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TENSOR PRODUCT BASES AND TENSOR DIAGONALS

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Abstract. Let X and Y denote Banach spaces with bases (x_i) and (y_i), respectively, and let $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ denote the completion in the ε and π crossnorms of the algebraic tensor product $X \otimes Y$.

The purpose of this paper is to study the structure of the tensor product spaces $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ through a consideration of the properties of the tensor product basis $(x_i \otimes y_j)$ for these spaces and the tensor diagonal $(x_i \otimes y_i)$ of such bases.

1. **Introduction.** Let X and Y be Banach spaces and denote by $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ the completion of the algebraic tensor product of X and Y in the ε and π topologies respectively (see §2 for definitions). If (x_i) is a basis for X and (y_i) a basis for Y, then the sequence of tensors $(x_i \otimes y_j)$, ordered in a certain fashion, is a basis for both $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ and is called the *tensor product basis.* We call the subsequence $(x_i \otimes y_i)$ of $(x_i \otimes y_j)$ the *tensor diagonal* of the bases (x_i) and (y_i).

The purpose of this paper is to study the properties of tensor product bases and tensor diagonals and thereby also to study the structure of the spaces $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$.

In §3 and §4 the permanence properties of tensor product bases and tensor diagonals in the ε and π topologies are discussed.

In §5 conditions on the bases (x_i) and (y_i) which are sufficient to force the diagonal $(x_i \otimes y_i)$ to be similar to a given basis are determined. A complete characterization of the diagonals of the unit vector bases in $l^p \otimes_\varepsilon l^r$ is obtained.

2. **Preliminary results and notation.** The only spaces considered in this paper will be Banach spaces. If X is a given space we will denote its dual or conjugate space by X^*. The closed linear span of a sequence (x_i) in X is denoted by $[x_i]$.

A sequence (x_i) in X is called a *basis* for X (*basic sequence* in X) if for each x in X (for each x in $[x_i]$) there exists a unique sequence of scalars (a_i) such that $x = \sum_{i=1}^\infty a_i x_i$, convergence in the norm topology of X. If for each x in X (for each x in $[x_i]$) this convergence is unconditional, then (x_i) is called an *unconditional basis* (*unconditional*

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basic sequence). For a discussion of unconditional convergence in Banach spaces see [14].

Associated with a basis (x_i) in X is a sequence of linear functionals (f_i) in X^* defined by $f_n(x)=f_n(\sum_i a_i x_i)=a_n$ and called the *associated sequence of coefficient functionals*. A basis (x_i) having coefficient functionals (f_i) is denoted by (x_i, f_i). It is well known that if (x_i, f_i) is a (unconditional) basis for X then (f_i) is a (unconditional) basic sequence in X^*.

A basis (x_i) is called *seminormalized* if $0<\inf_i \|x_i\| \leqq \sup_i \|x_i\| < +\infty$. Throughout this paper, unless specific mention is made to the contrary, *all bases will be assumed to be seminormalized.*

The following internal characterization of bases is due to M. M. Grinblyum [9] and is called the "K-condition".

THEOREM. *Let (x_i) be a sequence in X such that $[x_i]=X$. Then (x_i) is a basis for X if and only if there exists a $K \geqq 1$ such that for all $p \leqq q$ and all sequences (a_i), $\|\sum_{i=1}^{p} a_i x_i\| \leqq K \|\sum_{i=1}^{q} a_i x_i\|$.*

In the case $K=1$, (x_i) is called a *monotone basis*.

A similar criterion called the *unconditional form of the K-condition* characterizes unconditional bases. One has only to replace initial segments in the above theorem with arbitrary finite sets of integers.

If (x_i, f_i) is a basis for X and (y_i, g_i) is a basis for Y, then (x_i) and (y_i) are said to be *similar* if $\sum_i a_i x_i$ converges if and only if $\sum_i a_i y_i$ converges. It is a consequence of the Banach-Steinhaus theorem and the open mapping theorem that the bases (x_i) and (y_i) are similar if and only if there exists a linear homeomorphism $T: X \to Y$ such that $T(x_i)=y_i$.

It follows from the norm-determining property of $[f_i]$ over X [21] that the sequence of coefficient functionals (g_i) in $[f_i]^*$ associated with the basic sequence (f_i) is similar to the basis (x_i). Hence we write (f_i, x_i) is a basic sequence in X^*.

Let (x_i, f_i) be a basis for X. Then (x_i) is said to be

(i) *shrinking* [4, p. 69] if (f_i) is a basis for X^* (in the norm topology on X^*),

(ii) *boundedly complete* [4, p. 69] if $\sup_n \|\sum_{i=1}^{n} a_i x_i\| < +\infty$ implies $\sum_i a_i x_i$ converges,

(iii) *of type* wc_0 [7] if $\{x_i\}$ converges weakly to zero, (this property has also been called *semishrinking* [15], [17]). (Note: we denote weak convergence of a sequence (x_n) to x by $x_n \xrightarrow{w} x$.)

(iv) *of type* swc_0 [7] if there is a subsequence (x_{n_i}) of (x_i) such that $x_{n_i} \xrightarrow{w} 0$.

(v) of type $(\mathrm{wc}_0)^*$ [7] if $f_i \xrightarrow{w} 0$,

(vi) of type P [20] if $\sup_n \|\sum_{i=1}^{n} x_i\| < +\infty$,

(vii) of type P* [20] if there is an f in X^* such that $f(x_i)=1$ for all i.

Throughout the paper the notation $X=Y$ will mean X is linearly homeomorphic (isomorphic) to Y and $X \subset Y$ will mean X is isomorphic to a closed subspace of Y.

Finally, $B(X, Y)$ denotes the bounded bilinear forms on $X \times Y$ and $\mathscr{L}(X, Y)$

denotes the bounded linear operators from X to Y. The sequence (e_i) will *always* denote the usual unit vector basis for one of the spaces c_0 or l^p $(1 \leqq p < +\infty)$.

Let X and Y be Banach spaces. We will denote by $X \otimes_\varepsilon Y$ the completion of the algebraic tensor product $X \otimes Y$ in the norm

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\| = \sup_{\|f\| \leqq 1, f \in X^*; \|g\| \leqq 1, g \in Y^*} \left| \sum_{i=1}^{n} f(x_i) g(y_i) \right|,$$

and we denote by $X \otimes_\pi Y$ the completion of $X \otimes Y$ in the norm

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\| = \inf \left\{ \sum_{j=1}^{k} \|x_j'\| \, \|y_j'\| : \sum_{j=1}^{k} x_j' \otimes y_j' = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$

A *crossnorm* α on a tensor product space is one for which $\alpha(x \otimes y) = \|x\| \, \|y\|$. A crossnorm α is said to be uniform [19, p. 9] if for any bounded linear operators T on X and S on Y,

$$\alpha\Big(\sum_i S(x_i) \otimes T(y_i)\Big) \leqq \|S\| \, \|T\| \alpha\Big(\sum_i x_i \otimes y_i\Big).$$

It is well known that both ε and π are uniform crossnorms.

The following results on tensor products will be used frequently, most often without specific reference.

(1) *If M is a closed subspace of X and N is a closed subspace of Y, then $M \otimes_\varepsilon N$ is a closed subspace of $X \otimes_\varepsilon Y$* [19, p. 35]. As we have mentioned above, we will denote this by $M \otimes_\varepsilon N \subset X \otimes_\varepsilon Y$.

