ISOTOPY INVARIANTS IN QUASIGROUPS

BY

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Abstract. The purpose of this paper is to investigate quasigroup and loop identities that are invariant under isotopy. The varieties of quasigroups and the varieties of loops that are closed under isotopy form isomorphic lattices. Some methods of generating isotopically closed varieties of loops are given.

1. **Introduction.** A major problem in the theory of loops is the isotopy-isomorphism problem—When does isotopy imply isomorphism? A closely related, more general problem is the isotopy-invariance problem—What properties are invariant under isotopy? The purpose of this paper is to investigate loop and quasigroup identities that are invariant under isotopy. The associative law and the Moufang identity $(xy \cdot x)z = x(y \cdot xz)$ are well-known identities that are invariant with respect to loop isotopy, [3], [7]. Since varieties are equational classes, an equivalent problem is the determination of all varieties of quasigroups and loops that are closed under isotopy.

§§3 to 6 are devoted to general considerations of the problem. It is shown that a variety of quasigroups is closed under isotopy if and only if it is the class of all quasigroups all of whose loop isotopes lie in some variety of loops. As a result we obtain a set of quasigroup identities such that every identity invariant under isotopy is either in this set or is equivalent to a member of the set. We next show that the varieties of quasigroups and the varieties of loops that are closed under isotopy form isomorphic lattices, $\mathscr{Q}^*$ and $\mathscr{L}^*$, respectively, with $\mathscr{L}^*$ a sublattice of the lattice of loop varieties. An atom of $\mathscr{Q}^*$ that contains groups has the property that all of its free quasigroups of rank $\aleph_0$ or less are isotopic. We also give a variety of loops, which contains no subvariety that is closed under isotopy except the trivial one.

The latter part of the paper is devoted to methods of generating varieties of loops that are isotopically closed. It is shown that the classes of central, left nuclear, right nuclear, and middle nuclear extensions of loops belonging to an isotopically closed loop variety are also isotopically closed varieties. In particular, for each integer $n \geq 1$, the variety of centrally nilpotent loops of class $m \leq n$ is isotopically closed.

In addition, we show that any class of loops containing all loop isotopes of its members generates an isotopically closed loop variety. Because of the relationship between isotopically closed quasigroup and loop varieties we can generate an isotopically closed quasigroup variety from any of these new loop varieties by taking all isotopes of the members of the loop variety.

There has been a recent vigorous move in the study of the problem. Evans in [8] showed that a quasigroup is isotopic to a group if and only if it satisfies a law,

$$((xP_1 \cdot yP_2)P_3 \cdot zP_4)P_5 = (xQ_1 \cdot (yQ_2 \cdot zQ_3)Q_4)Q_5$$

where P_i, Q_i, $i = 1, 2, 3, 4, 5$ are permutations on the quasigroup. Belousov in [4] gave identities that are necessary and sufficient conditions that a quasigroup be isotopic to a group or to an abelian group. These identities are invariant under isotopy. Belousov and Ryžkov in [5] stated that there is a one-one correspondence between closure figures in the net determined by a loop and the identities invariant with respect to loop isotopy that are satisfied by the loop. In [12] Osborn studied the structure of loops, all of whose loop isotopes satisfied the identity, $x(xy)^{-1} = y^{-1}$. Aczél in [1] proved that the identities $y(z \cdot yx) = (y \cdot zy)x$ and $(xy \cdot z)y = x(yz \cdot y)$ are invariant with respect to loop isotopy. Other special results have been obtained by Sade and Hosszu, [2], [13].

2. **Definitions.** A *quasigroup* is a nonempty set, Q, which is closed with respect to three binary operations, multiplication $(\cdot)$, left division $(\backslash)$, right division $(/)$, and such that for all $x, y \in Q$,

1. $x \cdot (x \backslash y) = y$;
2. $x \backslash (x \cdot y) = y$;
3. $(y/x) \cdot x = y$;
4. $(y \cdot x)/x = y$.

A *loop* is a quasigroup which satisfies $x/x = y \backslash y$ for all x, y. Here we follow Evans, [11].

A *variety of quasigroups* is the class of all quasigroups satisfying some given set of quasigroup identities. A *variety of loops* is likewise defined. Loop varieties, free loops and free quasigroups are discussed in [9], [10], and [11].

A subloop H of a loop G is called a *normal subloop* of G if for all $x, y \in G$, we have $xH = Hx$, $(Hx)y = H(xy)$ and $y(xH) = (yx)H$. H is said to be *fully invariant* in G, if in addition to being normal, $H\theta \leqq H$ for each endomorphism θ of G. There is a one-one correspondence between the class of loop varieties and the set of fully invariant subloops of the free loop on $\aleph_0$ generators [10].

The *inner mapping group* I of a loop G is the group generated by the mappings $R(x, y) = R_x R_y R_{xy}^{-1}$, $L(x, y) = L_x L_y L_{yx}^{-1}$, $S(x) = R_x L_x^{-1}$ for all $x, y \in G$, where R_x and L_x are the mappings given by $yR_x = xy$ and $yL_x = yx$, for all $y \in G$. A nonempty subset S of a loop G is called *self-conjugate* in G if $SI = S$ [6].

The quasigroup $(G, \cdot, \backslash, /)$ is said to be *isotopic to* or an *isotope* of the quasigroup

$(H, \circ, \backslash, /)$ if there exist one-one mappings, α, β, γ of G upon H such that $x\alpha \circ y\beta$ $=(x \cdot y)\gamma$. If γ is the identity mapping on G, then G is said to be a *principal isotope* of H. These definitions and basic theorems on isotopy are to be found in Bruck, [6].

3. **Isotopically L- and Q-closed varieties.** A variety V of quasigroups having the property that every isotope of a quasigroup in V also belongs to V is said to be an *isotopically Q-closed variety*. Similarly, an *isotopically L-closed variety* is a variety of loops which contains all the loop isotopes of its members. The varieties of groups and Moufang loops are isotopically L-closed varieties. The variety V of quasigroups satisfying $(xy)/z=(xz)/y$ is an isotopically Q-closed variety. By a result of Belousov, [4], any loop in V is an abelian group, clearly of exponent 2. Since groups satisfy the isotopy-isomorphism property, [6], the variety of loops contained in V is isotopically L-closed. Moreover, a quasigroup is isotopic to a loop in this variety if and only if all its loop isotopes satisfy $x^2 = 1$, which holds, if and only if the quasigroup satisfies $(x/y) \cdot (z\backslash x)=zy$, or equivalently, $(xy)/z$ $=(xz)/y$.

