T. Since $N \cap B = \varnothing$, $B \neq S$, so that $A \neq B$ by connectedness of S. Let $x \in B-A$. Then $T(x) \cap M \neq \varnothing$, but $T(x) \cap M \cap (S-O) = \varnothing$. Let $p \in T(x) \cap M$. Then $T(p) \subset \overline{M} \cap T(x)$; in particular,

$$T(p) \cap (S-O) \subseteq \overline{M} \cap T(x) \cap (S-O) = \varnothing,$$

since $\overline{M} - M \subseteq O$. Thus, $T(p) \subseteq O$.

The next lemma is due to Eugene Vanden Boss.

LEMMA 11. *A semilocally connected T-additive continuum S is connected im kleinen.*

Proof. For $A \subseteq S$, A closed,

$$T(A) = \bigcup_{p \in A} T(p) = \bigcup_{p \in A} \{p\} = A.$$

Hence S is connected im kleinen at each point, [2].

LEMMA 12. *If S is a T-additive continuum for which T is continuous, and $W \subseteq S$ is a continuum domain, then $T(W) = W$.*

Proof. Let $L = \{p : T(p) \subseteq W\}$. Let $x \in W$. Let M be an arbitrary continuum neighborhood of x. Then $\operatorname{Int} M \cap \operatorname{Int} W \neq \varnothing$. Let $y \in \operatorname{Int} M \cap \operatorname{Int} W$. Then by idempotency and Lemma 2,

$$y \notin T(S - \operatorname{Int} M) \cup T(S - \operatorname{Int} W);$$

by additivity,

$$y \notin T((S - \operatorname{Int} M) \cup (S - \operatorname{Int} W)), \qquad y \notin T(S - (\operatorname{Int} M \cap \operatorname{Int} W)).$$

Hence there is a continuum N with $y \in \operatorname{Int} N$ and $N \subseteq \operatorname{Int} M \cap \operatorname{Int} W$. Then, by Lemmas 10 and 2, there is a $p \in \operatorname{Int} N$ such that $T(p) \subseteq N$. Then $T(p) \subseteq W$ so that

$p \in L$. Hence $M \cap L \neq \varnothing$ and $x \in T(L)$, so that $W \subseteq T(L)$. By definition of L and additivity, $T(L) \subseteq W$. Thus, $T(W) = T^2(L) = T(L) = W$.

LEMMA 13. *If S is a continuum for which T is continuous, S is T-additive iff S is T-symmetric.*

Proof. Since T-symmetry always implies T-additivity by Theorem 7 of [2], it suffices to prove the converse. Suppose S is T-additive and let A, B be closed subsets of S with $A \cap T(B) = \varnothing$. Then by definition of T, compactness, and Corollary 2, there exists a finite collection $\{W_i\}_{i=1}^n$ such that each W_i is a continuum domain, $A \subseteq \bigcup \operatorname{Int} W_i$, and $B \cap (\bigcup W_i) = \varnothing$. Then by additivity and Lemma 12, $T(\bigcup W_i) = \bigcup W_i$. Hence $T(A) \subseteq \bigcup W_i$, so that $T(A) \cap B = \varnothing$.

4. Principal results.

THEOREM 1. *If S is a continuum for which T is continuous and S is almost connected im kleinen at $p \in S$, then S is semilocally connected at p.*

Proof. Let

$$\mathscr{L} = \{A : A \text{ is closed in } S \text{ and } p \in \operatorname{Int} A\}.$$

By Lemma 10 and the almost connectedness im kleinen, the set $B(A) = \{x : T(x) \subseteq A\}$ is nonempty for each $A \in \mathscr{L}$. By continuity of T, $B(A)$ is closed for each A. Hence $\{B(A) : A \in \mathscr{L}\}$ is a filterbase of closed sets, and $\bigcap_{A \in \mathscr{L}} B(A) \neq \varnothing$. But,

$$\bigcap_{A \in \mathscr{L}} B(A) \subseteq \bigcap \mathscr{L} = \{p\}.$$

Thus, $T(p) \subseteq \bigcap \mathscr{L} = \{p\}$ and the proof is complete.

THEOREM 2. *If T is both additive and continuous for the continuum S and $p \in S$, then the following are equivalent.*

(1) *S is semilocally connected at p.*

(2) *S is almost connected im kleinen at p.*

(3) *S is connected im kleinen at p.*

Proof. (2) implies (1) by Theorem 1.

(3) implies (2). This is trivial.

(1) implies (3). Let O be any open set containing p. Since $T(p) \cap (S - O) = \varnothing$, $T(S - O) \cap \{p\} = \varnothing$ by Lemma 13. Thus p has a continuum neighborhood W which misses $S - O$, that is, $W \subseteq O$.

THEOREM 3. *If S is a continuum for which T is continuous and S is semilocally connected at each point, then S is connected im kleinen.*

Proof. Since $p \in T(q)$ iff $p = q$, S is point T-symmetric, and thus is T-additive by Lemma 9 and connected im kleinen by Theorem 2.

COROLLARY 3. *If S is a continuum for which T is continuous and S is almost connected im kleinen at each point, then S is connected im kleinen.*

5. The effect of mappings.

DEFINITION. A continuous function $f: S \to Z$ is called T-continuous provided that always $fT(A) \subseteq Tf(A)$ for $A \subseteq S$, or equivalently $f^{-1}T(A) \supseteq Tf^{-1}(A)$ for $A \subseteq Z$, where T is computed with respect to whichever of S, Z its argument is contained in. The simplest examples of T-continuous maps are continuous monotone maps.

The next result is due to H. S. Davis.

LEMMA 14. *If $f: S \to Z$ is a continuous surjection and $A \subseteq Z$, then $fTf^{-1}(A) \supseteq T(A)$.*

Proof. Suppose $x \notin fTf^{-1}(A)$. Then $f^{-1}(x) \cap Tf^{-1}(A) = \varnothing$. By definition of T and the compactness of $f^{-1}(x)$, there is a finite collection of continua, $\{W_i\}_{i=1}^n$ such that $f^{-1}(x) \subseteq \bigcup_{i=1}^n \operatorname{Int} W_i$, while $f^{-1}(A) \cap (\bigcup_{i=1}^n W_i) = \varnothing$ and for each W_i, $W_i \cap f^{-1}(x) \neq \varnothing$. Then, $A \cap f(\bigcup_{i=1}^n W_i) = \varnothing$, and $f(\bigcup W_i)$ is a continuum since each component of it contains x. Since $Z - f(S - \bigcup \operatorname{Int} W_i)$ is an open set containing x and contained in $f(\bigcup W_i)$, it follows that $x \notin T(A)$, and the proof is complete.

This leads to the final result about mappings which preserve continuity of T.

THEOREM 4. *If S is a continuum for which T is continuous, and $f: S \to Z$ is a continuous, T-continuous, open surjection, then T is continuous for Z also.*

Proof. By Lemma 14 and the definition of a T-continuous map,

$$fTf^{-1}(A) = T(A) \quad \text{for every } A \subseteq Z.$$

Since f is closed and open, both $f: \mathscr{F}(S) \to \mathscr{F}(Z)$ and $f^{-1}: \mathscr{F}(Z) \to \mathscr{F}(S)$ are continuous. Hence the T function for Z is a composition of three continuous functions.

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