The proposition obtained by replacing ε by π in (1) is in general *not* true [19, p. 35].

(2) $X^* \otimes_\varepsilon Y^* \subset (X \otimes_\pi Y)^*$ [19, p. 34].

(3) *The space $(X \otimes_\pi Y)^*$ has a representation as the space of all continuous linear maps from X to Y^*; i.e. $(X \otimes_\pi Y)^* = \mathscr{L}(X, Y^*)$* [19, p. 45].

(4) *The space $(X \otimes_\varepsilon Y)^*$ consists exactly of those v in $B(X, Y)$ that can be represented in the form*

$$v(w) = \int_{S \times T} w_0(x', y') \, d\mu(x', y')$$

where S and T are suitable closed equicontinuous subsets of X_σ^* and Y_σ^* and where w_0 is the restriction of the bilinear form w on $X^* \times Y^*$ to $S \times T$ [18, p. 168].

(5) *If X^* has a basis then $(X \otimes_\varepsilon Y)^* = X^* \otimes_\pi Y^*$* [12].

(6) *If (x_i, f_i) is a basis for X and (y_i, g_i) is a basis for Y then the sequence $(x_i \otimes y_j)$ in $X \otimes Y$ ordered in the following fashion*

$$\begin{array}{ccc|c} x_1 \otimes y_1 & x_1 \otimes y_2 & x_1 \otimes y_3 & \cdots \\ x_2 \otimes y_1 & x_2 \otimes y_2 & x_2 \otimes y_3 & \cdots \\ x_3 \otimes y_1 & x_3 \otimes y_2 & x_3 \otimes y_3 & \cdots \\ \hline \cdots & \cdots & \cdots & \end{array}$$

is a basis for both $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ and the sequence of coefficient functionals associated with $(x_i \otimes y_j)$ is the sequence $(f_i \otimes g_j)$ [8].

We call the basis $(x_i \otimes y_j)$ the *tensor product of the bases* (x_i) *and* (y_i), or the *tensor product basis.*

The subsequence $(x_i \otimes y_i)$ of the basis $(x_i \otimes y_j)$ will be called the *tensor diagonal of the bases* (x_i) *and* (y_i).

3. **The ε-tensor product.** In this section we will be concerned mainly with determining those properties of the bases (x_i) and (y_i) which carry over to the basis $(x_i \otimes y_j)$ or to the tensor diagonal $(x_i \otimes y_i)$ in $X \otimes_\varepsilon Y$. We will also deal with the converse problem.

We begin with several lemmas. The proofs are simple and will be omitted.

LEMMA 3.1. *Let $(x_i \otimes y_j)$ be a tensor product basis for $X \otimes_\alpha Y$ where α is any crossnorm on $X \otimes Y$. Then $(x_i \otimes y_1)$ is similar to (x_i) and $(x_1 \otimes y_i)$ is similar to (y_i).*

LEMMA 3.2. *Let (x_i, f_i) be a shrinking basis for X. Then any subsequence (x_{n_i}) is shrinking.*

LEMMA 3.3. *Let (x_i) be a boundedly complete basis for X. Then any subsequence (x_{n_i}) is boundedly complete.*

Our first theorem is a consequence of Grothendieck's characterization of $(X \otimes_\varepsilon Y)^*$ in the case where X^* has a basis (see §2).

THEOREM 3.4. *Let (x_i) and (y_i) be bases for X and Y respectively. Then the tensor product basis $(x_i \otimes y_j)$ for $X \otimes_\varepsilon Y$ is shrinking if and only if (x_i) and (y_i) are shrinking bases.*

Proof. If $(x_i \otimes y_j)$ is shrinking then each of $(x_i \otimes y_1)$ and $(x_1 \otimes y_i)$ is shrinking by Lemma 3.2. It follows from Lemma 3.1 that (x_i) and (y_i) are shrinking.

If (x_i, f_i) and (y_i, g_i) are both shrinking then by definition (f_i) is a basis for X^* and (g_i) is a basis for Y^*. Since $(f_i \otimes g_j)$ is the sequence of coefficient functionals associated with the basis $(x_i \otimes y_j)$ and $(f_i \otimes g_j)$ is a basis for $X^* \otimes_\pi Y^* = (X \otimes_\varepsilon Y)^*$, the basis $(x_i \otimes y_j)$ is shrinking.

It follows from Theorem 3.4 and Lemma 3.2 that if (x_i) and (y_i) are shrinking bases for X and Y then the diagonal $(x_i \otimes y_i)$ is shrinking in $X \otimes_\varepsilon Y$. The next theorem shows that the same result is true assuming only that *one* of (x_i) and (y_i) is shrinking. It is also our first example (others will be seen later) of how strongly some properties of bases carry over to the tensor diagonal.

THEOREM 3.5. *Let (x_i, f_i) be a shrinking basis for X and (y_i, g_i) a basis for Y. Then the tensor diagonal in $X \otimes_\varepsilon Y$ is shrinking.*

Proof. Let $(z_n) = (\sum_j b_j^{(n)} x_j \otimes y_j)$ be a bounded sequence in $[x_i \otimes y_i]$ with the property that $f_k \otimes g_k(z_n) \xrightarrow{n} 0$ for $k = 1, 2, \ldots$. That is, $b_k^{(n)} \xrightarrow{n} 0$ for $k = 1, 2, \ldots$.

By virtue of the characterization of $(X \otimes_\varepsilon Y)^*$ given in §2, we need only show that

$$\int_{S\times T}\left(\sum_{j=1}^{\infty} b_j^{(n)}x_j \otimes y_j\right)(x', y')\,d\mu(x', y') \xrightarrow{n} 0$$

for S and T equicontinuous subsets of X_σ^* and Y_σ^* [16].

Since $S\times T$ is a compact metric space [18, p. 87] and z_n is in $C(S\times T)$ for each n [18, p. 168] it is sufficient to show that

$$\sum_{j=1}^{\infty} b_j^{(n)}x_j \otimes y_j(x' \otimes y')$$

goes to zero with n for each (x', y') in $S\times T$ [6, p. 265].

To do this, fix x' in S and y' in T. Then

$$\sum_{j=1}^{\infty} b_j^{(n)}x_j \otimes y_j(x' \otimes y') = x'\left(\sum_{j=1}^{\infty} b_j^{(n)}y'(y_j)x_j\right).$$

Now $f_k(\sum_j b_j^{(n)}y'(y_j)x_j)=b_k^{(n)}y'(y_k) \xrightarrow{n} 0$ for $k=1, 2, \ldots$ since (y_k) is a bounded set and we have seen that $b_k^{(n)} \xrightarrow{n} 0$. Therefore since (x_i) is shrinking and the set $\{\sum_{j=1}^{\infty} b_j^{(n)}y'(y_j)x_j\}$ is bounded in X it follows that the sequence $\{\sum_{j=1}^{\infty} b_j^{(n)}y'(y_j)x_j\}$ converges weakly to zero in X [16]. In particular $x'(\sum b_j^{(n)}y'(y_j)x_j) \xrightarrow{n} 0$, and as we have noted above this implies $(x_i \otimes y_i)$ is shrinking.