A quasigroup identity is called a *universal identity* if it is invariant under isotopy. Belousov and Ryžkov have given the name "universal identity" to loop identities having this property with reference to loop isotopy, [5]. It is clear that any identity that is universal under quasigroup isotopy will also be universal under loop isotopy.

Let $(F, \cdot, \backslash, /)$ be the free quasigroup on $y, z, x_1, x_2, \ldots$ and let $(L, \circ, \backslash, /)$ be the free loop on $x_1, x_2, \ldots$. If w is a reduced word in L, we define its *isotopic-related word* in F, $\bar{w}$, by

(i) if $w=1$, then $\bar{w}=zy$;

(ii) if $w=x$, then $\bar{w}=x$;

(iii) if $w=u \circ v$, then $\bar{w}=(\bar{u}/y) \cdot (z\backslash\bar{v})$;

(iv) if $w=u \, \phi \, v$, then $\bar{w}=[\bar{u}/(z\backslash\bar{v})] \cdot y$;

(v) if $w=u \, \backslash \, v$, then $\bar{w}=z \cdot [(\bar{u}/y)\backslash\bar{v}]$.

If $w_1=w_2$ is a loop identity, then $\bar{w}_1=\bar{w}_2$ is called the *derived identity* of $w_1=w_2$.

LEMMA 3.1. *Let $(G, \cdot, \backslash, /)$ be a quasigroup with principal loop isotope $(G, \circ, \backslash, \phi)$ given by $x \circ y=(x/a)(b\backslash y)$ for elements $a, b \in G$. Let $g_1, g_2, \ldots, g_n$ be any elements of G. If $w(x_1, x_2, \ldots, x_n)$ is a reduced word in the free loop $(L, \circ, \backslash, \phi)$ with isotopic-related word $\bar{w}(x_1, x_2, \ldots, x_n, y, z)$ in the free quasigroup $(F, \cdot, \backslash, /)$, then*

$$w(g_1, g_2, \ldots, g_n) = \bar{w}(g_1, g_2, \ldots, g_n, a, b).$$

Proof. The proof is by induction on the length of w. If $w=1$, then $\bar{w}=zy$. Hence the neutral element of $(G, \circ)$ is $ba=\bar{w}(a, b)$. If $w=x$, then $\bar{w}=x$. Thus $w(g_1)=g_1$ $=\bar{w}(g_1, a, b)$. Assume that the assertion holds for words of length less than k. Consider $w(x_1, x_2, \ldots, x_n)$ of length $k>1$. We have three cases: (i) $w=u \circ v$, (ii) $w=u \, \phi \, v$, (iii) $w=u \, \backslash \, v$. By the induction hypothesis

$$u(g_1, g_2, \ldots, g_n) = \bar{u}(g_1, g_2, \ldots, g_n, a, b)$$

and

$$v(g_1, g_2, \ldots, g_n) = v(g_1, g_2, \ldots, g_n, a, b),$$

since the length of u and the length of v are both less than the length of w. In case (i),

$$\bar{w}(x_1, x_2, \ldots, x_n, y, z) = [\bar{u}(x_1, x_2, \ldots, x_n, y, z)/y][z\backslash\bar{v}(x_1, x_2, \ldots, x_n, y, z)].$$

Let θ be a mapping of the generators of F into $(G, \circ)$ which takes x_i into g_i, z into b, and y into a. Since θ can be extended into a homomorphism of F into $(G, \circ)$, we have

$$\begin{aligned}
\bar{w}(g_1, g_2, \ldots, g_n, a, b) &= [\bar{u}(g_1, g_2, \ldots, g_n, a, b)/a][b\backslash\bar{v}(g_1, g_2, \ldots, g_n, a, b)] \\
&= [u(g_1, g_2, \ldots, g_n)/a][b\backslash v(g_1, g_2, \ldots, g_n)] \\
&= u(g_1, g_2, \ldots, g_n) \circ v(g_1, g_2, \ldots, g_n) \\
&= w(g_1, g_2, \ldots, g_n).
\end{aligned}$$

The proofs of the other cases are analogous to that of the first case.

THEOREM 3.2. *V is a variety of quasigroups closed under isotopy if and only if V is the class of all quasigroups, all of whose loop isotopes lie in some variety of loops.*

Proof. Let V be an isotopically Q-closed variety and let L be the class of loops contained in V. L is clearly a loop variety for it is defined by $x/x=y\backslash y$ and the identities of V. If $G \in L$ and H is a loop isotope of G, then H is a quasigroup isotope of a member of V and hence belongs to V. But since H is a loop, $H \in L$. Therefore, L is an isotopically L-closed variety.

For the converse, we assume that V is the class of all quasigroups, all of whose loop isotopes lie in the variety of loops, L. Let I be a defining set of identities for L and let I' be the set of derived identities of I. Then I' defines a variety of quasigroups $Q(I')$. We will show that $V = Q(I')$.

Let G be a quasigroup in $Q(I')$ with principal loop isotope H given by $x \circ y = (x/a)(b\backslash y)$ for some elements $a, b \in G$. Let $w_1(x_1, x_2, \ldots, x_n) = w_2(x_1, x_2, \ldots, x_n)$ be an identity in I and $g_1, g_2, \ldots, g_n \in G$. G satisfies $\bar{w}_1 = \bar{w}_2$. Hence

$$\bar{w}_1(g_1, g_2, \ldots, g_n, a, b) = \bar{w}_2(g_1, g_2, \ldots, g_n, a, b)$$

and so, H is a member of L. Now, any loop isotope of G is isomorphic to a principal loop isotope of G and since varieties are closed under homomorphic images, it follows that every loop isotope of G belongs to L. But this means that G is in V. Hence $Q(I') \subseteq V$.

Next, we suppose that $G \in V$ and that

$$\bar{w}_1(x_1, x_2, \ldots, x_n, y, z) = \bar{w}_2(x_1, x_2, \ldots, x_n, y, z)$$

is an identity in I'. Let $g_1, g_2, \ldots, g_n, a, b \in G$ and let H be the principal loop isotope of G given by $x \circ y = (x/a)(b\backslash y)$. Since $H \in L$, H satisfies $w_1 = w_2$ and so,

$w_1(g_1, g_2, \ldots, g_n) = w_2(g_1, g_2, \ldots, g_n)$. It follows from Lemma 3.1 that

$$\bar{w}_1(g_1, g_2, \ldots, g_n, a, b) = \bar{w}_2(g_1, g_2, \ldots, g_n, a, b).$$

Since G satisfies the defining identities of $Q(I')$, $G \in Q(I')$ and so $V \subseteq Q(I')$. Thus, V is a variety.

