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ON GENERALIZED COMMUTING ORDER OF AUTOMORPHISMS WITH QUASI-DISCRETE SPECTRUM

BY
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0.0. **Introduction.** Abramov [1] has defined the notions of an automorphism of a finite measure space with quasi-discrete spectrum, using the concepts of quasi-proper function and quasi-proper value introduced by Halmos and von Neumann [12]. For this class of quasi-proper functions Abramov defines an ascending sequence of abelian groups, which turns out to be a complete set of invariants for the classification of automorphisms with quasi-discrete spectrum. In addition he proves a representation theorem. In [11] an analogous theorem was proved for a homeomorphism of a compact space. Adler [2] has introduced the generalized commuting order of an automorphism on a finite measure space. The generalized commuting order is conjugacy invariant for automorphisms. [2] proves that the generalized commuting order of a totally ergodic translation of the measure space consisting of a compact metric abelian group is two. Furthermore, [5] gives conditions that every member of the generalized commuting order 2 have quasi-discrete spectrum.

In this paper we discuss the result first obtained by Abramov [1] in §1. In §2 we show a result stronger than the representation theorem of Abramov, Hahn and Parry. Our main result is to know an answer to the following question raised by Adler [2]. Let T be an automorphism of a finite measure space, then are there automorphisms T for $CN(T)=n$ for an integer n including $CN(T)=\infty$? In §3, we mention a few examples concerning this question. Furthermore, for a totally ergodic automorphism of a finite measure space with quasi-discrete spectrum, we generalize the obtained examples.

I benefited from reading the papers by Adler [2], Hahn [10] and Hoare and Parry [14].

0. **Preliminaries.** By a dynamical system we mean a pair (X, T) where X is a compact Hausdorff space and T is a homeomorphism of X onto itself. We say that (X, T) is *minimal* if X contains no nonempty closed T-invariant set, and *totally minimal* if (X, T^m) is minimal for any integer $m\neq 0$. Throughout, a homeomorphism T is bicontinuous of X onto itself. Let $C(X)$ be the Banach algebra of continuous complex valued functions on a compact Hausdorff space X. A homeomorphism T induces an isometric isomorphism V_T of the Banach algebra $C(X)$, $V_Tf(x)=f(Tx)$.

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Let (X, T) be totally minimal. We recall the following definition of quasi-proper function [10]. Let $G(T)_0$ be a group $\{\alpha \in K : V_T f(x) = \alpha f(x), |f(x)| = 1 \text{ for } f \in C(X)\}$ where K is the unit circle in the complex plane. For $i > 0$ let $G(T)_i \subset C(X)$ be the group of all functions f such that $V_T f = gf$, $|f(x)| = 1$ where $g \in G(T)_{i-1}$. We put $G(T) = \bigcup_{i>0} (T)_i$. (X, T) is said to have *quasi-discrete spectrum* if $G(T)$ spans $C(X)$, and have *discrete spectrum* if $G(T)_1$ spans $C(X)$. If it ever happens that $G(T)_n = G(T)_{n+1}$, then $G(T)_n = G(T)_{n+k}$ for all k and in this case we define $GN(T) = \min\{n : G(T)_n = G(T)_{n+1}\}$ and otherwise $GN(T) = \infty$. It follows that $G(T) = K \times O(T)$ where $O(T)$ is a subgroup of $G(T)$ isomorphic to the factor group $G(T)/K$ and the elements of $O(T)$ are linearly independent. If (X, T) is totally minimal and has quasi-discrete spectrum, then there exists a unique T-invariant finite Borel measure [10]. Two compact Hausdorff spaces X and Y are homeomorphic if and only if their corresponding Banach algebras $C(X)$ and $C(Y)$ are isomorphic [9]. Whenever a compact Hausdorff space X is metrizable, Halmos and von Neumann [12] have proved that if (X, T) is minimal and T an isometric homeomorphism on X then it is possible to introduce into X a multiplication so that X becomes (with the original topology of X) a compact metric abelian group and T becomes a translation. But a homeomorphism T of the circle such that no power of T has a fixed point is homeomorphic to a translation [8]. Hahn and Parry [11] have proved that if (X, T) is totally minimal and has quasi-discrete spectrum then there exist a compact abelian group with the normalized Haar measure and a totally ergodic affine transformation A on the space, and T is homeomorphic to A. Let X_n be the n-dimensional torus, i.e., $X_n = R^n/\sim$ where R^n is the Euclidean plane and $\sim$ is the equivalence relation which identifies n-points in the plane if their corresponding coordinates differ by integers. A metric on X_n can be defined in terms of the metric on R^n by taking the distance between n-points of X_n to be the minimal distance between any representatives of these points in R^n. The set of functions $[\psi_{p_1, p_2, \ldots, p_n}]$:

$$\psi_{p_1, p_2, \ldots, p_n}(x_1, x_2, \ldots, x_n) = \exp[2\pi i(p_1 x_1 + p_2 x_2 + \cdots + p_{n-1} x_{n-1} + p_n x_n)]$$

where $p_i = 0, \pm 1, \pm 2, \ldots$ and $i = 1, 2, \ldots, n$, forms a complete system of $C(X_n)$. The set of generators of a compact metric connected abelian group has a positive measure with respect to its Haar measure [13]. It is known that a translation $T_r : x \to x + r$ on a compact abelian group X is ergodic if and only if r is a generator of X. Let (Ω, Σ, μ) be a finite measure space where Ω is a set of elements, Σ a σ-field of measurable subsets of X, and μ a finite measure on Σ. We denote by $\Sigma(\mu)$ the Boolean σ-algebra by identifying sets in whose symmetric difference has zero measure, and μ is induced on the elements of $\Sigma(\mu)$ in the natural way. Let $L^2(\Sigma)$ be the Hilbert space of complex-valued square integrable functions defined on (Ω, Σ, μ), but sometimes we use two symbols $L^2(\Omega)$ and $L^2(\Sigma(\mu))$ instead of $L^2(\Sigma)$. Let T be an automorphism of (Ω, Σ, μ) and we denote by $V_T : f(x) \to f(Tx)$ $(f \in L^2(\Sigma))$ the linear isometry induced by T. An automorphism of the measure algebra is called a *metric automorphism*. An automorphism T of (Ω, Σ, μ) induces a metric automorphism in the natural way and sometimes we denote by T' an induced metric automorphism. T is said to be *totally ergodic* if T^n is ergodic for every integer $n \neq 0$. We recall the following definition of quasi-proper function for a totally ergodic automorphism of (Ω, Σ, μ) [1]. This definition is an analogue to that of a totally minimal dynamical system. Let $G_\mu(T)_0 = \{\alpha \in K : V_T f = \alpha f$ a.e., $\|f\|_2 = 1$ for $f \in L^2(\Sigma)\}$, and for $i > 0$ let $G_\mu(T)_i \subset L^2(\Sigma)$ be the set of all normalized functions f such that $V_T f = gf$ a.e. where $g \in G_\mu(T)_{i-1}$. Then $G_\mu(T)_i$ is the set of quasi-proper functions of order i. T is said to have *quasi-discrete spetrum* if $G_\mu(T) = \bigcup_{i>0} G_\mu(T)_i$ spans $L^2(\Sigma)$. Since $G_\mu(T)$ is a group, we follow that $G_\mu(T) = K \times O_\mu(T)$ where $O_\mu(T)$ is a subgroup of $G_\mu(T)$. We denote by $G_\mu N(T)$ the least positive integer n for which $G_\mu(T)_n = G_\mu(T)_{n+1}$ does happen and otherwise $G_\mu N(T) = \infty$. Halmos and von Neumann [12] shows that a linear isometry V on $L^2(\Sigma)$ onto itself is induced by an automorphism of the measure algebra if and only if both V and V^{-1} send every bounded function onto a bounded function and $V(fg) = Vf \cdot Vg$ whenever f and g are bounded functions. A necessary and sufficient condition that a closed subspace H of $L^2(\Sigma)$ be of the form $H = L^2(\Phi(\mu))$ where $\Phi(\mu)$ is the smallest σ-algebra of $\Sigma(\mu)$ with respect to which all functions in H are measurable is that H contains a dense subalgebra consisting of bounded functions, constant functions and their complex conjugations [6]. If G is any group, and a any element of G, then we define subsets $C_n(a)$ $(n = 0, 1, 2, \ldots)$ of G in the following way:

$$C_0(a) = \{e\},$$
$$C_n(a) = \{b \in G : bab^{-1}a^{-1} \in C_{n-1}(a)\} \qquad (n = 1, 2, \ldots).$$

It is clear that $C_n(a) \subset C_{n+1}(a)$, $n = 0, 1, 2, \ldots$. The least n for which $C_n(a) = C_{n+1}(a)$ is called the *generalized commuting order* of a in G, and we denote by $CN(a)$ such an integer n. If $bab^{-1}a^{-1} = a'$ where $b, a, a' \in G$ then it is clear that $CN(a) = CN(a')$. But the converse does not hold. Adler [2] has shown the following results: let T_a be the translation by a in a compact separable abelian group. If T_a is totally ergodic, then $CN(T_a) = 2$ and $C_1(T_a)$ is the group of translations, and $C_2(T_a)$ is the group consisting of translations composed with continuous group automorphisms; let r be an irrational number of the 1-dimensional torus X_1, and let n be an integer, then for $T_{r,n}(x_1, x_2) = (x_1 + r, x_2 + nx_1)$ (additions mod 1), $CN(T_{r,n}) = 3$, and $C_1(T_{r,n})$, $C_2(T_{r,n})$ and $C_3(T_{r,n})$ are groups. By Adler's ideas, [4] has proved, without the representation theorem due to Halmos and von Neumann, that the generalized commuting order of a totally ergodic metric automorphism with discrete spectrum on the measure algebra associated with a finite measure space is two.

1. **Properties of automorphisms with quasi-discrete spectrum.** If X is a compact abelian group, $r \in X$ and β is a continuous group automorphism of X, then $T(x) = T_r\beta(x)$ is called an *affine transformation* of X onto itself. For a totally ergodic, (with respect to the Haar measure) affine transformation on X, both definitions of

the word "quasi-discrete spectrum," introduced by Abramov [1] and Hahn and Parry [11], coincide.

The next result was first obtained by Hahn and Parry [11].

LEMMA 1.1. *Let X be a compact connected abelian group with Haar measure on X. If a totally ergodic affine transformation* $T(x)=T_r\beta(x)$, $x \in X$, *has quasi-discrete spectrum, then* (X, T) *is a totally minimal dynamical system.*

Proof. Since the totally ergodic affine transformation T has quasi-discrete spectrum, we see that $O_\mu(T)$ (μ is Haar measure on X) is equal to the character group of X. Let C_n $(n=1, 2, \ldots)$ be a set

$$\{g : B^n g = 1, g \text{ a character of } X\}$$

where B is a homomorphism on the character group of X defined by $Bg=g^{-1}V_\beta g$. Then $\bigcup_{k=1}^\infty C_k$ is equal to the character group of X if and only if T has quasi-discrete spectrum [14]. Suppose that x, y, z are in X. Suppose that $\{n_j : j \in \Delta\}$ is a net of integer such that

$$\lim T^{n_j}x = \lim T^{n_j}y = z.$$

Then $\lim g(T^{n_j}x)=\lim g(T^{n_j}y)=g(z)$ for every character g of X. Here we prove by induction that if g is a quasi-proper function belonging to C_n for any integer n then $g(x)=g(y)$. If $n=1$, then $g(x)=g(y)$ since $Bg=1$ and

$$\lim g(\beta^{n_j}(xy^{-1})) = 1.$$

Suppose now that all characters which are quasi-proper functions belonging to C_n annihilate xy^{-1}. Let g be a quasi-proper function of C_{n+1}. Then $B^{n+1}g=1$. Thus $B^n(Bg)=1$ and $Bg \in C_n$. Therefore $Bg(x)=Bg(y)$ and $g(\beta(xy^{-1}))=g(xy^{-1})$ which gives $g(x)=g(y)$. We have shown that the character group is equal to $\bigcup_{k=1}^\infty C_k$ and every character g satisfies $g(x)=g(y)$. By the duality theorem, we have $x=y$. But this is a definition of distal. Let N be the smallest β-invariant subgroup of the character group containing characters $f_{t_1}, f_{t_2}, \ldots, f_{t_n}$ and let ann (N) be the annihilator of N. Then $X/\text{ann}(N)$ is metrizable. If T' is the affine transformation on $X/\text{ann}(N)$ induced by T, we see that T' is totally ergodic and distal. From ergodicity of T', there is an element $x' \in X/\text{ann}(N)$ such that $\{T'^n x' : n=0, \pm 1, \pm 2, \ldots\}$ is dense in $X/\text{ann}(N)$. Moreover, since T' is distal, $(X/\text{ann}(N), T')$ is minimal [7]. This fact and connectedness of X guarantee that (X, T) is totally minimal.

The idea of the following theorem is essentially contained in Abramov [1].

THEOREM 1.2. *Let* (Ω, Σ, μ) *be a normalized measure space, and let Q be a totally ergodic automorphism of* (Ω, Σ, μ) *with quasi-discrete spectrum. Then there exist a compact connected abelian group X with the normalized Haar measure and affine transformation* $T(x)=T_a\beta(x)$, $x \in X$, *where* $a \in X$ *and* β *is a continuous group automorphism of X, and Q is conjugate to T. Furthermore, the dynamical system* (X, T) *is totally minimal. If* (Ω, Σ, μ) *is separable, then X is metrizable.*

Proof. We denote by X the character group of $O_\mu(Q)$ imposed by the discrete topology. If (Ω, Σ, μ) is separable, $O_\mu(Q)$ is countable so that X is metrizable. X is a compact abelian group with the normalized Haar measure. Let $\langle \cdot, \cdot \rangle$ denote the pairing between $O_\mu(Q)$ and its dual. To define the linear isometry, we put

$$V\left(\sum_{k=1}^{n} r_k f_k\right) = \sum_{k=1}^{n} r_k \langle \cdot, f_k \rangle, \qquad f_k \in O_\mu(Q).$$