Simple examples show that the tensor diagonal (and hence the tensor product) of boundedly complete bases need not be boundedly complete in $X \otimes_\varepsilon Y$. However, in certain special cases this result is true.

Proposition 3.6. *Let (x_i) be a basis for a reflexive space X and let (y_i) be a boundedly complete basic sequence in l^1. Then the tensor diagonal $(x_i \otimes y_i)$ in $X \otimes_\varepsilon [y_i]$ is boundedly complete.*

Proof. Suppose $\sup_n \|\sum_{i=1}^n a_ix_i \otimes y_i\| < +\infty$. Then

$$\sup_n \sup_{\substack{\|f\|\leqq 1\\ f\in X^*}} \left\|\sum_{i=1}^{n} a_if(x_i)y_i\right\| < +\infty,$$

implying that $\sum_{i=1}^{\infty} a_if(x_i)y_i$ converges for each f in X^*. By the Banach-Steinhaus theorem the linear mapping $T: X^* \to [y_i]\subset l^1$ defined by $T(f)=\sum_{i=1}^{\infty} a_if(x_i)y_i$ is continuous and therefore compact since X^* is reflexive [6, p. 515]. But then the set $\{\sum_i a_if(x_i)y_i \mid f\in X^*, \|f\|\leqq 1\}$ is relatively compact in $[y_i]$ and so $\sum_{i=1}^{\infty} a_if(x_i)y_i$ converges uniformly over $\|f\|\leqq 1$. It now follows by definition of the ε-norm that $\sum_{i=1}^{\infty} a_ix_i \otimes y_i$ converges in $X \otimes_\varepsilon [y_i]$ and $(x_i \otimes y_i)$ is boundedly complete.

Theorem 3.7. *Let (x_i) be a basis for X and (y_i) a basis for Y. Then the basis $(x_i \otimes y_i)$ for $x \otimes_\varepsilon Y$ is of type* wc_0 *if and only if each of (x_i) and (y_i) is of type* wc_0.

Proof. The proof of the first implication follows immediately from Lemma 1.1. The proof of the second is similar to that of Theorem 3.5. We give a brief sketch.

Let g be in $(X \otimes_\varepsilon Y)^*$. Then

$$g(w) = \int_{S \times T} w_0(x', y')\, d\mu(x', y')$$

where S and T are equicontinuous subsets of X^*_σ and Y^*_σ and w_0 is the restriction of the bilinear form w on $X^* \times Y^*$ to $S \times T$ (recall that every w in $X \otimes Y$ can be viewed as a bilinear form on $X^* \times Y^*$). Again, $S \times T$ is a metric space so since $x_i \otimes y_j(x', y') = x_i(x')y_j(y')$ for all (x', y') in $S \times T$, and since by assumption $x_i(x') \xrightarrow{i} 0$ and $y_j(y') \xrightarrow{j} 0$, we have $(x_i \otimes y_j)$ (as a sequence in $C(S \times T)$) converges to zero at each point in $S \times T$ and hence converges weakly to zero in $C(S \times T)$.

Therefore

$$g(x_i \otimes y_j) = \int_{S \times T} x_i \otimes y_j(x', y')\, d\mu(x', y') = \mu(x_i \otimes y_j) \to 0$$

(since μ is in $C(S \times T)^*$) and $(x_i \otimes y_j)$ is of type wc_0.

An inspection of the proof of Theorem 3.7 establishes the following result which should be compared with Theorem 3.5.

THEOREM 3.8. *Let (x_i) be a basis of type* wc_0 *for X and let (y_i) be any basis for Y. Then the tensor diagonal $(x_i \otimes y_i)$ in $X \otimes_\varepsilon Y$ is of type* wc_0.

COROLLARY 3.9. *Let (x_i) and (y_i) be bases for X and Y respectively. Then $(x_i \otimes y_i)$ is of type* swc_0 *in $X \otimes_\varepsilon Y$ if and only if (x_i) or (y_i) is of type* swc_0.

Proof. Suppose $(x_{n_i}) \xrightarrow{w} 0$. By Theorem 3.8 $(x_{n_i} \otimes y_{n_i}) \xrightarrow{w} 0$ in $[x_{n_i}] \otimes_\varepsilon [y_{n_i}] \subset X \otimes_\varepsilon Y$. Hence $(x_i \otimes y_i)$ is of type swc_0.

Suppose $(x_{n_i} \otimes y_{n_i}) \xrightarrow{w} 0$ in $X \otimes_\varepsilon Y$. If $(y_{n_i}) \not\xrightarrow{w} 0$ there is an f_0 in Y^* and a subsequence (y_{m_k}) of (y_{n_i}) such that $|f_0(y_{m_k})| \geq 1$ for $k = 1, 2, \ldots$. Since $(x_{m_k} \otimes y_{m_k}) \xrightarrow{w} 0$, $f \otimes f_0(x_{m_k} \otimes y_{m_k}) \xrightarrow{k} 0$ for every f in X^* and so $|f(x_{m_k})|\,|f_0(y_{m_k})| \to 0$. It follows that $|f(x_{m_k})| \xrightarrow{k} 0$ for each f in X^* and (x_i) is of type swc_0.

REMARK 3.10. The converse of Theorem 3.8 is false. In fact, there exist bases (x_i) and (y_i) for $C[0, 1]$ neither of which is of type wc_0, but whose diagonal $(x_i \otimes y_i)$ in $C[0, 1] \otimes_\varepsilon C[0, 1]$ is similar to the unit vector basis of c_0 [11].

In view of this situation the next proposition is interesting.

PROPOSITION 3.11. *Let (x_i) be a basis for X. Then (x_i) is of type* wc_0 *if and only if the tensor diagonal $(x_i \otimes x_i)$ in $X \otimes_\varepsilon X$ is of type* wc_0.

Proof. The first implication follows immediately from Theorem 3.8.

Let f be an element of X^*. Then $f \otimes f$ is in $(X \otimes_\varepsilon X)^*$ and $f \otimes f(x_i \otimes x_i) = [f(x_i)]^2$. Since $(x_i \otimes x_i) \xrightarrow{w} 0$ we have $[f(x_i)]^2 \to 0$ and (x_i) is of type wc_0.

Gelbaum and de Lamadrid have shown that the tensor product $(x_i \otimes y_i)$ of unconditional bases (x_i) and (y_i) need not be unconditional in either $X \otimes_\varepsilon Y$ or in $X \otimes_\pi Y$ [8].

In contrast to this result we prove the following theorem which illustrates again the permanence properties of the tensor diagonal.

THEOREM 3.12. *Let (x_i) be an unconditional basis for X, (y_i) a basis for Y, and α a uniform crossnorm. Then the tensor diagonal $(x_i \otimes y_i)$ in $X \otimes_\alpha Y$ is unconditional.*

Proof. Suppose $\sum_{i=1}^\infty a_i x_i \otimes y_i \in X \otimes_\alpha Y$ and (ε_i) is any ± 1 sequence.

Then $(\varepsilon_i x_i)$ is a basis for X similar to (x_i), implying there exists a continuous linear mapping $T: X \to X$ for which $Tx_i = \varepsilon_i x_i$. Now α is a uniform crossnorm so the tensor product map $T \otimes I: X \otimes_\alpha Y \to X \otimes_\alpha Y$ (where I is the identity map) is continuous. It follows that

$$T \otimes I\left(\sum_{i=1}^{\infty} a_i x_i \otimes y_i\right) = \sum_{i=1}^{\infty} \varepsilon_i a_i x_i \otimes y_i$$

converges in $X \otimes_\alpha Y$. Since the sequence (ε_i) was arbitrary, the basic sequence $(x_i \otimes y_i)$ is unconditional.