It remains to show that V is isotopically Q-closed. Let $G \in V$ and let H be any isotope of G. Then, any loop isotope of H is a loop isotope of G and hence, belongs to L. Since every loop isotope of H is in L, $H \in V$. Therefore, V is isotopically Q-closed.

COROLLARY 3.3. *A derived identity is a universal identity.*

Proof. By Theorem 3.2 and its proof, the variety V defined by $\bar{w}_1 = \bar{w}_2$ is isotopically Q-closed. But then, if Q is any quasigroup satisfying $\bar{w}_1 = \bar{w}_2$, so is every isotope of Q. Therefore, $\bar{w}_1 = \bar{w}_2$ is a universal identity.

4. The lattices of isotopically L- and Q-closed varieties. We now introduce the following notation. If L is a loop variety, then $Q^*(L)$ will denote the variety of quasigroups, all of whose loop isotopes lie in L, and L^*, the variety of loops contained in $Q^*(L)$. We have seen in the preceding section that $Q^*(L)$ is the variety defined by the derived identities of any defining set of identities for L and that both $Q^*(L)$ and L^* are closed under isotopy.

THEOREM 4.1. *L^* is the maximum isotopically L-closed subvariety of L.*

Proof. L^* is clearly a subvariety of L, for if $G \in L^* \subseteq Q^*(L)$, then every loop isotope of G is in L. But since isotopy is an equivalence relation, this means that G, itself, is in L. Now, if L' is any isotopically L-closed subvariety of L and G is any loop in L', then every loop isotope of G is in L' and hence, in L. Therefore G is a loop in $Q^*(L)$ and hence, belongs to L^*.

THEOREM 4.2. *$\mathscr{L}^*$, the class of all isotopically L-closed varieties, is a sublattice of the lattice of loop varieties.*

Proof. Let $\cup$ and $\cap$ denote the join and meet operations in the lattices of loop and quasigroup varieties. We need only show that the join and meet of two isotopically L-closed varieties L_1^* and L_2^* are also isotopically L-closed varieties. This is trivial for the meet, $L_1^* \cap L_2^*$, for if $G \in L_1^* \cap L_2^*$, then every loop isotope of G belongs to L_1^* and to L_2^*, and hence to their meet.

To show that the same is true of the join, we first note that if $L^* \subseteq L_a^*$, then $Q^*(L^*) \subseteq Q^*(L_a^*)$. This holds since a quasigroup in $Q^*(L^*)$ must have all its loop isotopes in L^* and hence, in L_a^*. But $Q^*(L_a^*)$ is the variety of all quasigroups all of whose loop isotopes lie in L_a^*. Therefore $Q^*(L^*) \subseteq Q^*(L_a^*)$. In view of this fact, we have $L_1^* \cup L_2^* \subseteq Q^*(L_1^*) \cup Q^*(L_2^*) \subseteq Q^*(L_1^* \cup L_2^*)$. But the variety of loops in $Q^*(L_1^* \cup L_2^*)$ is $(L_1^* \cup L_2^*)^*$. Hence $L_1^* \cup L_2^* \subseteq (L_1^* \cup L_2^*)^*$. By Theorem 4.1 $(L_1^* \cup L_2^*)^* \subseteq L_1^* \cup L_2^*$ and hence, equality follows. Since $(L_1^* \cup L_2^*)^*$ is isotopically L-closed, so is $L_1^* \cup L_2^*$.

THEOREM 4.3. $\mathscr{Q}^*$, *the class of isotopically Q-closed varieties, is a lattice, isomorphic to $\mathscr{L}^*$.*

Proof. $\mathscr{Q}^*$ is clearly ordered under $\subseteq$. The isomorphism between $\mathscr{Q}^*$ and $\mathscr{L}^*$ is the obvious one, $L^* \xrightarrow{\theta} Q^*(L^*)$. θ is order preserving for $L_1^* \subseteq L_2^*$ if and only if $Q^*(L_1^*) \subseteq Q^*(L_2^*)$. Moreover, since $Q^*(L^*)$ consists of all quasigroups isotopic to loops in L^*, we have $L_1^* \subset L_2^*$ if and only if $Q^*(L_1^*) \subset Q^*(L_2^*)$. Therefore θ is a lattice isomorphism between $\mathscr{Q}^*$ and $\mathscr{L}^*$.

The variety V_L of all loops and the variety V_Q of all quasigroups are the greatest elements in $\mathscr{L}^*$ and $\mathscr{Q}^*$, respectively. The least element in both lattices is E, the trivial variety. Since every group variety belongs to $\mathscr{L}^*$, $\mathscr{Q}^*$ contains a sublattice which is isomorphic to the lattice of group varieties.

Although $\mathscr{Q}^*$ is a lattice isomorphic to a sublattice of loop varieties, we have been unable to establish that $\mathscr{Q}^*$ is a sublattice of the lattice of quasigroup varieties. One difficulty lies in the fact that even though our defining set of identities of a variety in $\mathscr{Q}^*$ consists only of universal identities, it is not the case that every identity implied by these identities and hence holding in all members of the variety, is a universal identity. The universal identity $(x/y)(z\backslash x)=zy$ implies the identity $x(x^2\backslash x^2)=x^2 \cdot x$ which is satisfied by the loop G below, but not by its isotope H.

G	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	3	4
3	3	4	1	5	2
4	4	5	2	1	3
5	5	3	4	2	1

H	1	2	3	4	5
1	4	5	2	1	3
2	5	3	4	2	1
3	1	2	3	4	5
4	3	4	1	5	2
5	2	1	5	3	4

G is a loop satisfying $x^2=1$ and hence, clearly satisfies the identity given above. In H, the element 1 fails to satisfy the identity. Therefore, the identity is not invariant under isotopy.