Then V is an isometry which can be extended uniquely to an isometry of $L^2(\Sigma)$ onto $L^2(X)$. We suppose that V is an extended linear isometry. Since V satisfies the conditions of the multiplication theorem, there exists a metric isomorphism φ such that $V = V_\varphi$. Now define V' on $L^2(X)$ by $V' = V_\varphi V_Q V_\varphi^{-1}$ and put $O(Q) = \{\langle \cdot, f \rangle : f \in O_\mu(Q)\}$. Then V' has quasi-discrete spectrum and $K \times O(Q)$ is invariant under V'. Here we show that V' is an operator induced by an affine transformation on X. V' is an automorphism of $K \times O(Q)$ onto itself and a subgroup $K \times 1$ is mapped identically onto itself. We define maps

$$P\colon O(Q) \to O(Q), \qquad r\colon O(Q) \to K$$

by $V'g = r(g)P(g)$, $g \in O(Q)$. We have $r(fg) = r(f)r(g)$ and $P(fg) = P(f)P(g)$ for $f, g \in O(Q)$. Therefore $r(\cdot)$ and $P(\cdot)$ are homomorphisms of $O(Q)$. To show that $P(\cdot)$ is one-to-one, let us put $P(f) = P(g)$ for $f, g \in O(Q)$, then we have $V'(fg^{-1}) = r(fg^{-1})$ and $fg^{-1} = r(fg^{-1}) \in O(Q)$, i.e., $f(x) = g(x)$ for all $x \in X$. It is clear that $P(\cdot)$ is onto. We have shown that $P(\cdot)$ is an automorphism of $O(Q)$. $P(\cdot)$ therefore induces a continuous group automorphism β of X. Since r is a homomorphism of $O(Q)$ into K, r is an element of X. Therefore

$$V'g(x) = r(g)P(g) = g(r)g(\beta x) = g(T_r \beta x)$$

for all $x \in X$ and all g. We have proved that V' is an operator induced by $T_r\beta$, and Q is conjugate to $T_r\beta$. Since $T_r\beta$ is totally ergodic and has quasi-discrete spectrum, it follows that X is connected. It is clear from Lemma 1.1 that (X, T) is totally minimal.

The next corollary is the result of Hahn and Parry [11] and Hoare and Parry [14].

COROLLARY 1.3. *Let X be a compact connected abelian group with Haar measure on X. An ergodic affine transformation T has quasi-discrete spectrum if and only if (X, T) is totally minimal.*

2. **Behavior of affine transformations with quasi-discrete spectrum.** We see that the continuous group automorphisms of X_n are in correspondence with the invertible linear transformations of R^n which preserve subset Z_n of R^n consisting of points with integer coordinates. Therefore if a fixed base is chosen in X_n, the automorphisms of X_n are in one-to-one correspondence with $n \times n$ unimodular matrices. Let β be a continuous group automorphism of X_n and let $[\beta]$ denote the

corresponding matrix. If $[\beta]=[a_{ij} : i,j=1,2,\ldots,n]$ then the automorphism β is given by

$$\beta((x_1, x_2, \ldots, x_n)+Z_n) = \left(\left(\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \ldots, \sum_{j=1}^{n} a_{nj}x_j\right)+Z_n\right).$$

This equation is denoted by

$$\beta(x_1, x_2, \ldots, x_n) = \left(\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \ldots, \sum_{j=1}^{n} a_{nj}x_j\right) \quad \text{(additions mod 1).}$$

THEOREM 2.1. *Let T be a homeomorphism of X_n onto itself. If a dynamical system (X_n, T) is totally minimal and has quasi-discrete spectrum, then there is an affine transformation $T_r\beta$ of X_n homeomorphic to T. Furthermore, $T_r\beta$ is homeomorphic some affine transformation given by some matrix*

$$\begin{bmatrix} 1 & & & & \\ a_{21} & 1 & & 0 & \\ a_{31} & a_{32} & 1 & & \\ \cdots & & & \ddots & \\ a_{n1} & & \cdots & a_{n\,n-1} & 1 \end{bmatrix} \quad \textit{and} \quad r' = \begin{bmatrix} r'_1 \\ r'_2 \\ \vdots \\ r'_n \end{bmatrix}.$$

In particular, if $a_{ij}=0$ for $i\neq j$ such that $2\leqq i\leqq l$ and $1\leqq j\leqq n$, then the numbers $r'_1, r'_2, \ldots, r'_l$ are integrally independent.

Proof. Since (X_n, T) is a totally minimal dynamical system with quasi-discrete spectrum, elements of $O(T)$ are linearly independent and $O(T)$ spans $C(X_n)$. We denote by X the character group of $O(T)$ imposed by the discrete topology. X is homeomorphic to X_n. Thus the rank of the character group $O(T)$ of X is equal to the number n since X is the n-dimensional topological space. Since X is connected and locally connected, $O(T)$ is the direct product of the free cyclic groups C_j, $j=1,2,\ldots,n$. Therefore X is isomorphic to the n-dimensional torus X_n. If T' is the homeomorphism on X induced by T, (X, T') is a totally minimal dynamical system with quasi-discrete spectrum. Since X is the n-dimensional torus, we may suppose that $X=X_n$. Then

$$V_{T'}(K\times[\psi_{p_1,p_2,\ldots,p_n}]) = (K\times[\psi_{p_1,p_2,\ldots,p_n}]).$$

As we did in Theorem 1.2, we follow that T' is an affine transformation $T_r\beta$ such that T_r is a translation of X_n and β a continuous group automorphism of X_n. Thus $G(T_r\beta)=K\times[\psi_{p_1,p_2,\ldots,p_n}]$ and $GN(T_r\beta)$ is finite. If $GN(T_r\beta)=m$ and

$$G(T_r\beta)_i = [\psi_{p_1,p_2,\ldots,p_{l_i}}], \qquad i = 1,2,\ldots,m, \tag{1}$$

we choose a base dependent on (1) in X_n. For the base in X_n, there exists some unimodular matrix

$$[\beta'] = [a_{ij} : i,j = 1,2,\ldots,n]$$

so that the continuous group automorphism given by the matrix $[\beta']$ is isomorphic to β. Thus $T_r\beta$ is homeomorphic to the affine transformation $T_{r'}\beta'$ where

$$r' = \begin{bmatrix} r'_1 \\ \vdots \\ r'_n \end{bmatrix}.$$

Since the operator $V_{\beta'}$ is identical on $[\psi_{p_1,p_2,\dots,p_{l_1}}]$, it follows that

$$[\beta'] = \left[\begin{array}{c|c} E_1 & 0 \\ \hline \multicolumn{2}{c}{*} \end{array}\right]$$

where E_1 is the identity matrix of order $l_1 \times l_1$. For every $g \in [\psi_{p_1,p_2,\dots,p_{l_2}}]$,

$$V_{\beta'}g(x_1, x_2, \dots, x_{l_2}) = g'(x_1, x_2, \dots, x_n)g(x_1, x_2, \dots, x_{l_2})$$

where

$$\begin{aligned} g'(x_1, x_2, \dots, x_n) &\\ = \exp &\left[2\pi i\left(\left(\sum_{k=1}^{l_2} p_k a_{k1}\right)x_1 + \left(\sum_{k=1}^{l_2} p_k a_{k2}\right)x_2 + \cdots + \left(\sum_{k=1}^{l_2} p_k a_{kn}\right)x_n\right)\right] \\ &\cdot \exp\left[-2\pi i(p_1x_1 + p_2x_2 + \cdots + p_{l_1}x_{l_1})\right], \end{aligned}$$