In particular, Theorem 3.12 holds when $\alpha = \varepsilon$ or $\alpha = \pi$ [19].

The converse to Theorem 3.12 in the case $X \otimes_\varepsilon Y$ is false, as is shown by the following example.

EXAMPLE 3.13. *Let (x_i) denote the conditional basis for l^1 defined by $x_1 = e_1$, $x_n = -e_n + e_{n-1}$ for $n = 2, 3, \ldots$. Then the tensor diagonal $(x_i \otimes x_i)$ in $l^1 \otimes_\varepsilon l^1$ is similar to the unit vector basis (e_i) of l^1.*

Proof. Suppose $\sum_{i=1}^\infty a_i x_i \otimes x_i$ is in $l^1 \otimes_\varepsilon l^1$. Then $\sup_n \|\sum_{i=1}^n a_i x_i \otimes x_i\|_\varepsilon \leqq M$. From the definition of the ε-topology it follows that $\sup_{\|f\| \leqq 1} \|\sum_{i=1}^n a_i f(x_i) x_i\| \leqq M$ for all n. That is,

$$\begin{aligned}\sup_{\|f\|\leqq 1} &\left\| a_1 f(x_1) e_1 + \sum_{i=2}^{n} a_i f(x_i)(e_{i-1} - e_i)\right\| \\ &= \sup_{\|f\|\leqq 1} \left\{ |a_1 f(x_1) + a_2 f(x_2)| + \sum_{i=2}^{n} |a_{i+1} f(x_{i+1}) - a_i f(x_i)| + |a_n f(x_n)| \right\} \leqq M.\end{aligned}$$

Let f in l^∞ be defined by $f = (1, 1, 0, 0, 1, 1, 0, 0, 1, 1, \ldots)$. One easily checks that

$$\begin{aligned}|f(x_i)| &= 1 \quad \text{if } i \text{ is odd,}\\ &= 0 \quad \text{if } i \text{ is even.}\end{aligned}$$

Hence since $\|f\| = 1$ we have $\sum_{\sigma_1} |a_i| \leqq M$ (where σ_1 is the subset of $\{1, 2, \ldots, n\}$ consisting of odd integers). Similarly, defining g in l^∞ by $g = (0, 1, 1, 0, 0, 1, 1, \ldots)$ we see

$$\begin{aligned}|g(x_i)| &= 1 \quad \text{if } i \text{ is even,}\\ &= 0 \quad \text{if } i \text{ is odd,}\end{aligned}$$

so again $\sum_{\sigma_2} |a_i| \leqq M$ (where σ_2 is the set of even integers in $\{1, 2, \ldots, n\}$). Since these results are independent of n, we then have $\sum_{i=1}^\infty |a_i| \leqq 2M$ and it follows that $(x_i \otimes x_i)$ in $l^1 \otimes_\varepsilon l^1$ is similar to (e_i) in l^1.

As we have mentioned, the tensor product of unconditional bases need not be unconditional. However, in special cases the tensor product basis is always unconditional.

PROPOSITION 3.14. *Let (x_i) be an unconditional basis for X and (e_i) the unit vector basis for c_0. Then the tensor product basis $(x_i \otimes e_j)$ is unconditional in $X \otimes_\varepsilon c_0$.*

Proof. Suppose $\sum a_{ij}x_i \otimes e_j$ converges in $X \otimes_\varepsilon c_0$. Then given $\delta > 0$ there is a pair (i_0, j_0) such that for $(i_1, j_1) \geqq (i_0, j_0)$ (in the tensor product ordering of these pairs),

$$\left\| \sum_{(i_0,j_0)}^{(i_1,j_1)} a_{ij}x_i \otimes e_j \right\| < \frac{\delta}{K}$$

(where K is given by the K-condition for (x_i)).

By definition of the ε-norm we have, grouping together those tensors $a_{ij}x_i \otimes e_j$ having the same j index and summing over the designated indices,

$$\sup_{\|f\| \leqq 1} \left\| \sum_j \left(\sum_i a_{ij} f(x_i) \right) e_j \right\| < \frac{\delta}{K}.$$

That is, $\sup_j \|\sum_i a_{ij}x_i\| < \delta/K$. Now if (b_{ij}) is a 0, 1 sequence then in exactly the same way

$$\left\| \sum_{(i_0,j_0)}^{(i_1,j_1)} b_{ij}a_{ij}x_i \otimes e_j \right\| = \sup_j \left\| \sum_i b_{ij}a_{ij}x_i \right\| \leqq \sup_j K \left\| \sum_i a_{ij}x_i \right\| < K \cdot \frac{\delta}{K} = \delta,$$

and $(x_i \otimes e_j)$ is unconditional.

We have used repeatedly the theorem of Gelbaum and de Lamadrid that if (x_i) and (y_i) are bases for X and Y then $(x_i \otimes y_j)$ is a basis for $X \otimes_\varepsilon Y$ and for $X \otimes_\pi Y$. If (x_i) and (y_i) are only assumed to be basic sequences in X and Y it is natural to ask whether $(x_i \otimes y_j)$ is basic in $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$.

In the case $X \otimes_\varepsilon Y$ the question is easily answered since (x_i) is a basis for $[x_i]$, (y_i) is a basis for $[y_i]$, and so $(x_i \otimes y_j)$ is a basis for $[x_i] \otimes_\varepsilon [y_i]$, a closed subspace of $X \otimes_\varepsilon Y$.

In connection with this idea the next proposition is of interest.

PROPOSITION 3.15. *Let (x_i) be a basic sequence in X and let (y_i) be any sequence of nonzero elements of Y. Then the sequence $(x_i \otimes y_i)$ in $X \otimes_\varepsilon Y$ is a basic sequence.*

Proof. Let g be in Y^* and $\|g\| \leqq 1$. Then there is a $K \geqq 1$ such that

$$\left\| \sum_{i=1}^{n} a_i g(y_i)x_i \right\| \leqq K \left\| \sum_{i=1}^{n+p} a_i g(y_i)x_i \right\| \leqq K \sup_{\|g\| \leqq 1} \left\| \sum_{i=1}^{n+p} a_i g(y_i)x_i \right\| = K \left\| \sum_{i=1}^{n+p} a_i y_i \otimes x_i \right\|.$$

Hence

$$\left\| \sum_{i=1}^{n} a_i y_i \otimes x_i \right\| \leqq K \left\| \sum_{i=1}^{n+p} a_i y_i \otimes x_i \right\|$$

and the proposition is proved.

If (x_i) is a monotone basis for X, then the K of Proposition 3.15 is 1 and we have

COROLLARY 3.16. *Let (x_i) be a monotone basis for X and (y_i) a basis for Y. Then the tensor diagonal $(x_i \otimes y_i)$ is monotone in $X \otimes_\varepsilon Y$.*

In contrast to Corollary 3.16 we show that the tensor product of monotone bases need not be monotone.

Recall that if M and N are compact metric spaces then

$$C(M) \otimes_\varepsilon C(N) = C(M \times N)$$

and $\|\sum_{i=1}^n z_i \otimes w_i\| = \sup_{(s,t)} |\sum_{i=1}^n z_i(s)w_i(t)|$ [2].