Another difficulty lies in the fact that if Q and L are quasigroup and loop varieties such that $L^* \subset Q \subseteq Q^*(L)$, then it is not necessarily the case that $Q = Q^*(L)$. The quasigroup isotope K of $C_2 \times C_2$

K	1	2	3	4
1	1	3	4	2
2	3	1	2	4
3	4	2	1	3
4	2	4	3	1

satisfies $x^2 = y^2$, $xy = yx$ and belongs to the variety $Q^*(A_2)$ where A_2 is the variety of abelian groups satisfying $x^2 = 1$. Therefore, the subvariety, V of $Q^*(A_2)$ determined by $x^2 = y^2$, $xy = yx$ is not A_2, since it contains K. We have $A_2 \subset V \subseteq Q^*(A_2)$ but $V \neq Q^*(A_2)$.

THEOREM 4.4. *A quasigroup identity is invariant under isotopy if and only if it is equivalent to a derived identity.*

Proof. Let $w_1 = w_2$ be a quasigroup identity that is equivalent to a derived identity, $\bar{w}_3 = \bar{w}_4$. By Corollary 3.3, $\bar{w}_3 = \bar{w}_4$ is a universal identity. But a quasigroup satisfies $w_1 = w_2$ if and only if it satisfies $\bar{w}_3 = \bar{w}_4$. Hence, $w_1 = w_2$ is invariant under isotopy.

On the other hand, if $w_1 = w_2$ is invariant under isotopy and is not a derived identity, let V_1 and V_2 be the quasigroup varieties defined by $w_1 = w_2$ and $\bar{w}_1 = \bar{w}_2$, respectively. These varieties are closed under isotopy with respect to their defining identities. If G is a loop in V_1, then G satisfies $\bar{w}_1 = \bar{w}_2$ since all its loop isotopes lie in V_1. If G is a loop in V_2, then G satisfies

$$\bar{w}_1(x_1, x_2, \ldots, x_n, y, z) = \bar{w}_2(x_1, x_2, \ldots, x_n, y, z).$$

Setting $y = z = 1$, we see that G satisfies

$$\bar{w}_1(x_1, x_2, \ldots, x_n, 1, 1) = \bar{w}_2(x_1, x_2, \ldots, x_n, 1, 1).$$

A simple induction argument shows that this is the identity $w_1 = w_2$. Thus, if L_1 and L_2 are the varieties of loops in V_1 and V_2, then $L_1 = L_2$. But by Theorem 3.2, L_1 and L_2 are isotopically L-closed varieties. Therefore, it follows from Theorem 4.3 that $V_1 = V_2$ which establishes the equivalence of $w_1 = w_2$ and its derived identity.

In view of the preceding theorem, the study of universal identities reduces to the study of derived identities. In fact, if S is the set of all loop identities $w_1 = w_2$ on $x_1, x_2, \ldots$ with w_1 and w_2 reduced words, then Theorem 4.4 implies that the set of derived identities of S, S', has the property that every universal identity is either in S' or is equivalent to a member of S'. It can also be shown that any loop identity that is universal for loop isotopy is either a derived identity or equivalent to its derived identity. Thus, for example, the Moufang identity, $(xy \cdot z)y = x(y \cdot zy)$ is equivalent to its derived identity,

$$([[[(x/u)(v \backslash y)]/u)(v \backslash z)]/u)(v \backslash y) = (x/u)(v \backslash [(y/u)(v \backslash [(z/u)(v \backslash y)])]).$$

5. The atoms $Q^*(A_p)$. The lattice $\mathscr{G}$ of group varieties is a sublattice of the lattice of isotopically L-closed varieties. By Theorem 4.3, the sublattice $\mathscr{S}$ of isotopically Q-closed varieties consisting of all varieties $Q^*(L)$ where L is a group variety is isomorphic to $\mathscr{G}$. Since the atoms of $\mathscr{G}$ are known to be the varieties A_p defined by $x \cdot yz = xy \cdot z$, $xy = yx$, $x^p = 1$, for p, a prime, the atoms of $\mathscr{S}$ will be the varieties of quasigroups $Q^*(A_p)$ defined by

(i) $x(v \backslash [(y/u)z]) = ([x(v \backslash y)]/u)z$;

(ii) $x[z\backslash(yu)] = y[z\backslash(xu)]$;

(iii) $((\cdots((((yx)/z)x)z)\cdots x)/z)x = yz$ where x occurs p times.

For each cardinal n, there is at most one group within isomorphism of order n in A_p. In other words, all groups in A_p with cardinality n are isomorphic. Every quasigroup in $Q^*(A_p)$ with cardinality n must be isotopic to a group in A_p of the same cardinality. Therefore we have the following result.

LEMMA 5.1. *All quasigroups in $Q^*(A_p)$ of cardinality n are isotopic.*

The preceding lemma implies that free quasigroups in $Q^*(A_p)$ are isotopic if they have the same cardinality. It is a well-known fact that two free quasigroups in a variety on α and β generators where α and β are different infinite cardinals cannot be isotopic since the cardinality of the free quasigroups will be α and β respectively. However, it is possible for two free quasigroups in a variety to have the same cardinality if each has a finite or countably infinite number of free generators.

THEOREM 5.2. *All free quasigroups in $Q^*(A_p)$ on a countably infinite or finite number of free generators are isotopic.*

Proof. We will show that the free quasigroup on one generator in $Q^*(A_p)$ is infinite. Let C_p be the group in A_p of order p and let C be the restricted direct product of $\aleph_0$ copies of C_p, where we take the generators of the copies of C_p to be $a_1, a_2, a_3, \ldots$. Let α, β be permutations on C given by

(i) for $x \in C$, $x \neq a_i$, $x\alpha = x\beta = x$, $a_1\beta = a_1$, $a_2\beta = a_2$, $a_3\beta = a_4$;

(ii) for $n \geq 0$, $a_{3n+1}\alpha = a_{3n+2}$, $a_{3n+2}\alpha = a_{3n+3}$, $a_{3n+3}\alpha = a_{3n+1}$;

(iii) for $n \geq 1$, $a_{3n+1}\beta = a_{3n+2}$, $a_{3n+2}\beta = a_{3n}$, $a_{3n+3}\beta = a_{3n+4}$.