and g' is an element of $G(T_{r'}\beta') = [\psi_{p_1,p_2,\dots,p_{l_1}}]$. From this fact, the form of $[\beta']$ is the following matrix

$$[\beta'] = \left[\begin{array}{c|c|c} E_1 & \multicolumn{2}{c}{0} \\ \hline & E_2 & \\ \hline \multicolumn{3}{c}{*} \end{array}\right]$$

where E_1 is the identity matrix of order $l_1 \times l_1$ and E_2 the identity matrix of order $(l_2 - l_1) \times (l_2 - l_1)$. From such an argument we see that the form of $[\beta']$ is the following triangular matrix

$$[\beta'] = \begin{bmatrix} E_1 & & & & 0 \\ & E_2 & & & \\ & & E_3 & & \\ & * & & \ddots & \\ & & & & E_m \end{bmatrix}$$

where E_j is the identity matrix of order $(l_j - l_{j-1}) \times (l_j - l_{j-1})$ (but $l_0 = 0$) for $j = 1, 2, \ldots, m$. The fact that $r'_1, r'_2, \ldots, r'_{l_1}$, are integrally independent follows immediately from the fact that $(X, T_{r'}\beta')$ is minimal.

It is well known that on the 1-dimensional torus X_1 there exist only two continuous group automorphisms, the identical automorphism and another automorphism β for which $\beta x = -x$, $x \in X_1$.

We have the next corollary here.

COROLLARY 2.2. *Let T be a homeomorphism of X_1 onto itself and if a dynamical system (X_1, T) is totally minimal and has quasi-discrete spectrum, then there is a translation of X_1 homeomorphic to T.*

Let T be a homeomorphism of the 2-dimensional torus X_2, and defined by

$$T: (x_1, x_2) \to (x_1 + r, x_2 + nx_1) \quad \text{(additions mod 1)}$$

where r is a real number and n an integer. Such a transformation is called a skew product transformation of X_2 [3].

COROLLARY 2.3. *If (X_2, T) is totally minimal and has quasi-discrete spectrum, then there is a skew product transformation of X_2 homeomorphic to T.*

This is direct from Theorem 2.1.

COROLLARY 2.4. *Let (X_n, T) be a totally minimal dynamical system with quasi-discrete spectrum and let $CN(T) = 2$. Then there is a following affine transformation $T_r\beta$ homeomorphic to T,*

$$T_r\beta: (x_1, x_2, \ldots, x_n)$$
$$\to \left(x_1 + r_1, \ldots, x_l + r_l, x_{l+1} + \sum_{j=1}^{l} a_{l+1\,j}x_j + r_{l+1}, \ldots, x_n + \sum_{j=1}^{l} a_{nj}x_j + r_n\right)$$

(*additions* mod 1)

where each a_{ij} is some integer and each r_j some real number, and moreover the numbers $r_1, r_2, \ldots, r_l$ are integrally independent.

COROLLARY 2.5. *Let (X_n, T) be a totally minimal dynamical system with quasi-discrete spectrum and let $GN(T) = n$. Then there is a following affine transformation $T_r\beta$ homeomorphic to T,*

$$T_r\beta: (x_1, x_2, \ldots, x_n)$$
$$\to \left(x_1 + r_1, x_2 + a_{21}x_1 + r_2, \ldots, x_{n-1} + \sum_{j=1}^{n-2} a_{n-1\,j}x_j + r_{n-1}, x_n + \sum_{j=1}^{n-1} a_{nj}x_j + r_n\right)$$

(*additions* mod 1)

where each a_{ij} is some integer, but $a_{j\,j-1}$ is nonzero for $j = 2, \ldots, n$, and each r_j some real number, but r_1 irrational.

Proof. From Theorem 2.1, T is homeomorphic to some affine transformation $T_r\beta$ such that the matrix $[\beta]$ is of the following form

$$[\beta] = \begin{bmatrix} 1 & & & & \\ a_{21} & 1 & & 0 & \\ a_{31} & a_{32} & 1 & & \\ \cdots & & & \ddots & \\ a_{n1} & & \cdots & a_{n\,n-1} & 1 \end{bmatrix} \quad \text{and} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

where each number a_{ij} is some integer and each number r_j some real. But $a_{j\,j-1}$ is nonzero for $j=2,\ldots,n$ since $GN(T)=n$ and from (totally) minimality of $(X_n, T_r\beta)$, r_1 is irrational.

COROLLARY 2.6. *Let T be a totally ergodic automorphism of a finite measure space (Ω, Σ, μ) with quasi-discrete spectrum. Then there exist metric automorphisms W and S such that W has each function of $O_\mu(T)$ as a proper function and V_S maps $O_\mu(T)$ onto itself, and the metric automorphism of T is equal to SW.*

The proof of Corollary 2.6 is similar to [5].

3. **Generalized commuting order of transformations.** As pointed out in Adler [2], it is interesting to know an answer to the following question: are there examples for $CN(T)=n$ for an integer n including $CN(T)=\infty$? The next example shows that this question has a positive answer.

THEOREM 3.1. *Let T be an affine transformation of X_n and let (X_n, T) be a totally minimal dynamical system (with quasi-discrete spectrum). If $GN(T)=n$, then $CN(T)=n+1$.*

Proof. From Theorem 2.1 we may suppose that the affine transformation T is written as follows

$$T: (x_1, x_2, \ldots, x_n)$$
$$\to \left(x_1+r_1,\, x_2+a_{21}x_1+r_2,\, \ldots,\, x_{n-1}+\sum_{j=1}^{n-2} a_{n-1\,j}x_j+r_{n-1},\, x_n+\sum_{j=1}^{n-1} a_{nj}x_j+r_n\right)$$

(additions mod 1)

where each number a_{ij} is an integer and $r_j, j=1,2,\ldots,n$, are real. Since $GN(T)=n$, from Corollary 2.5, the integer $a_{j\,j-1}$ is nonzero for $j=2,\ldots,n$ and r_1 irrational. We show by induction that members of $C_n(T)$, $n=1,2,\ldots$, are affine transformations. If $n=1$, then for $S_1 \in C_1(T)$ we have $S_1T=TS_1$ and therefore

$$V_{S_1}(K\times[\psi_{p_1,p_2,\ldots,p_n}]) = K\times[\psi_{p_1,p_2,\ldots,p_n}].$$

From this relation and the proof of Theorem 1.2, we see that S_1 is an affine transformation. Let members of $C_n(T)$ be affine transformations and let $S_{n+1} \in C_{n+1}(T)$. Then $S_{n+1}TS_{n+1}^{-1} = S_nT$ where $S_n \in C_n(T)$. T and S_nT are affine transformations, and (X_n, T) and (X_n, S_nT) are totally minimal dynamical systems (with quasi-discrete spectrum). Therefore it follows that

$$V_{S_{n+1}}(K \times [\psi_{p_1, p_2, \ldots, p_n}]) = K \times [\psi_{p_1, p_2, \ldots, p_n}].$$