EXAMPLE 3.17. Let $(\phi_i)_{i=0}^\infty$ denote the usual Schauder basis for $C[0, 1]$. One verifies easily that $\|\phi_0 \otimes \phi_0 + \phi_0 \otimes \phi_1 + \phi_1 \otimes \phi_1\| = 3$ while

$$\|\phi_0 \otimes \phi_0 + \phi_0 \otimes \phi_1 + \phi_1 \otimes \phi_1 - \phi_1 \otimes \phi_0\| = 2.$$

Hence $(\phi_i \otimes \phi_j)$ is not monotone (although (ϕ_i) is monotone).

The proof of the next proposition makes use of the same ideas as that of Proposition 3.14 and will be omitted. The proposition itself should be compared with Proposition 3.14.

PROPOSITION 3.18. *Let (x_i) be a monotone basis for X and (e_i) the unit vector basis for c_0. Then the basis $(x_i \otimes e_j)$ for $X \otimes_\varepsilon c_0$ is monotone.*

REMARK 3.19. Joiner has shown, in a more general context than Banach spaces, that if (x_i) and (y_i) are both of type P (type P*) then $(x_i \otimes y_j)$ is of type P (type P*) in $X \otimes_\varepsilon Y$ [13]. Since every subsequence of a type P* basis is of type P* it follows that the tensor diagonal of type P* bases is of type P*. However this is in general not the case for bases of type P.

In fact, if (x_i) is the basis of type P for $C[0, 1]$ constructed by Singer and Foiaş [7] then $x_i = \delta_i \phi_i$, where $\delta_i = \pm 1$ and (ϕ_i) is the usual Schauder basis for $C[0, 1]$. However one can show that $(\phi_i \otimes \phi_i)$, and hence $(x_i \otimes x_i)$ is similar to the basis (ϕ_i^2) in $C[0, 1]$ [11] and this last is easily seen to be nontype P.

4. **The π-tensor product.** This section is concerned with the same general subject matter as §3 except that we consider here the π-topology rather than the ε-topology. A number of the theorems proved in §1 for the space $X \otimes_\varepsilon Y$ have analogues in the space $X \otimes_\pi Y$. However, there is generally a strong contrast in the results obtained in these two topologies.

Our first result is a consequence of Theorem 3.4.

THEOREM 4.1. *Let (x_i) and (y_i) be bases for X and Y respectively. Then the tensor product basis $(x_i \otimes y_j)$ for $X \otimes_\pi Y$ is boundedly complete if and only if (x_i) and (y_i) are boundedly complete.*

Proof. The first implication is obvious from Lemmas 3.1 and 3.3.

If (x_i, f_i) and (y_i, g_i) are both boundedly complete then each of (f_i) and (g_i) are shrinking basic sequences. By Theorem 3.4, $(f_i \otimes g_j)$ is a shrinking basis for $[f_i] \otimes_\varepsilon [g_i]$, a closed subspace of $(X \otimes_\pi Y)^*$. Hence $(x_i \otimes y_j, f_i \otimes g_j)$ is boundedly complete.

Simple examples show that both (x_i) and (y_i) may be of type wc_0 and yet $(x_i \otimes y_i)$ (hence certainly $(x_i \otimes y_j)$) may not be of type wc_0 in $X \otimes_\pi Y$. However the next proposition follows immediately from Theorem 3.7.

PROPOSITION 4.2. *Let (x_i, f_i) and (y_i, g_i) be bases of type $(\mathrm{wc}_0)^*$ for X and Y respectively. Then the basis $(x_i \otimes y_j)$ for $X \otimes_\pi Y$ is of type $(\mathrm{wc}_0)^*$.*

As we have remarked following Theorem 3.12, if (x_i) is an unconditional basis for X and (y_i) is a basis for Y, then $(x_i \otimes y_i)$ is an unconditional basic sequence in $X \otimes_\pi Y$. The converse is false.

EXAMPLE 4.3. *Let (x_i) be the conditional basis for l^1 defined by $x_1 = e_1$, $x_n = e_{n-1} - e_n$ for $n = 2, 3, \ldots$. Then the tensor diagonal $(x_i \otimes x_i)$ in $l^1 \otimes_\pi l^1$ is similar to (e_i) in l^1.*

Proof. Using the well-known result that in $l^1 \otimes_\pi W$ $\|\sum_{i=1}^n e_i \otimes w_i\| = \sum_{i=1}^n \|w_i\|$ [2], one easily computes that for $n \geqq 3$

$$\left\| \sum_{i=1}^{n} a_i x_i \otimes x_i \right\| = \sum_{i=1}^{n-1} |a_i + a_{i+1}| + 2 \sum_{i=2}^{n-1} |a_i| + 3|a_n|.$$

Hence if $\sum_i a_i x_i \otimes x_i$ converges, then $\sum_i |a_i| < +\infty$ and $(x_i \otimes x_i)$ is similar to (e_i) in l^1.

As we have mentioned in §3 the tensor product of unconditional bases need not be unconditional in $X \otimes_\pi Y$. However we can show, in a manner analogous to that of Proposition 3.14, that in some cases the tensor product is always unconditional.

PROPOSITION 4.4. *Let (x_i) be an unconditional basis for X and (e_i) the unit vector basis of l^1. Then $(x_i \otimes e_j)$ is an unconditional basis for $X \otimes_\pi l^1$.*

Proof. Suppose $\sum a_{ij} x_i \otimes e_j$ converges in $X \otimes_\pi l^1$. Then given $\delta > 0$ there is a pair (i_0, j_0) such that if $(i_1, j_1) \geqq (i_0, j_0)$ (in the ordering on the indices from the tensor product basis) then

$$\left\| \sum_{(i_0, j_0)}^{(i_1, j_1)} a_{ij} x_i \otimes e_j \right\| < \frac{\delta}{K},$$

where K is given by the K-condition on (x_i). By grouping together those tensors with the same e_j term in them and summing over the designated indices we may write

$$\left\| \sum_{(i_0, j_0)}^{(i_1, j_1)} a_{ij} x_i \otimes e_j \right\| = \left\| \sum_j \left(\sum_i a_{ij} x_i \right) \otimes e_j \right\| = \sum_j \left\| \sum_i a_{ij} x_i \right\| < \frac{\delta}{K}.$$

Now if (b_{ij}) is any 0, 1 sequence, then

$$\left\| \sum_{(i_0, j_0)}^{(i_1, j_1)} b_{ij} a_{ij} x_i \otimes e_j \right\| = \sum_j \left\| \sum_i b_{ij} a_{ij} x_i \right\| \leqq \sum_j K \left\| \sum_i a_{ij} x_i \right\| < K \cdot \frac{\delta}{K} = \delta.$$

Hence $\sum b_{ij}a_{ij}x_i \otimes e_j$ converges and $(x_i \otimes e_j)$ is an unconditional basis for $X \otimes_\pi l^1$.

Our next result is an analogue of Proposition 3.15. However we will need to impose the additional conditions that (x_i) be a basis rather than just a basic sequence and that Y have a basis. These assumptions are probably superfluous.