Let G be the isotope of C given by $x \circ y = x\alpha \cdot y\beta$ and let B be the subquasigroup of G generated by a_1. First we note that $a_1 \circ a_1 = a_2a_1$; $a_2a_1 \circ a_2a_1 \circ \cdots \circ a_2a_1 = 1$ (p times); $1 \circ a_1 = a_1$; $a_1 \circ 1 = a_2$; $a_2 \circ 1 = a_3$; $1 \circ a_3 = a_4$. Therefore for $n=0$, a_{3n+1}, a_{3n+2}, a_{3n+3} are generated by a_1. Assume that for $n \leq k$, that this same triple is generated by a_1. Then

$$1 \circ a_{3k+3} = a_{3k+4} = a_{3(k+1)+1}; \quad a_{3(k+1)} \circ 1 = a_{3(k+1)+2}; \quad a_{3(k+1)+2} \circ 1 = a_{3(k+1)+3}.$$

Therefore, for each nonnegative integer i, $a_i \in B$. Since B is an infinite one-generator subquasigroup and belongs to $Q^*(A_p)$, the free quasigroup on one generator in this variety is infinite. Therefore every free quasigroup in $Q^*(A_p)$ with $\aleph_0$ or fewer generators has cardinality $\aleph_0$. By Lemma 5.1, these quasigroups are isotopic.

COROLLARY 5.3. *Every free quasigroup in an isotopically Q-closed variety containing groups is infinite.*

6. **Anti-associative loop varieties.** Every variety of loops, L, determines an isotopically L-closed variety L^*. However, L^* may be the trivial loop variety in which case $Q^*(L)$ is also trivial. L^* is nontrivial if and only if the derived identities

defining $Q^*(L)$ are consistent. T. Evans in [9] defines an *anti-associative loop variety* to be a variety of loops containing no nontrivial groups and shows that every anti-associative loop variety has as one of its defining identities, an *anti-associative loop identity*, that is, an identity which is incompatible with associativity in loops. It is easy to see that if L is not anti-associative, then L^* is nontrivial, for in this case L must contain a nontrivial variety of groups. Since every variety of groups is an isotopically L-closed variety, and L^* is the maximum subvariety of L that is closed under isotopy, L^* cannot be the trivial variety.

We have been unable to prove the existence of an anti-associative isotopically L-closed variety that is nontrivial. However, the following example does establish the existence of a variety of loops containing no isotopically L-closed subvariety except the trivial one.

Let $(G, \cdot)$ be the nine element loop satisfying $x[x(1/x)]=1$ with isotope $(H, \circ)$ given by $x \circ y = (x/2)(8\backslash y)$.

G	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	6	1	3	8	5	9	4	7
3	3	5	8	6	1	7	4	9	2
4	4	1	9	7	2	8	6	5	3
5	5	7	4	1	9	3	2	6	8
6	6	9	5	8	3	2	1	7	4
7	7	8	6	5	4	9	3	2	1
8	8	4	2	9	7	1	5	3	6
9	9	3	7	2	6	4	8	1	5

H	1	2	3	4	5	6	7	8	9
1	8	9	5	1	6	3	2	4	7
2	6	3	8	2	7	9	5	1	4
3	4	7	1	3	8	5	6	9	2
4	1	2	3	4	5	6	7	8	9
5	7	8	9	5	4	2	1	3	6
6	5	1	4	6	9	7	8	2	3
7	3	4	6	7	2	8	9	5	1
8	9	6	2	8	3	1	4	7	5
9	2	5	7	9	1	4	3	6	8

G is anti-associative, since no group except the trivial one can satisfy the identity $x[x(1/x)]=1$, but H is not, since H contains the subgroup $\{4, 5\}$. Therefore the variety L generated by G is not closed under isotopy.

We will construct the polynomial loop F on one generator in the variety L. Using functional notation, F is generated by the function $f \in G^9$ defined by $f(x)=x$. We will use the notation x^n and ^nx which are defined by

(i) $x^1 = {}^1x = x$;

(ii) $x^{n+1} = x^n \cdot x$; $^{n+1}x = x \cdot {}^nx$.

With some computation we can find the following elements in F:

$$f_9 = {}^{210}f = (1, 1, 1, 1, 1, 1, 1, 1, 6),$$
$$f_7 = {}^{126}f = (1, 1, 1, 1, 1, 1, 7, 1, 1),$$
$$f_3 = {}^{315}f = (1, 1, 9, 1, 1, 1, 1, 1, 1),$$

$$f_5 = (f_7^4)f^{36} = (1, 1, 1, 1, 5, 1, 1, 1, 1),$$

$$g = (f_5^4(f_7^4(f \cdot {}^5f_9)))f_3^2 = (1, 2, 1, 4, 1, 6, 1, 8, 1),$$

$$f_8 = ({}^4g \cdot g^2)^5 = (1, 1, 1, 1, 1, 1, 1, 4, 1),$$

$$f_4 = (g \cdot {}^3f_8)^9 = (1, 1, 1, 4, 1, 1, 1, 1, 1),$$

$$f_2 = ((f_4^3(g \cdot {}^3f_8))^3)^5 = (1, 4, 1, 1, 1, 1, 1, 1, 1),$$

$$f_6 = f_2^2 \cdot (f_4^3(g \cdot {}^3f_8))^3 = (1, 1, 1, 1, 1, 5, 1, 1, 1).$$

Since G is generated by each of its elements different from 1, f_i, $i = 2, 3, \ldots, 9$ generates an isomorphic copy of G. Therefore $F = G_1 \times G_2 \times \cdots \times G_8$ where $G_i \cong G$, $i = 1, 2, \ldots, 8$.

Now, suppose that L contains a nontrivial subvariety L' that is isotopically L-closed. Let F' be the free loop in L' on one generator. Since F was constructed to be the free loop in L on one generator, there is a homomorphism θ of F onto F'. Since $F' = F\theta = (G_1 \times G_2 \times \cdots \times G_8)\theta$, F' is generated by $G_1\theta, G_2\theta, \ldots, G_8\theta$. It is easily verified that the only homomorphic images of G are 1 and G itself. Thus, for $i = 1, 2, \ldots, 8$, $G_i\theta = 1$ or G_i. Let x_i be the element of F whose ith component is x and whose other components are 1. If $F' \neq 1$, then there exists an integer i, $1 \leq i \leq 8$ such that for all $x, y \in G_i$, $x\theta \neq y\theta$, for if not, then $G_i\theta = 1$, $i = 1, 2, \ldots, 8$, from which it follows that $F' = 1$. Consider the isotope of F, K, given by $x \circ y = (x/2_i)(8_i \backslash y)$. The identity element of K is 4_i and $5_i \circ 5_i = (5_i/2_i)(8_i \backslash 5_i) = 4_i$. Hence K contains the nontrivial subgroup $\{4_i, 5_i\}$. Now θ induces a homomorphism $\theta\circ$ of K onto K' where K' is an isotope of F' and $x\theta\circ = x\theta$. Since $4_i\theta \neq 5_i\theta$, $4_i\theta\circ \neq 5_i\theta\circ$. Therefore $\{4_i\theta\circ, 5_i\theta\circ\}$ is a two element subgroup of K'. Since K' is not anti-associative, the variety L' is not isotopically L-closed. Therefore, L contains no isotopically L-closed subvariety except the trivial one.