We see easily that S_{n+1} is an affine transformation. We have shown that the members of $\bigcup_{n=0}^{\infty} C_n(T)$ are affine transformations. If $S_{n+2} \in C_{n+2}(T)$ then we can write $S_{n+2}TS_{n+2}^{-1} = S_{n+1}T$ for some $S_{n+1} \in C_{n+1}(T)$. Since $S_{n+1}T$ is homeomorphic to T, $(X_n, S_{n+1}T)$ is totally minimal and $GN(S_{n+1}T) = n$. The affine transformation S_{n+1} has a representation as follows:

$$S_{n+1} = T_{r_{n+1}}\beta_{n+1}$$

where $T_{r_{n+1}}$ is a translation of X_n and $\beta_{r_{n}+1}$ is a continuous group automorphism of X_n. We put $T = T_r\beta$ for convenience. Since $S_{n+1}T$ is homeomorphic to T and since

$$S_{n+1}T = T_{r_{n+1}+\beta_{n+1}(r)}\beta_{n+1}\beta$$

has quasi-discrete spectrum and $GN(S_{n+1}T) = n$, we see by induction that the matrix $[\beta_{n+2}]$ is lower triangular. $[\beta_{n+1}\beta]$ is a lower triangular matrix such that the numbers 1 appear throughout the diagonal, because the spectrum of $S_{n+1}T$ is quasi-discrete. Thus we follow that $[\beta_{n+1}]$ is a matrix such that

$$\begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ & & \ddots & \\ & * & & 1 \end{bmatrix}.$$

We show now that $C_{n+1}(T) = C_{n+2}(T)$. Let $S_{n+2} \in C_{n+2}(T)$. Then $S_{n+2}TS_{n+2}^{-1}T^{-1} = S_{n+1}$ where $S_{n+1} \in C_{n+1}(T)$. Furthermore, for the affine transformation $S_{n+1} = T_{r_{n+1}}\beta_{n+1}$,

$$S_{n+1}TS_{n+1}^{-1}T^{-1} = S_n \tag{1}$$

where $S_n \in C_n(T)$ and $S_n = T_{r_n}\beta_n$. Then from (1),

$$\beta_{n+1}\beta = \beta_n\beta\beta_{n+1}, \tag{2}$$

$$r_{n+1} + \beta_{n+1}(r) = r_n + \beta_n(r) + \beta_n\beta(r_{n+1}). \tag{3}$$

From the fact that the spectrum of S_nT is quasi-discrete, the matrix $[\beta_n\beta]$ is a lower triangular matrix such that the numbers 1 appear throughout the diagonal.

Furthermore, since the numbers 1 appear throughout the diagonal of $[\beta_{n+1}]$ and from the relation (2), we see that

$$(4) \qquad [\beta_n] = \begin{bmatrix} 1 & & & & & \\ 0 & & & & & \\ b_{31} & 0 & 1 & & 0 & \\ \vdots & \ddots & & \ddots & \ddots & \\ b_{n1} & \cdots & b_{n\,n-2} & & 0 & 1 \end{bmatrix}$$

where each b_{ij} is some integer. Therefore, from the relations (3) and (4), it is easily to see that

$$r_n = \begin{bmatrix} 0 \\ r_{n2} \\ \vdots \\ r_{nn} \end{bmatrix}.$$

Next, for the affine transformation $S_n = T_{r_n}\beta_n$, we have $S_n T S_n^{-1} = S_{n-1}T$ where $S_{n-1} \in C_{n-1}(T)$, and therefore S_{n-1} is equal to an affine transformation $T_{r_{n-1}}\beta_{n-1}$ such that β_{n-1} has a unimodular matrix

$$[\beta_{n-1}] = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & 1 & & 0 & & \\ c_{41} & 0 & 0 & 1 & & & \\ \vdots & \ddots & & \ddots & \ddots & \ddots & \\ c_{n1} & \cdots & c_{n\,n-3} & & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad r_{n-1} = \begin{bmatrix} 0 \\ 0 \\ r_{n-1\,3} \\ \vdots \\ r_{n-1\,n} \end{bmatrix}.$$

Following this argument step-by-step, we get that $S_3 T S_3^{-1} = S_2 T$ where $S_2 \in C_2(T)$, and the behavior of S_2 is the following

$$S_2(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{n-1}, x_n + r_{2n}) \quad \text{(addition mod 1)}$$

where r_{2n} is some real number. Therefore it follows that S_2 commutes with T, i.e., $S_2 \in C_1(T)$. From this fact,

$$S_3 \in C_2(T), \ldots, S_{n-1} \in C_{n-2}(T),\ S_n \in C_{n-1}(T)$$

and, from $S_{n+1} T S_{n+1}^{-1} T^{-1} = S_n$,

$$S_{n+1} \in C_n(T) \quad \text{and} \quad S_{n+2} \in C_{n+1}(T).$$

Thus we have shown that $C_{n+1}(T) = C_{n+2}(T)$.

We give a translation T_d:

$$(x_1, x_2, \ldots, x_n) \to (x_1 + d_1, x_2 + d_2, \ldots, x_n + d_n) \quad \text{(additions mod 1)}$$

where $d_1, d_2, \ldots, d_{n-1}$ and d_n are nonzero real numbers, but not integers. Then we have the following equation

$$T_d T T_d^{-1} T^{-1}(x_1, x_2, \ldots, x_n)$$
$$= (x_1, x_2+d_2^{(2)}, x_3+d_3^{(2)}, \ldots, x_n+d_n^{(2)}) \quad \text{(additions mod 1)}$$

where

$$d_k^{(2)} = -\sum_{j=1}^{k-1} a_{kj} d_j, \qquad k = 2, 3, \ldots, n.$$

Putting

$$T_{d^{(2)}}(x_1, x_2, \ldots, x_n) = (x_1, x_2+d_2^{(2)}, x_3+d_3^{(2)}, \ldots, x_n+d_n^{(2)}) \quad \text{(additions mod 1)},$$

we have that

$$T_{d^{(2)}} T T_{d^{(2)}}^{-1} T^{-1}(x_1, x_2, \ldots, x_n) = (x_1, x_2, x_3+d_3^{(3)}, x_4+d_4^{(3)}, \ldots, x_n+d_n^{(3)})$$
$$\text{(additions mod 1)}$$

where

$$d_k^{(3)} = -\sum_{j=2}^{k-1} a_{kj} d_j^{(2)}, \qquad k = 3, 4, \ldots, n.$$

We obtain from the argument that

$$T_{d^{(n)}} T(x_1, x_2, \ldots, x_n) = T T_{d^{(n)}}(x_1, x_2, \ldots, x_n)$$

where $d_n^{(n)} = -a_{n\, n-1} d_{n-1}^{(n-1)}$ and

$$T_{d^{(n)}}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n+d_n^{(n)}) \quad \text{(addition mod 1)}.$$

Since $a_{j\, j-1}$ is a nonzero integer for $j=2, \ldots, n$, we can choose nonzero real numbers (but not integers) $d_1, d_2, \ldots, d_{n-1}$ and d_n such that

$$d_k^{(2)} \neq 0 \quad \text{for } k = 2, 3, 4, \ldots, n,$$
$$d_k^{(3)} \neq 0 \quad \text{for } k = 3, 4, 5, \ldots, n,$$
$$d_k^{(4)} \neq 0 \quad \text{for } k = 4, 5, 6, \ldots, n,$$
$$\cdots$$
$$d_k^{(n-1)} \neq 0 \quad \text{for } k = n-1, n,$$
$$d_n^{(n)} \neq 0.$$