PROPOSITION 4.5. *Let (x_i) be a basis for X and suppose Y has a basis. Then for any nonzero sequence (z_i) in Y, $(x_i \otimes z_i)$ is a basic sequence in $X \otimes_\pi Y$.*

Proof. Let (y_i) be a basis for Y and suppose

$$\sum_{i=1}^{n+p} a_i x_i \otimes z_i = \sum_{k=1}^{m} v_k \otimes w_k = \sum_{k=1}^{m} \left(\sum_j b_j^{(k)} x_j\right) \otimes \left(\sum_l c_l^{(k)} y_l\right)$$
$$= \sum_j \sum_l \left(\sum_{k=1}^{m} b_j^{(k)} c_l^{(k)}\right) x_j \otimes y_l.$$

Since

$$\sum_{j=1}^{n+p} a_j x_j \otimes z_j = \sum_{j=1}^{n+p} a_j x_j \otimes \left(\sum_l d_l^{(j)} y_l\right) = \sum_{j=1}^{n+p} \sum_l a_j d_l^{(j)} x_j \otimes y_l$$

and since $(x_i \otimes y_j)$ is a basis for $X \otimes_\pi Y$ we must have

$$\sum_j \sum_l \left(\sum_{k=1}^{m} b_j^{(k)} c_l^{(k)}\right) x_j \otimes y_l = \sum_{j=1}^{n+p} \sum_l \left(\sum_{k=1}^{m} b_j^{(k)} c_l^{(k)}\right) x_j \otimes y_l,$$

and $\sum_{k=1}^m b_j^{(k)} c_l^{(k)} = a_j d_l^{(j)}$ for $j = 1, 2, \ldots, n+p$ and any $l = 1, 2, \ldots$.

Now $\sum_{j=1}^n a_j x_j \otimes z_j = \sum_{j=1}^n \sum_l a_j d_l^{(j)} x_j \otimes y_l$, which by the above is equal to

$$\sum_{j=1}^{n} \sum_l \left(\sum_{k=1}^{m} b_j^{(k)} c_l^{(k)}\right) x_j \otimes y_l = \sum_{k=1}^{m} \left(\sum_{j=1}^{n} b_j^{(k)} x_j\right) \otimes \left(\sum_l c_l^{(k)} y_l\right) = \sum_{k=1}^{m} v_k' \otimes w_k$$

where $v_k' = \sum_{j=1}^n b_j^{(k)} x_j$. Now $\|v_k'\| \leq K\|v_k\|$ so $K \sum_{k=1}^m \|v_k\|\,\|w_k\| \geq \sum_{k=1}^m \|v_k'\|\,\|w_k\|$. Since this is true for every representation of $\sum_{i=1}^{n+p} a_i x_i \otimes z_i$, we have

$$K\left\|\sum_{i=1}^{n+p} a_i x_i \otimes z_i\right\| \geq \left\|\sum_{i=1}^{n} a_i x_i \otimes z_i\right\|$$

and by the K-condition $(x_i \otimes z_i)$ is basic in $X \otimes_\pi Y$.

The proof of Proposition 4.5 shows that the K corresponding to the basic sequence $(x_i \otimes z_i)$ is the same K which corresponds to the basis (x_i). Hence we have

COROLLARY 4.6. *Let (x_i) be a monotone basis for X and (y_i) any basis for Y. Then the tensor diagonal $(x_i \otimes y_i)$ in $X \otimes_\pi Y$ is monotone.*

Here again, as in the case of the ε-topology, the tensor product of monotone bases may not be monotone.

EXAMPLE 4.7. Let (x_i) be the (monotone) basis for l^1 defined in Example 4.3. Then $\|x_1 \otimes x_1 + x_2 \otimes x_2\| = 5$ while $\|x_1 \otimes x_1 + x_2 \otimes x_2 - x_2 \otimes x_1\| = 3$. Hence $(x_i \otimes x_j)$ is not monotone in $l^1 \otimes_\pi l^1$.

In contrast to this example we state the following result. The proof is omitted since it follows closely that of Proposition 4.4. The proposition should be compared with Propositions 4.4, 3.14, and 3.18.

PROPOSITION 4.8. *Let (x_i) be a monotone basis for X and (e_i) the unit vector basis for l^1. Then $(x_i \otimes e_j)$ is a monotone basis for $X \otimes_\pi l^1$.*

REMARK. In §3 we showed that if (x_i) is a basic sequence in X and (y_i) is a basic sequence in Y then $(x_i \otimes x_j)$ is a basic sequence in $X \otimes_\varepsilon Y$. However the method of proof is not applicable to the case $X \otimes_\pi Y$ since $[x_i] \otimes_\pi [y_i]$ need not be a closed subspace of $X \otimes_\pi Y$.

PROBLEM. *If (x_i) is a basic sequence in X and (y_i) is a basic sequence in Y, is $(x_i \otimes y_j)$ a basic sequence in $X \otimes_\pi Y$?*

A negative answer to this problem would also answer negatively the unsolved problem of whether every basic sequence in a Banach space may be extended to a basis for the whole space. For, if (x_i) and (y_i) are basic in X and Y respectively and (z_n) and (w_m) are extensions of each which are bases for X and Y, then $(x_i \otimes y_j)$ is basic in $X \otimes_\pi Y$ being a subsequence of the basis $(z_n \otimes w_m)$ for $X \otimes_\pi Y$.

5. **Properties of the tensor diagonal.** As the results of this section show, a consideration of the tensor diagonal $(x_i \otimes y_i)$ of a tensor product basis is often useful in determining properties of the spaces $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$.

We will be concerned mainly with imposing conditions on the bases (x_i) and (y_i) so that the diagonal $(x_i \otimes y_i)$ in $X \otimes_\varepsilon Y$ or $X \otimes_\pi Y$ will be similar to some given basis and with characterizing the diagonals of certain well known bases.

Our first results in this direction also demonstrate additional permanence properties of tensor products of arbitrary bases with the unit vector bases of c_0 and l^1.

PROPOSITION 5.1. *Let (x_i) be a basis for X and (e_i) the unit vector basis for c_0. Then the tensor diagonal $(x_i \otimes e_i)$ in $X \otimes_\varepsilon c_0$ is similar to (e_i) in c_0.*

Proof. Since (e_i) in c_0 is unconditional, $(x_i \otimes e_i)$ in $X \otimes_\varepsilon c_0$ is unconditional by Theorem 3.12. By definition of the ε-norm,

$$\left\| \sum_{i=1}^n x_i \otimes e_i \right\| = \sup_{\|f\| \leqq 1} \left\| \sum_{i=1}^n f(x_i) e_i \right\| = \sup_{1 \leqq i \leqq n} \sup_{\|f\| \leqq 1} |f(x_i)| \leqq \sup_i \|x_i\| < +\infty.$$

Hence $(x_i \otimes e_i)$ is of type P and is therefore similar to (e_i) in c_0.

A dual result is

PROPOSITION 5.2. *Let (x_i) be a basis for X and (e_i) the unit vector basis for l^1. Then the tensor diagonal $(x_i \otimes e_i)$ in $X \otimes_\pi l^1$ is similar to (e_i) in l^1.*

Proof. If $\sum_i a_i x_i \otimes e_i$ converges in $X \otimes_\pi l^1$ then

$$\left\| \sum_{i=m}^n a_i x_i \otimes e_i \right\| = \sum_{i=m}^n |a_i| \, \|x_i\| \xrightarrow{m,n} 0.$$

Since $\inf_i \|x_i\| > 0$ we must have $\sum_i |a_i| < +\infty$ and $(x_i \otimes e_i)$ is similar to (e_i) in l^1.