7. Some special examples of isotopically L-closed varieties. If we are given two isotopically L-closed varieties, neither containing the other, then we can find two new isotopically L-closed varieties by taking the join and meet of the two varieties. In this section we will give a method of constructing new isotopically L-closed varieties from a single isotopically L-closed variety.

If x, y and z are elements of a loop G, then the *commutator* of x and y is $[x, y] = xy/yx$. The *associator* of x, y, and z is $[x, y, z] = (xy \cdot z)/(x \cdot yz)$. The *left nucleus* of G is the set of all $a \in G$ such that $(ax)y = a(xy)$ for all $x, y \in G$. The *middle nucleus* of G is similarly defined in terms of $(xa)y = x(ay)$, and the *right nucleus* in terms of $(xy)a = x(ya)$. The *semicenter* is the subloop generated by all $a \in G$ such that $ax = xa$ for all $x \in G$. The *center* $Z(G)$ is the set of all elements in the intersection of the three nuclei that commute with every element of G. The three nuclei are subgroups of G and the center is an abelian subgroup, [6].

Let F be the free loop on a countably infinite number of generators, $x_1, x_2, \ldots$. Let I be the inner mapping group of F and N, a fully invariant subloop of F. We

consider the following subloops of F.

$$U_1 = \langle [n, u]\alpha \mid n \in N, u \in F, \alpha \in I \rangle,$$
$$U_2 = \langle [n, u, v]\alpha \mid n \in N, y, v \in F, \alpha \in I \rangle,$$
$$U_3 = \langle [u, n, v]\alpha \mid n \in N, u, v \in F, \alpha \in I \rangle,$$
$$U_4 = \langle [u, v, n]\alpha \mid n \in N, u, v \in F, \alpha \in I \rangle,$$
$$U_5 = \langle U_1, U_2, U_3, U_4 \rangle.$$

LEMMA 7.1. U_1–U_5 *are fully invariant subloops of* F.

Proof. The first four subloops are normal in F, since they are generated by self-conjugate subsets, [6, p. 63]. Let θ be an endomorphism of F. Since N is fully invariant, if $[n, u]\alpha$ is a generator of U_1, then $([n, u]\alpha)\theta = [n\theta, u\theta]\alpha' \in U_1$, where $\alpha' \in I$ is obtained by replacing every x_i involved in α by $x_i\theta$. Because $U_1\theta$ is generated by the images of the generators of U_1, we have $U_1\theta \leq U_1$.

A similar argument shows that U_2, U_3, and U_4 are also fully invariant. As a subloop generated by four fully invariant subloops, U_5 is fully invariant.

We will denote the varieties determined by the fully invariant subloops, U_1, U_2, U_3, U_4, U_5, N by L_1, L_2, L_3, L_4, L_5, and L, respectively. We note that any identity satisfied in one of the varieties is loop equivalent to an identity $w = 1$, where w is an element of the corresponding fully invariant subloop of F. The elements of the first four subloops are easily seen to be words in the generators. For example, an identity in L_1 is equivalent to an identity of the form,

$$w([n_1(x), w_1(x)]\alpha_1 \cdots [n_t(x), w_t(x)]\alpha_t) = 1,$$

$$\text{where } \alpha_i \in I, n_i \in N, w_i \in F, i = 1, 2, \ldots, t,$$

and where we are abbreviating a word $w(x_1, x_2, \ldots, x_n)$ by $w(x)$. Since U_5 is generated by normal subloops, $U_5 = U_1 U_2 U_3 U_4$. Thus, any identity satisfied in L_5 is equivalent to an identity of the form $(w_1 w_2 \cdot w_3)w_4 = 1$ where w_i is a word in U_i. It is clear in view of the fact that the lattice of loop varieties is isomorphic to the lattice of fully invariant subloops of F, that L_5 is the intersection of the varieties L_1, L_2, L_3, and L_4. Moreover, $L \subseteq L_5$, since $U_5 \subseteq N$. It is known that in certain cases, L_2, L_3, L_4, and L_5 are one and the same variety, for example, if L is the variety of Moufang loops.

THEOREM 7.2. *The varieties,* L_1–L_5, *are the classes* C_1–C_5 *of all loops containing a normal subloop in the semicenter, left nucleus, middle nucleus, right nucleus or center, respectively, with quotient loop in* L.

Proof. We will show first that the free loops in L_1 are in C_1. Let n be any cardinal. Since N and U_1 are fully invariant subloops of F, the free loops on n generators $s_1, s_2, \ldots$ in L_1 and L are given by $F_n/U_1(F_n)$ and $F_n/N(F_n)$, respectively. Since $U_1(F_n) \lhd N(F_n)$, there is a homomorphism θ from $F_n/U_1(F_n)$ onto $F_n/N(F_n)$ defined by $wU_1(F_n)\theta = wN(F_n)$ where $w \in F_n$. The kernel of θ is $N(F_n)/U_1(F_n)$. If $a \in \ker \theta$ and $b \in F_n/U_1(F_n)$, then $[a, b] = [n(s), w(s)]U_1(F_n) = U_1(F_n)$ since $[n(x), w(x)] \in U_1$.

Thus, ker θ is a normal subloop of the semicenter of $F_n/U_1(F_n)$. Since

$$(F_n/U_1(F_n))/(N(F_n)/U_1(F_n)) \cong F_n/N(F_n) \in L,$$

the quotient loop is in the variety L. Therefore the free loop on n generators in L_1 is in the class C_1.

Now, if G is any loop in L_1, then there exists a free loop in L_1 with free generators $x_1, x_2, \ldots$ such that the mapping ψ given by $x_i \to g_i$, for all $g_i \in G$, can be extended to a homomorphism ψ of F onto G. Let N be a normal subloop of the semicenter of F such that $F/N \in L$. Then $N\psi$ is a normal subloop of the semicenter of G. Define $\alpha: F/N \to G/N\psi$ by

$$w(x_1, x_2, \ldots, x_n)N\alpha = w(g_1, g_2, \ldots, g_n)N\psi.$$

α is clearly a homomorphism from F/N onto $G/N\psi$ which implies that $G/N\psi \in L$. Therefore $L_1 \subseteq C_1$.