In particular, we can choose d_1 such that the number $-\frac{1}{2} \cdot d_1$ is the first coordinate r_1 of $r=(r_1, r_2, \ldots, r_n)$. For such real numbers d_j, $j=1, 2, \ldots, n$, and the translation T_d where $d=(d_1, d_2, \ldots, d_n)$, it follows that $T_{d^{(n)}}T = TT_{d^{(n)}}$, and since $d_n^{(n)} \neq 0$ (mod 1), $T_{d^{(n)}}$ is not the identical map. Thus we have that $T_d \in C_n(T) - C_{n-1}(T)$. Here we put $S = T_b \beta'$ where β' is a group automorphism of X_n such that $\beta' x = -x$, $x \in X_n$, and T_b is a translation determined by the element b satisfying the equation $b + [\beta'][r] = d + r + [\beta][b]$. Then $STS^{-1}T^{-1} = T_d$ and therefore $S \in C_{n+1}(T) - C_n(T)$. We have shown that $CN(T) = n+1$.

COROLLARY 3.2. *Let T be an affine transformation and let (X_1, T) be a totally minimal dynamical system (with quasi-discrete spectrum). If $GN(T)=1$, then $CN(T)=2$.*

The corollary was shown by Adler [2].

COROLLARY 3.3. *Let T be a skew product transformation of X_2 and let (X_2, T) be a totally minimal dynamical system (with quasi-discrete spectrum) and let $GN(T)=2$. Then $CN(T)=3$.*

The statement is direct from Theorem 3.1.

The following theorem is a result better than Theorem 3.1.

THEOREM 3.4. *Let T be an affine transformation of X_n and let (X_n, T) be a totally minimal dynamical system (with quasi-discrete spectrum) with $GN(T)=m$. Then $CN(T)=m+1$.*

Proof. We put $T=T_r\beta$ where T_r is a translation of X_n and β a continuous group automorphism of X_n. From Theorem 2.1 and the analogous argument of Corollary 2.5, we may suppose that since $GN(T)=m$, the affine transformation T is written as follows

$$T_r\beta\colon (x_1, x_2, \ldots, x_n)$$

$$\to \Bigl(x_1+r_1, \ldots, x_{l_1}+r_{l_1}, x_{l_1+1}+\sum_{j=1}^{l_1} a_{l_1+1\,j}x_j+r_{l_1+1}, \ldots,$$

$$x_{l_2}+\sum_{j=1}^{l_1} a_{l_2j}x_j+r_{l_2}, \ldots, x_{l_{m-1}+1} \tag{1}$$

$$+\sum_{j=1}^{l_{m-1}} a_{l_{m-1}+1\,j}x_j+r_{l_{m-1}+1}, \ldots, x_{l_m}+\sum_{j=1}^{l_{m-1}} a_{l_mj}x_j+r_{l_m}\Bigr)$$

(additions mod 1).

Here each a_{ij} is an integer and each l_j an integer such that $l_m=n$ and there is at the least one nonzero integer in

$$\{a_{l_{i-1}+1\,l_{i-2}+1}, a_{l_{i-1}+1\,l_{i-2}+2}, \ldots, a_{l_{i-1}+1\,l_{i-1}}\} \qquad (l_0 = 0)$$

for $i=2, 3, \ldots, m$. Since T is totally ergodic, the real numbers r_k, $k=1, 2, \ldots, l_1$, are integrally independent. Thus the matrix $[\beta]$ is of the form

$$\begin{bmatrix} E_{11} & & & & \\ A_{21} & E_{22} & & 0 & \\ A_{31} & A_{32} & E_{33} & & \\ & \cdots & & \ddots & \\ A_{m1} & & \cdots & A_{m\,m-1} & E_{mm} \end{bmatrix} \tag{2}$$

where E_{ii}, $i=1,2,\ldots,m$, are the identity matrices of order $(l_i-l_{i-1})\times(l_i-l_{i-1})$

and A_{ij}, $i\neq j$, $i=2,3,\ldots,m$, $j=1,2,\ldots,m$, are matrices of order $(l_i-l_{i-1})\times(l_j-l_{j-1})$ (but $l_0=0$). Since T has quasi-discrete spectrum and $GN(T)=m$, the blocks of $[\beta]$, $A_{i\,i-1}$, $i=2,3,\ldots,m$, are nonzero matrices and there is at the least one nonzero integer in the first row

$$(a_{l_{i-1}+1\ l_{i-2}+1}, a_{l_{i-1}+1\ l_{i-2}+2}, \ldots, a_{l_{i-1}+1\ l_{i-1}})$$

of the matrix $A_{i\,i-1}$. For each positive integer k and each $S_k\in C_k(T)$ with $S_k=T_{r_k}\beta_k$, it follows by induction that the matrix $[\beta_k]$ is of the form

$$\begin{bmatrix} B_{11} & & & & \\ B_{21} & B_{22} & & 0 & \\ B_{31} & B_{32} & B_{33} & & \\ & \cdots & & \ddots & \\ B_{m1} & & \cdots & B_{m\,m-1} & B_{mm} \end{bmatrix} \tag{3}$$

where B_{ij}, $i,j=1,2,\ldots,m$, are matrices of order $(l_i-l_{i-1})\times(l_j-l_{j-1})$. Because, if $k=1$, then $\beta_1\beta=\beta\beta_1$. Since the spectrum of $T=T_r\beta$ is quasi-discrete, the matrix $[\beta_1]$ is of the form (3). Let $[\beta_k]$ be of the form (3). Then the rank of the group $O(T)\cap G(T)_i$ is equal to the rank of the group $O(S_kT)\cap G(S_kT)_i$ since $V_{S_{k+1}}G(T)_i=G(S_kT)_i$ for $i=1,2,\ldots,m$. Thus we see that the matrix $[\beta_k\beta]$ is of the form (2). Since $\beta_{k+1}\beta=\beta_k\beta\beta_{k+1}$, the matrix $[\beta_{k+1}]$ is of the form (3). We show that $C_{m+1}(T)=C_{m+2}(T)$. If $S_{m+2}\in C_{m+2}(T)$ with $S_{m+2}=T_{r_{m+2}}\beta_{m+2}$, then we have $S_{m+2}TS_{m+2}^{-1}=S_{m+1}T$ where $S_{m+1}\in C_{m+1}(T)$ with $S_{m+1}=T_{r_{m+1}}\beta_{m+1}$. From the equation above, we have $\beta_{m+2}\beta=\beta_{m+1}\beta\beta_{m+2}$. Since the matrix $[\beta_{m+1}\beta]$ is of the form (2), the matrix $[\beta_{m+1}]$ is also of the form (2). For the affine transformation $S_{m+1}=T_{r_{m+1}}\beta_{m+1}$, $S_{m+1}TS_{m+1}^{-1}=S_mT$ where $S_m\in C_m(T)$ with $S_m=T_{r_m}\beta_m$. Therefore $\beta_{m+1}\beta=\beta_m\beta\beta_{m+1}$ and

$$r_{m+1}+\beta_{m+1}(r) = r_m+\beta_m(r)+\beta_m\beta(r_{m+1}). \tag{4}$$

Since $[\beta_m]$ and $[\beta_{m+1}]$ are of the form (2) and $[\beta_{m+1}]=[\beta_m\beta\beta_{m+1}]$, it follows that the blocks of $[\beta_m]$, $A_{i\,i-1}$, $i=2,3,\ldots,m$, are zero matrices. From (4) and the form of the matrix $[\beta_m]$ obtained above, we see that T_{r_m} is of the form

$$r_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r_{m\,l_1+1} \\ \vdots \\ r_{mn} \end{bmatrix}.$$