We have shown in Theorem 3.12 that the tensor diagonal $(x_i \otimes y_i)$ of an unconditional basis (x_i) for X and a basis (y_i) for Y is unconditional. In the case where (y_i) is of type P* we can say much more.

PROPOSITION 5.3. *Let (x_i) be an unconditional basis for X and (y_i) a basis of type* P* *for Y. Then the tensor diagonal $(x_i \otimes y_i)$ in $X \otimes_\varepsilon Y$ is similar to (x_i).*

Proof. If $\sum_i a_i x_i \otimes y_i$ converges in $X \otimes_\varepsilon Y$ then

$$\sup_{\|g\| \leq 1} \left\| \sum_{i=m}^{n} a_i g(y_i) x_i \right\| \xrightarrow{m,n} 0.$$

Since (y_i) is of type P*, there is a g in Y^* such that $\|g\| = 1$ and $g(y_i) = b > 0$ for all i. Hence by the above $\|\sum_{i=m}^{n} b a_i x_i\| \xrightarrow{m,n} 0$, implying $\sum_i a_i x_i$ converges in X.

Conversely, if $\sum_i a_i x_i$ converges then since (x_i) is unconditional, $\sum_i c_i a_i x_i$ converges uniformly for (c_i) in any bounded subset of l^∞ [14]. Since $\{(g(y_i)) \mid g \text{ in } Y^*, \|g\| \leq 1\}$ is a bounded subset of l^∞, it follows that $\sum_i a_i g(y_i) x_i$ converges uniformly over $\|g\| \leq 1$, and by definition of the ε-norm $\sum_i a_i x_i \otimes y_i$ converges in $X \otimes_\varepsilon Y$. Therefore (x_i) is similar to $(x_i \otimes y_i)$.

Proposition 5.3 offers an interesting contrast between the ε and π topologies since an analogous proposition with ε replaced by π is not true.

EXAMPLE 5.4. Let (e_i) denote the unit vector basis in l^p $(1 < p < +\infty)$. Then (e_i) in l^p is unconditional while (e_i) in l^1 is of type P*. However, by Proposition 5.2, $(e_i \otimes e_i)$ in $l^p \otimes_\pi l^1$ is similar to (e_i) in l^1 (and hence not to (e_i) in l^p).

Notice, however, that since $\|\cdot\|_\varepsilon \leq \|\cdot\|_\pi$, it follows that if (y_i) is of type P* and $\sum_i a_i x_i \otimes y_i$ converges in $X \otimes_\pi Y$, then $\sum_i a_i x_i$ converges in X.

The next theorem provides an interesting characterization of the tensor diagonal $(e_i \otimes e_i)$ in $l^p \otimes_\varepsilon l^r$ $(1 < p, r < \infty)$ where (e_i) denotes the unit vector basis.

THEOREM 5.5. *Let $1 < p, r < +\infty$ and set $q = p/(p-1)$. Then*

(i) *If $r \geq q$ the tensor diagonal $(e_i \otimes e_i)$ in $l^p \otimes_\varepsilon l^r$ is similar to (e_i) in c_0.*

(ii) *If $r < q$ the tensor diagonal $(e_i \otimes e_i)$ in $l^p \otimes_\varepsilon l^r$ is similar to (e_i) in $l^{qr/(q-r)}$.*

Proof. (i) By Theorem 3.13, $(e_i \otimes e_i)$ in $l^p \otimes_\varepsilon l^r$ is unconditional. Now

$$\left\| \sum_{i=1}^{n} e_i \otimes e_i \right\| = \sup_{\|x^*\| \leq 1; x^* \text{ in } l^q} \left\| \sum_{i=1}^{n} x^*(e_i) e_i \right\|_{l^r} \leq \sup_{\|x^*\| \leq 1; x^* \text{ in } l^r} \left\| \sum_{i} x^*(e_i) e_i \right\|_{l^r} \leq 1$$

(since $q \leq r$ implies any x^* in l^q is in l^r and $\|x^*\|_{l^q} \leq 1$ implies $\|x^*\|_{l^r} \leq 1$). Hence $(e_i \otimes e_i)$ is of type P and must then be similar to (e_i) in c_0.

(ii) Suppose $r < q$ and $\sum a_i e_i \otimes e_i$ converges in $l^p \otimes_\varepsilon l^r$. Then

$$\sup_{\|x^*\| \leq 1; x^* \text{ in } l^q} \left[\sum_{i=m}^{n} |a_i|^r |x^*(e_i)|^r \right]^{1/r} \to 0,$$

implying $\sum_i |a_i|^r |x^*(e_i)|^r$ converges for all x^* in l^q.

For any sequence (b_i) in $l^{q/r}$ (note $q/r>1$), define $x^*(e_i)=|b_i|^{1/r}$. Clearly the sequence $(x^*(e_i))$ defined this way is an element of l^q which we denote by x^*. By the above we then have

$$\sum_i |a_i|^r(|b_i|^{1/r})^r = \sum_i |a_i|^r|b_i|$$

converges for all (b_i) in $l^{q/r}$, and by a well known result it follows that $|a_i|^r$ is in $l^{q/(q-r)}=(l^{q/r})^*$. That is, (a_i) is in $l^{qr/(q-r)}$.

Conversely, if (a_i) is in $l^{qr/(q-r)}$, then for any m, n

$$\sup_{\|x^*\|\leqq 1;x^*\text{in } l^q}\left[\sum_{i=m}^{n} |a_i|^r|x^*(e_i)|^r\right]^{1/r}$$
$$\leqq \sup_{\|x^*\|\leqq 1}\left[\left\{\sum_{i=m}^{n}(|a_i|^r)^{q/(q-r)}\right\}^{(q-r)/q}\cdot\left\{\sum_{i=m}^{n}(|x^*(e_i)|^r)^{q/r}\right\}^{r/q}\right]^{1/r}$$

by Hölder's inequality. But this last is equal to

$$\sup_{\|x^*\|\leqq 1}\left[\sum_{i=m}^{n}|a_i|^{qr/(q-r)}\right]^{(q-r)/qr}\cdot\left[\sum_{i=m}^{n}|x^*(e_i)|^q\right]^{1/q} \leqq \left[\sum_{i=m}^{n}|a_i|^{qr/(q-r)}\right]^{(q-r)/qr} \xrightarrow{m,n} 0.$$

Hence if (a_i) is in $l^{qr/(q-r)}$, then $\sum_i a_ie_i \otimes e_i$ converges in $l^p \otimes_\varepsilon l^r$ and we see $(e_i \otimes e_i)$ is similar to (e_i) in $l^{qr/(q-r)}$.

NOTE. The special case $q=r$ of part (i) of Theorem 5.5 was proved by Dunford and Schatten [5].

COROLLARY 5.6. *Given any $1<s<+\infty$ there is a $1<p<2$ such that $(e_i \otimes e_i)$ in $l^p \otimes_\varepsilon l^p$ is similar to (e_i) in l^s.*

Proof. For $1<s<+\infty$ set $q=2s/(s-1)$. Then $q>2$ so $p=q/(q-1)<2$. By Theorem 5.5, $(e_i \otimes e_i)$ in $l^p \otimes_\varepsilon l^p$ is similar to (e_i) in $l^{pq/(q-p)}$. It is easily seen that $pq/(q-p)=s$.