To show that the converse is true, let G be a loop with normal subloop M in its semicenter and $G/M \in L$. Let $w_1 = w_2$ be any identity in L_1. Then $w_1 = w_2$ is equivalent to an identity $w([n_1, u_1]\alpha_1 \cdots [n_t, u_t]\alpha_t) = 1$ where $n_i, u_i, i = 1, 2, \ldots, t$ are words in the free generators $x_1, x_2, \ldots, x_n$. Let $g_1, g_2, \ldots, g_n \in G$. Then for $i = 1, 2, \ldots, t$, $n_i(g_1 M, g_2 M, \ldots, g_n M) = M$ since $G/M \in L$. Therefore, $n_i(g_1, g_2, \ldots, g_n) \in M \subseteq$ semicenter of G. So, $[n_i(g), u_i(g)] = 1$. Since, for any $\alpha \in I$, $1\alpha = 1$, we have

$$w([n_1(g), u_1(g)]\alpha_1, \ldots, [n_t(g), u_t(g)]\alpha_t) = 1.$$

It follows that $C_1 \subseteq L_1$.

The proofs for the other cases are analogous to the one given.

THEOREM 7.3. *If L is closed under isotopy, then so are L_2, L_3, L_4 and L_5.*

Proof. Let $G \in L_2$ with principal loop isotope H given by $x \circ y = (x/a)(b\backslash y)$. By Theorem 7.2, G contains a normal subloop N in its left nucleus with $G/N \in L$. By a result of Bruck, [6, p. 57, (v)], G/N is isotopic to H/baN. Since L is closed under isotopy and $G/N \in L$, $H/baN \in L$ also. By Bruck [6, p. 57, (iv)], the two left nuclei of the principal loop isotopes, G and H, are isomorphic with the isomorphism given by $R_b R_a$. Therefore $NR_b R_a = (Nb)a = baN$ is normal in the left nucleus of H. Hence H has a normal subloop baN in the left nucleus such that $H/baN \in L$, which means that $H \in L_2$. Since every loop isotope of G is isomorphic to a principal loop isotope of G, it follows that L_2 is closed under isotopy.

The proofs for L_3, L_4, and L_5 are similar.

If we call the loops in the classes C_2–C_5, left nuclear, middle nuclear, right nuclear, and central extensions by loops in L, then we can restate the results in Theorems 7.2 and 7.3 as

THEOREM 7.4. *The classes of central, left nuclear, middle nuclear and right nuclear extensions by loops in an isotopically L-closed variety are isotopically L-closed varieties.*

8. Nilpotent varieties. If G, H, and K are subsets of a loop M, then we define $(G, H) = \{[x, y]\alpha \mid x \in G, y \in H, \alpha \in I\}$ and

$$(G, H, K) = \{[x, y, z]\alpha \mid x \in G, y \in H, z \in K, \alpha \in I\}$$

where I is the inner mapping group of M. We consider the following chain of subloops of a loop G.

$$G^1 = G,$$
$$G^2 = [G, G] = \langle t \mid t \in (G, G) \cup (G, G, G)\rangle,$$
$$G^3 = [G^2, G] = \langle t \mid t \in (G^2, G) \cup (G^2, G, G) \cup (G, G^2, G) \cup (G, G, G^2)\rangle,$$
$$\vdots$$
$$G^{n+1} = [G^n, G] = \langle t \mid t \in (G^n, G) \cup (G^n, G, G) \cup (G, G^n, G) \cup (G, G, G^n)\rangle,$$
$$\vdots$$

G is *nilpotent* if for some nonnegative integer n, $G^{n+1} = 1$; G is nilpotent of *class n* if n is the least integer such that $G^{n+1} = 1$.

The following theorem has a proof similar to the group case with the proper extension to the associators and the use of the inner mapping group to insure normality.

THEOREM 8.1. *$G/Z(G)$ is nilpotent of class $n-1$ if and only if G is nilpotent of class n.*

Let L_n be the class of all loops of nilpotency class $m \leq n$.

THEOREM 8.2. *L_n is an isotopically L-closed variety. If G has nilpotency class n, then so does every loop isotope of G.*

Proof. The proof is by induction on n. For $n=1$, L_1 is the variety of abelian groups which is closed under isotopy and whose members all have nilpotency class one, disregarding the trivial group. Assume that the assertion holds for $n < k$. Let V be the class of all central extensions by loops in L_{k-1}. By Theorem 8.1, if $G \in L_k$, then $G \in V$. Therefore $L_k \subseteq V$. Conversely, if $G \in V$, then G has a normal subloop N in its center and G/N is in L_{k-1}. Therefore, there exists a homomorphism from G/N onto $G/Z(G)$. By the induction hypothesis, L_{k-1} is a variety. Therefore $G/Z(G) \in L_{k-1}$ and it follows that $V \subseteq L_k$. By Theorem 7.4, $V = L_k$ is an isotopically L-closed variety.

Now consider a loop G of nilpotency class k with principal loop isotope H. Then $G/Z(G)$ is isotopic to $H/Z(H)$. By Theorem 8.1, $G/Z(G)$ is nilpotent of class $k-1$. By the induction hypothesis $H/Z(H)$ also has nilpotency class $k-1$, from which it follows that H is nilpotent of class k. Since every loop isotope of G is isomorphic to a principal loop isotope, every loop isotope of G has nilpotency class k.

T. Evans has given the set below of defining identities for L_2.

(i) $[[x_1, x_2], x_3] = 1$;

(ii) $[[x_1, x_2], x_3, x_4] = 1$;

(iii) $[[x_1, x_2, x_3], x_4, x_5] = 1$;

(iv) $[[x_1, x_2, x_3], x_4] = 1$;

(v) $[x_1, [x_2, x_3], x_4] = 1$;

(vi) $[x_1, [x_2, x_3, x_4], x_5] = 1$.