Continuing the argument m times, for the affine transformation $S_2=T_{r_2}\beta_2$ belonging to $C_2(T)$, there exists an affine transformation $S_1\in C_1(T)$ such that $S_2TS_2^{-1}T^{-1}=S_1$, but it follows that S_1 is the identity map since the continuous group automorphism β_2 is the identity map and $r_2=(0,\ldots,0,r_{2l_{m-1}+1},\ldots,r_{2n})$. Therefore $S_1\in C_0(T)$ and $S_{m+2}\in C_{m+1}(T)$. Consequently, $C_{m+1}(T)=C_{m+2}(T)$. Since the blocks $A_{i\,i-1}$, $i=2,3,\ldots,m$, of the matrix $[\beta]$ are nonzero and moreover the first row in $A_{i\,i-1}$, $(a_{l_{i-1}+1\;l_{i-2}+1},a_{l_{i-1}+1\;l_{i-2}+2},\ldots,a_{l_{i-1}+1\;l_{i-1}})$, is a nonzero vector. By the same manner used in Theorem 3.1, we can construct an affine transformation $S=T_d\beta'$ such that $S\in C_{m+1}(T)$, but $S\notin C_m(T)$. Therefore $CN(T)=m+1$.

THEOREM 3.5. *Let T be an affine transformation of X_n and let (X_n,T) be a totally minimal dynamical system (with quasi-discrete spectrum) with $GN(T)=m$. Then $C_{m+1}(T)$ is a subgroup of the group consisting of all homeomorphisms of X_n.*

Proof. As in Theorems 3.1 and 3.4, we may suppose that the affine transformation $T=T_r\beta$ is written as follows

$$T:(x_1,x_2,\ldots,x_n)\to\Big(x_1+r_1,\ldots,x_{l_1}+r_{l_1},x_{l_1+1}+\sum_{j=1}^{l_1}a_{l_1+1j}x_j+r_{l_1+1},\ldots,x_{l_2}+\sum_{j=1}^{l_1}a_{l_2j}x_j+r_{l_2},\ldots,x_{l_{m-1}+1}+\sum_{j=1}^{l_{m-1}}a_{l_{m-1}+1j}x_j+r_{l_{m-1}+1},\ldots,x_{l_m}+\sum_{j=1}^{l_{m-1}}a_{l_mj}x_j+r_{l_m}\Big)$$

(additions mod 1),

where each a_{ij} is an integer and indices l_j, $j=1,2,\ldots,m$, are integers such that $l_m=n$ and there is at the least one nonzero integer in

$$\{a_{l_{j-1}+1\;l_{j-2}+1},a_{l_{j-1}+1\;l_{j-2}+2},\ldots,a_{l_{j-1}+1\;l_{j-1}}\}\quad(\text{but } l_0=0).$$

Since (X_n,T) is (totally) minimal, real numbers r_k, $k=1,2,\ldots,l_1$ are integrally independent. We consider affine transformations $Q_k=T_{r_k}\beta_k$, $k=0,1,2,\ldots,m$, where T_{r_k} are translations of X_n and β_k a continuous group automorphism of X_n. Suppose now that the matrix $[\beta_m]$ is of the form

$$\text{(1)}\qquad\begin{bmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & 0 & \\ A_{31} & A_{32} & A_{33} & & \\ & \cdots & & \ddots & \\ A_{m1} & & \cdots & A_{m\,m-1} & A_{mm}\end{bmatrix}$$

where A_{ij}, $i,j=1,2,\ldots,m$, are matrices of order $(l_i-l_{i-1})\times(l_j-l_{j-1})$ (but $l_0=0$). The form of $[\beta]$ has the form (1) and in particular A_{ii}, $i=1,2,\ldots,m$, are the identity matrices. We put

$$\text{(2)}\qquad Q_{m-j}TQ_{m-j}^{-1}T^{-1}=Q_{m-1-j},\qquad j=0,1,2,\ldots,m-1.$$

From the relation (2), the matrices $[\beta_{m-j}]$, $j=1, 2, \ldots, m-1$, are of the form (1). Moreover, from $[\beta_{m-j}\beta]=[\beta_{m-1-j}\beta\beta_{m-j}]$, $[\beta_{m-j}]$, $j=1, 2, \ldots, m$, are the matrices such that

$$\begin{bmatrix} E_{11} & & & & & & \\ & \ddots & & 0 & & & \\ & 0 & & & & & \\ B_{j+2\,1} & 0 & \cdots & 0 & E_{j+2\,j+2} & & \\ \cdots & \ddots & \cdots & & \ddots & & \\ B_{m1} & \cdots & B_{m\,m-1-j} & 0 & \cdots & 0 & E_{mm} \end{bmatrix}$$

where each E_{ii} is the identity matrix of order $(l_i-l_{i-1})\times(l_i-l_{i-1})$ and each B_{ij} is the matrix of order $(l_i-l_{i-1})\times(l_j-l_{j-1})$ (but $l_0=0$). From this fact and the relation (2), the translation $T_{r_{m-j}}$ is of the form

$$r_{m-j} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r_{m-j\,l_{j-1}+1} \\ \vdots \\ r_{m-j\,n} \end{bmatrix}, \qquad j = 2, 3, \ldots, m-1.$$

Thus it follows that if an affine transformation S is a transformation consisting of a translation composed with continuous group automorphisms β' of X_n such that the matrix $[\beta']$ is of the form (1), then $S\in C_{m+1}(T)$. Here we show that $C_{m+1}(T)$ is a subgroup. For $S, S''\in C_{m+1}(T)$ with $S=T_{r'}\beta'$ and $S''=T_{r''}\beta''$. Let us put

$$D = (S''S^{-1})T(S''S^{-1})^{-1}T^{-1},$$

then $\beta^{(3)}$ $(=(\beta''\beta'^{-1})\beta(\beta''\beta'^{-1})^{-1}\beta^{-1})$ is a continuous group automorphism of X_n and $T_{r^{(3)}}$ a translation of X_n where

$$r^{(3)} = r''+\beta''\beta'^{-1}(r-r')-\beta''\beta'^{-1}\beta\beta'\beta''^{-1}(r'')$$
$$-\beta''\beta'^{-1}\beta(r')-\beta''\beta'^{-1}\beta\beta'\beta''^{-1}\beta^{-1}(r).$$

The affine transformation $D=T_{r^{(3)}}\beta^{(3)}$ belongs to $C_{m+2}(T)$, because the matrices $[\beta']$ and $[\beta'']$ are of the form (1). Consequently $S''S^{-1}\in C_{m+1}(T)$ since

$$D = (S''S^{-1})T(S''S^{-1})^{-1}T^{-1} \quad \text{and} \quad CN(T) = m+1.$$

We have shown that $C_{m+1}(T)$ is a subgroup.