In order to describe the diagonals $(e_i \otimes e_i)$ in $l^p \otimes_\pi l^r$ we will need the following lemma which is useful in duality type proofs.

LEMMA 5.7. *Let (x_i, f_i) be a basis for X having a subsequence (x_{n_i}) which is similar to (e_i) in c_0. Then the corresponding subsequence (f_{n_i}) of (f_i) in X^* is similar to (e_i) in l^1.*

Proof. The sequence (x_{n_i}) is similar to (e_i) in c_0 so the associated sequence of coefficient functionals (g_{n_i}) in $[x_{n_i}]^*$ is similar to (e_i) in l^1.

It is clear that $g_{n_i}=f_{n_i}|_{[x_{n_i}]}$ for each i. Hence if $\sum_i a_i f_{n_i}$ converges, then $\sum_i a_i f_{n_i}|_{[x_{n_i}]}$ $=\sum_i a_i g_{n_i}$ converges. But then by the above $\sum_i |a_i|<+\infty$ and (f_{n_i}), being semi-normalized, is similar to (e_i) in l^1.

NOTE. It follows from a previous comment we have made (§2) about $[f_i]$ being norm-determining over X that if (x_i, f_i) is a basis for X such that (f_{n_i}) is similar to (e_i) in c_0, then (x_{n_i}) is similar to (e_i) in l^1.

COROLLARY 5.8. *Let* $1<m, n<+\infty$ *and* $n/(n-1)\geqq m$. *Then* $(e_i \otimes e_i)$ *in* $l^m \otimes_\pi l^n$ *is similar to* (e_i) *in* l^1.

Proof. Since $n/(n-1)\geqq m$ it follows from Theorem 5.5 that $(e_i \otimes e_i)$ in $l^{m/(m-1)} \otimes_\varepsilon l^{n/(n-1)}$ is similar to (e_i) in c_0. Hence by Lemma 5.7 $(e_i \otimes e_i)$ in $l^m \otimes_\pi l^n = (l^{m/(m-1)} \otimes_\varepsilon l^{n/(n-1)})^*$ is similar to (e_i) in l^1.

It is a trivial corollary of Theorem 5.5 that $(e_i \otimes e_i)$ in $l^p \otimes_\varepsilon l^q$ $(p^{-1}+q^{-1}=1)$ is similar to (e_i) in c_0. The next theorem shows this is only a special case of a more general result.

THEOREM 5.9. *Let X be a space with an unconditional basis* (x_i, f_i). *Then the tensor diagonal* $(x_i \otimes f_i)$ *in* $X \otimes_\varepsilon [f_i]$ *is similar to* (e_i) *in* c_0.

Proof. By Theorem 3.12 the diagonal $(x_i \otimes f_i)$ is unconditional. We show it is also of type P and hence similar to (e_i) in c_0.

By definition

$$\left\| \sum_{i=1}^{n} x_i \otimes f_i \right\| = \sup_{\|x\|\leqq 1} \left\| \sum_{i=1}^{n} f_i(x)x_i \right\| \leqq K \sup_{\|x\|\leqq 1} \left\| \sum_{i} f_i(x)x_i \right\| = K$$

(where K is given by the K-condition on (x_i)). Hence $(x_i \otimes f_i)$ is of type P in $X \otimes_\varepsilon [f_i]$.

COROLLARY 5.10. *Let X be a space with an unconditional basic sequence* (y_i) *such that there is a continuous linear mapping from X onto* $[y_i]$ *(e.g. if* $[y_i]=X$ *or* $[y_i]$ *is complemented in X). Then c_0 can be embedded in* $X \otimes_\varepsilon X^*$.

Proof. Suppose $T: X \to [y_i]$ is a continuous linear mapping which is onto. Then $[y_i]=X/K$ where K is the kernel of T. It follows that $[y_i]^*$ can be embedded in X^* (since $[y_i]^*=[X/K]^*=K^0 \subset X^*$ [18, p. 161]). The conclusion now follows from Theorem 5.9 since $[y_i] \otimes_\varepsilon [y_i]^*$ is a closed subspace of $X \otimes_\varepsilon X^*$.

COROLLARY 5.11. *Let X be a space with an unconditional basis. Then* $X \otimes_\varepsilon X^*$ *is not weakly complete.*

COROLLARY 5.12. *Let X be a reflexive space with an unconditional basis. Then* $X \otimes_\varepsilon X^*$ *has a basis which is neither shrinking nor boundedly complete.*

Proof. If (x_i, f_i) is an unconditional basis for X then $(x_i \otimes f_j)$ is a basis for $X \otimes_\varepsilon X^*$ whose diagonal is similar to (e_i) in c_0. Since no separable conjugate space contains c_0 [1], the space $X \otimes_\varepsilon X^*$ has *no* boundedly complete basis.

If every basis were shrinking then $X \otimes_\varepsilon X^*$ would be reflexive [22], contradicting the fact that $c_0 \subset X \otimes_\varepsilon X^*$.

COROLLARY 5.13. *Let X be a reflexive space with an unconditional basis. Then* $X \otimes_\varepsilon X^*$ *is not quasi-reflexive of order n for any* $n\geqq 0$.

Proof. Every closed subspace of a quasi-reflexive space is quasi-reflexive [3]. Since c_0 is not quasi-reflexive, the result follows immediately from Theorem 5.9.

A dual result to Theorem 5.9 is

COROLLARY 5.14. *Let (x_i, f_i) be an unconditional basis for X. Then $(x_i \otimes f_i)$ in $X \otimes_\pi [f_i]$ is similar to (e_i) in l^1.*

Proof. By Theorem 3.12 the diagonal $(x_i \otimes f_i)$ in $X \otimes_\pi [f_i]$ is unconditional. Also, we have previously seen that the coefficient functionals associated with the basic sequence $(x_i \otimes f_i)$ in $X \otimes_\pi [f_i]$ are just the restrictions to $[x_i \otimes f_i]$ of the functionals $(f_i \otimes x_i)$ in $[f_i] \otimes_\varepsilon [x_i] \subset (X \otimes_\pi [f_i])^*$.

Now $(f_i \otimes x_i)$ in $[f_i] \otimes_\varepsilon [x_i]$ is similar to (e_i) in c_0 by Theorem 5.9, and hence is of type P. It is clear that the restrictions of a type P basic sequence will also be of type P. Therefore the coefficient functionals of the basic sequence $(x_i \otimes f_i)$ are of type P, implying $(x_i \otimes f_i)$ is of type P* and hence similar to (e_i) in l^1.

From Corollary 5.14 we obtain the following interesting result.

COROLLARY 5.15. *Let X be a reflexive space with an unconditional basis. Then $\mathscr{L}(X, X)$ is nonseparable.*

Proof. If (x_i) is unconditional, then, by Corollary 5.14, $(x_i \otimes f_i)$ in $X \otimes_\pi X^*$ is similar to (e_i) in l^1. Therefore l^∞ is isomorphic to a factor space of $(X \otimes_\pi X^*)^* = \mathscr{L}(X, X^{**}) = \mathscr{L}(X, X)$ and the conclusion follows.

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