By Theorem 3.2, the variety of quasigroups $Q^*(L_2)$ is closed under isotopy and consists of all quasigroups, all of whose loop isotopes are nilpotent of class $n \leq 2$. Since L_2 is isotopically L-closed, $Q^*(L_2)$ can be generated by taking all quasigroup isotopes of loops that have nilpotency class $n \leq 2$. Since, the derived identities of a set of defining identities for L_2 is a basis for the identities of $Q^*(L_2)$, this quasigroup variety is defined by the identities given below where $[x, y, u, v]$ denotes the expression $\{[(x/u)(v\backslash y)]/(v\backslash[(y/u)(v\backslash x)])\} \cdot u$ and $[x, t, z, u, v]$, the expression

$$\{[([(x/u)(v\backslash y)]/u)(v\backslash z)]/(v\backslash[(x/u)(v\backslash[(y/u)(v\backslash z)])])\} \cdot u.$$

(i) $[[x_1, x_2, x_3, x_4], x_5, x_3, x_4] = x_4 x_3$,

(ii) $[[x_1, x_2, x_3, x_4], x_5, x_6, x_3, x_4] = x_4 x_3$,

(iii) $[[x_1, x_2, x_3, x_4, x_5], x_6, x_7, x_4, x_5] = x_5 x_4$,

(iv) $[[x_1, x_2, x_3, x_4], x_5, x_3, x_4] = x_4 x_3$,

(v) $[x_1, [x_2, x_3, x_4, x_5], x_6, x_4, x_5] = x_5 x_4$,

(vi) $[x_1, [x_2, x_3, x_4, x_5, x_6], x_7, x_5, x_6] = x_6 x_5$.

9. **Varieties generated by classes of loops.** It is well known that given any class of loops there is a loop variety whose identities are exactly the identities common to the members of the class. We impose a special condition on a class of loops in order to generate an isotopically L-closed variety.

THEOREM 9.1. *If C is a class of loops containing all loop isotopes of the members of C, then the variety L of loops generated by C is closed under isotopy.*

Proof. Let $G = \prod_{i \in I} G_i$ where $G_i \subseteq C$. Let H be a principal loop isotope of G given by $x \circ y = (x/a)(b\backslash y)$ where $a = \prod_I a_i$, $b = \prod_I b_i$. We show that $H = \prod_I H_i$, where H_i is the isotope of G_i given by $x_i \circ y_i = (x_i/a_i)(b_i\backslash y_i)$, $i \in I$. Let θ be the identity mapping from H onto $\prod_I H_i$. If $x, y \in H$, then

$$(x \circ y)\theta = (x/a)(b\backslash y) = \left(\prod_I x_i / \prod_I a_i\right)\left(\prod_I b_i \backslash \prod_I y_i\right) = \prod_I (x_i/a_i) \prod_I (b_i\backslash y_i)$$

$$= \prod_I ((x_i/a_i)(b_i\backslash y_i)) = \prod_I (x_i \circ y_i) = \prod_I x_i \circ \prod_I y_i = x \circ y = x\theta \circ y\theta.$$

Hence θ is an isomorphism. Thus, a principal loop isotope of any cartesian product of members of C is a cartesian product of members of C and therefore belongs to $V(C)$, the variety generated by C.

Let A be a subloop of G and A', a principal loop isotope of A given by $x \circ y = (x/c)(d \backslash y)$ where $c, d \in A$. Then this isotopy induces an isotopy on G given in the same way and applied to all elements of G. Let H' be the isotope of G obtained in this way. Then H' is a principal isotope of G and therefore belongs to $V(C)$ by the first part. Since A' is a subset of H' and is closed under the operations $\circ$, $\wedge$, ϕ, A' is a subloop of H. Therefore, $A' \in V(C)$.

Now, let B be a homomorphic image of A and B', a principal loop isotope given by $x \circ y = (x\theta/n\theta)(m\theta \backslash y\theta)$ for some elements $m, n \in A$. Let A'' be the isotope of A given by $x \circ y = (x/n)(m \backslash y)$. θ is clearly a mapping from A'' onto B'. Moreover, $(x \cdot y)\theta = [(x/n)(m \backslash y)]\theta = (x\theta/n\theta)(m\theta \backslash y\theta) = x\theta \circ y\theta$. Hence θ is a homomorphism from A'' onto B'. Since $A'' \in V(C)$ by the second part and $V(C)$ is closed under homomorphic images, $B' \in V(C)$.

Let G be any loop in $V(C)$ with loop isotope K. Then K is isomorphic to some G', where G' is a principal loop isotope of G. G is a homomorphic image of a subloop of a cartesian product of members of C. By what has already been shown, $G' \in V(C)$. Since $K \cong G'$, $K \in V(C)$. Therefore $V(C)$ is closed under isotopy.

Let $\circ$ stand for the operation on classes of loops of closure under isotopy. Thus, if C is any class of loops, then $C\circ$ is the class of all loops isotopic to the loops in C. Since $C\circ \subseteq V(C)\circ$, we have $V(C\circ) \subseteq V((V(C))\circ)$. Now, if G is any loop in $V(C)$, then $G \in V(C\circ)$ which is closed under isotopy by Theorem 9.1. Hence, if H is a loop isotope of G, then $H \in V(C\circ)$ which implies that $(V(C))\circ \subseteq V(C\circ)$. Therefore, $V((V(C))\circ) \subseteq V(V(C\circ)) \subseteq V(C\circ)$. Thus, we have the following result.

COROLLARY 9.2. *If C is any class of loops, then $V(C\circ) = V((V(C))\circ)$.*

COROLLARY 9.3. *If there exists a finite anti-associative loop with every loop isotope anti-associative, then there is an anti-associative isotopically L-closed variety.*

Proof. If G_1 is a finite anti-associative loop, then G has only a finite number of nonisomorphic loop isotopes, $G_2, G_3, \ldots, G_n$. Evans in [9] has shown that every anti-associative loop satisfies an identity of the form $xp(x) = 1$, where $p(x)$ is a word in one variable in the subloop $[F, F]$ of the free loop F. Let $xp_i(x) = 1$ be an identity satisfied by G_i, $i = 1, 2, \ldots, n$. Then $G_1 \times G_2$ satisfies the identity $(xp_2(x)) \cdot [p_1(x \cdot p_2(x))] = 1$ which is again an anti-associative identity. Continuing in this way we can find an anti-associative identity $xp(x) = 1$ which is satisfied by $G_1 \times G_2 \times \cdots \times G_n$. Therefore, the variety V generated by G and its isotopes is anti-associative. By Theorem 9.1, V is isotopically L-closed.

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