As before, (Ω, Σ, μ) is a finite measure space and T is a totally ergodic automorphism of (Ω, Σ, μ) with quasi-discrete spectrum. We consider a normalized measure space (Ω, Σ, μ).

In order to prove the following lemma, we invoke properties of entropy.

LEMMA 3.6. *Let S and W be as in Corollary 2.6 and let T' be a metric automorphism induced by T such that $T'=SW$. If for any $f\in O_\mu(T)$, Y is a subgroup generated by an orbit of f under V_S, then Y is finitely generated.*

Proof. We denote by G the subgroup of Y generated by the set

$$\{V_S^j f : j = 1, 2, \ldots\}.$$

If $V_S G\neq G$. Then it is well known that $T'=SW$ has positive entropy. On the other hand, since T is totally ergodic and has quasi-discrete spectrum, the entropy of T is zero. Thus $V_S G=G$ and

$$f = V_S^{n_1} f^{q_1} \cdot V_S^{n_2} f^{q_2} \cdot \cdots \cdot V_S^{n_k} f^{q_k}$$

where each n_j is an integer and each q_j an integer. If G' is a subgroup generated by $\{V_S^j f : j=1, 2, \ldots, k\}$ where $k=\max\{n_1, n_2, \ldots, n_k\}$, it follows that $V_S G'=G'$. Thus $Y=G'$ and Y is finitely generated.

We denote by T' the metric automorphism induced by an automorphism T.

THEOREM 3.7. *If $G_\mu N(T)=m$, then $CN(T')=m+1$. Furthermore, if $G_\mu N(T)=+\infty$ then $CN(T')=+\infty$.*

Proof. Since T is totally ergodic and has quasi-discrete spectrum, $O_\mu(T)$ is an orthonormal base of $L^2(\Sigma(\mu))$, and, by Corollary 2.6, the metric automorphism T' has a representation $T'=SW$ on $\Sigma(\mu)$ for metric automorphisms S and W such that $V_S O_\mu(T)=O_\mu(T)$ and W has each function in $O_\mu(T)$ as a proper function. For any $f\in O_\mu(T)$, it follows from Lemma 3.6 that $Y(f)$, the smallest subgroup generated by an orbit of f under V_S, is finitely generated. Since T is totally ergodic $Y(f)$ is torsion free. Here we suppose that the number of generators of $Y(f)$ is n. Since there exists a nontrivial T-invariant sub σ-algebra $\Phi(\mu)$ such that

$$L^2(\Phi(\mu)) = \overline{\text{span } Y(f)}$$

and T' has quasi-discrete spectrum on $L^2(\Phi(\mu))$, it follows from the proof of Theorem 1.2 that there exists the n-dimensional torus X_n such that the dynamical system (X_n, A_f) is totally minimal, and that T' restricted to $\Phi(\mu)$ is isomorphic to the metric automorphism A'_f induced by A_f; in other words, $\varphi_f T'=A'_f\varphi_f$ where φ_f is a metric isomorphism from the measure algebra $\Phi(\mu)$ to the measure algebra associated with the measure space consisting of the Borel field of X_n and the normalized Haar measure. We show first that $CN(T')=m+1$ if $GN(T)=m$ and m is an integer. Suppose now that $GN(A_f)\leqq G_\mu N(T)-1$ $(=m-1)$ for each $f\in O_\mu(T)$. Then we have

$$G_\mu(T)\cap Y(f) = G_\mu(T)_{m-1}\cap Y(f),$$

and

$$\begin{aligned} G_\mu(T) &= \bigcup\{G_\mu(T)\cap(K\times Y(f)) : f\in O_\mu(T)\} \\ &= \bigcup\{G_\mu(T)_{m-1}\cap(K\times Y(f)) : f\in O_\mu(T)\} \\ &= G_\mu(T)_{m-1}. \end{aligned}$$

This contradicts $G_\mu N(T)=m$. Therefore there exists a function $f\in O_\mu(T)$ such that $GN(A_f)=m$. Let us put for each $f\in O_\mu(T)$

$$C_0(A_f) = \{A : A' = \varphi_f S'\varphi_f^{-1} \text{ for } S' \in C_0(T')\}$$

and

$$C_n(A_f) = \{A : A' = \varphi_f S'\varphi_f^{-1} \text{ for } S' \in C_n(T')\}, \qquad n = 1, 2, \ldots .$$

$C_n(A_f)$, $n=0, 1, 2, \ldots$, are the generalized commuting classes of affine transformations with respect to the affine transformation A_f. It follows from Theorem 3.4 that for $f\in O_\mu(T)$ with $GN(A_f)=m$, $CN(A_f)=m+1$. It is clear that $CN(A_g)\leqq m$ for $g\in O_\mu(T)$ with $GN(A_g)\leqq m-1$. Consequently we see that

$$m+1 = \max\{n : C_n(A_f) = C_{n+1}(A_f) \text{ for } f\in O_\mu(T)\}.$$

Therefore we have $CN(T')=m+1$. It remains to show that if $G_\mu N(T)=\infty$ then $CN(T')=\infty$. For the finitely generated group $Y(f)$ of a member f of $O_\mu(T')$, we denote by g_0 a function g such that $g\in O_\mu(T)$ and $g\notin Y(f)$, and by g_1 a function g such that $g\in O_\mu(T)$ and $g\notin Y(f)\cup Y(g_0)$ and so on. Then we can choose infinitely many set functions $\{g_j : j=0, 1, 2, \ldots\}$ such that g_j belongs to a group of distinct order for $j=0, 1, 2, \ldots$. Here we put

$$G_{l_m} = \prod_{j=0}^{m} Y(g_j), \qquad m = 0, 1, 2, \ldots,$$

and let the index l_m be the least integer such that $(K\times\prod_{j=0}^{m} Y(g_j))\cap G_\mu(T)\subset G_\mu(T)_{l_m}$. Then $V_T G_{l_m}=G_{l_m}$ and $l_k\uparrow +\infty$ as $k\to+\infty$. Let X be the dual space of a discrete group G_{l_m} and Q a transformation on X induced by T. Then Q is totally ergodic with respect to Haar measure μ' on X and has quasi-discrete spectrum and $G_{\mu'}N(Q)=l_m$. We see from Theorem 3.4 that the generalized commuting order of Q is l_m+1. From this, we follow that $CN(T')\geqq l_m+1$. Since m is an arbitrary positive integer, we have $CN(T')=\infty$.

THEOREM 3.8. *If $CN(T')=m$, then $G_\mu N(T)=m-1$. Furthermore, if $CN(T')=+\infty$ then $G_\mu N(T)=+\infty$.*

The proof is an application of Theorems 3.1, 3.4 and 3.7.

COROLLARY 3.9. *If $G_\mu N(T)=m$, then $C_{m+1}(T')$ is a subgroup of the group consisting of all metric automorphisms of $\Sigma(\mu)$ onto itself. Furthermore, if $G_\mu N(T)=+\infty$ then $C_\infty(T')=\bigcup_{n=0}^{\infty} C_n(T')$ is a subgroup.*

This corollary is proved by Theorems 3.5 and 3.7.

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