

A SURFACE IN E^3 IS TAME IF IT HAS ROUND TANGENT BALLS⁽¹⁾

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Abstract. R. H. Bing has asked if a 2-sphere S in E^3 is tame when it is known that for each point p in S there exist two round balls which are tangent to each other at p and which lie, except for p , in opposite complementary domains of S . The main result in this paper is that Bing's question has an affirmative answer.

A 2-sphere S in E^3 is said to have *tangent balls at a point p of S* if and only if there exist two round balls which are tangent to each other at p and which lie, except for p , in opposite complementary domains of S . Bing's question [3], [4] about the tameness of such a sphere was partially answered by H. C. Griffith [8]. He proved that S is tame if there exists a positive number δ such that, for each point p of S , S has tangent balls of radius δ at p . We make use of part of Griffith's proof to establish Theorem 1, our main result. Our proof also depends upon a paper by Cannon [6] and a resulting characterization of tameness which is closely related to the one given in Theorem 11 of [9]. We define below what we mean when we say that a 2-sphere S is "locally capped on tame continua" and Theorem 2 states that such spheres are tame. The essence of the proof of Theorem 1 will be to show that a 2-sphere is locally capped on tame continua at certain crucial points if the sphere has tangent balls at each of its points.

Let p be a point in a 2-sphere S in E^3 and let V be a component of $E^3 - S$. We define S to be *locally capped on tame continua at p from V* if for each $\xi > 0$ there exist a disk R in S and an open disk D in V such that $p \in \text{Int } R$, R lies in the boundary of an ξ -component of $V - D$, and the boundary of D is a tame continuum in $S - R$. We say that S is *locally capped on tame continua* if it has the above property independent of both p and V . A 2-sphere has been called locally capped if it has the above properties without the condition on the tameness of M [11]; such 2-spheres are not known to be tame.

THEOREM 1. *A 2-sphere S in E^3 is tame if it has tangent balls at each of its points.*

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Proof. We define a ball B to be *tangent to S at a point p* of S if $B \cap S = \{p\}$. For each $p \in S$ it follows from the hypothesis that there exist two balls B and B' , each having the same radius r , such that $B \cap B' = \{p\}$, $B - \{p\} \subset \text{Int } S$, and $B' - \{p\} \subset \text{Ext } S$. It is not difficult to see that the balls B and B' are the only balls of radius r which are tangent to S at p . For a third such ball would have to intersect both $\text{Int } B$ and $\text{Int } B'$, hence it would intersect both $\text{Int } S$ and $\text{Ext } S$ —an impossibility. We denote the unique ball B by $B_p(r)$ and the unique ball B' by $B'_p(r)$. It follows that the center $c_p(r)$ of $B_p(r)$ lies in $\text{Int } S$ while the center $c'_p(r)$ of $B'_p(r)$ lies in $\text{Ext } S$. Using this notation we define $M_p = \{r \mid S \text{ has tangent balls } B_p(r) \text{ and } B'_p(r)\}$, and we let $r_p = \min(1, \sup M_p)$. For each positive integer i we let $X_i = \{p \in S \mid r_p \geq 1/i\}$, and we note that $S = \bigcup X_i$ and $X_i \subset X_{i+1}$.

For each integer i , the set X_i is closed. To see this we identify, for each $p \in X_i$, two particular balls B_p and B'_p each of radius $1/i$ and tangent to each other at p such that $\text{Int } B_p \subset \text{Int } S$ and $\text{Int } B'_p \subset \text{Ext } S$. The straight line segment I_p joining the centers of B_p and B'_p has p as its midpoint. Suppose y is a limit point of X_i that is not in X_i . There must exist a sequence $\{x_j\}$ of points of X_i converging to y such that the sequence $\{I_{x_j}\}$ of segments converges to a straight line segment I_y containing y . The balls B_y and B'_y having radius $1/i$ and centers at the opposite endpoints of I_y are tangent to each other at y , since y is the midpoint of I_y , and they lie, except for their boundaries, in opposite complementary domains of S . Now it is clear that $\sup M_y \geq 1/i$, since for each positive number $r < 1/i$ there exist balls of radius r lying inside $B_y \cup B'_y$ and satisfying the definition of S having tangent balls at y . Consequently $y \in X_i$, and we see that X_i is closed.

We now assume that the set W of wild points of S is not empty. Since W is a compact set which is the countable union of the closed sets $W \cap X_i$, it follows that there exists an integer t such that $W \cap X_t$ contains an open subset of W . Thus there exists an open subset U of S such that $U \cap W$ is a nonempty subset of X_t . We let $p \in U \cap W$, we choose δ such that $0 < \delta < 1/t$, and in our notation we now let $B_p = B_p(\delta)$, $B'_p = B'_p(\delta)$, $c_p = c_p(\delta)$, and $c'_p = c'_p(\delta)$.

Assertion (A), which is stated below, was first established by Griffith [8], although, apparently because of lack of space in the journal to which his paper was submitted, his proof has not appeared. For this reason we give the essentials of his argument here.

(A) There exists a disk T in U containing p in its interior and having the property that if B is a ball of radius δ tangent to S at a point y of T , then the center c of B must lie in B_p or B'_p according as $B - \{y\} \subset \text{Int } S$ or $B - \{y\} \subset \text{Ext } S$, respectively.

Proof of (A). The disk T in (A) can be any disk in U having diameter less than $(\sqrt{6} - 2)\delta/2$ and containing p in its interior. For suppose B is a ball centered at c with radius δ such that B is tangent to S at a point y of T , $B - \{y\} \subset \text{Int } S$, and $c \notin B_p$. It follows that c is outside the ball $B(2\delta)$ of radius 2δ centered at c'_p , for otherwise the balls B and B'_p would intersect, yet they are, except for p and y , on opposite sides of S . Thus $c \in (E^3 - B(2\delta)) \cup (E^3 - B_p)$ and the distance from p to this set

small enough that $2r < \delta$ and $V \cap S \subset T \subset U$. From (A) we see that if $w \in W \cap V$, then $c_p \in B_w$, $c_w \in B_p$, $c'_p \in B'_w$, and $c'_w \in B'_p$.

Let B be a ball centered at p such that $B \cap S$ lies in a disk in $S \cap V$, and let S' be the 2-sphere boundary of B . We shall show how to use part of S' together with disks near S to construct the open disk D desired in the definition of *locally capped on tame continua*. If $S' \cap W = \emptyset$, then we could adjust S' and S to general position to insure that the component F of $S' - S$, which contains the disk $S' \cap B'_p$, is an open disk-with-holes. One of the simple closed curves in $\text{Bd } F$ would serve as the boundary M of D while the others would bound disks in $S \cap V$. If it should happen that none of these disks in $S \cap V$ intersects W , we could use disks near them to fill in the holes in F in forming D . Of course it is not at all apparent that W is so well behaved; nevertheless this idea motivated the proof. We shall see that the existence of round balls tangent to S at points of W greatly restricts the manner in which W intersects S' and also causes certain small disks in $V \cap S$ to be tame.

We assume for convenience that the z -axis contains $\{p, c_p\}$ and that p lies above c_p . We say that a point x is above (below) a point y if the z -coordinate of x is larger (smaller) than that of y . If $x \in V - (B_p \cup B'_p)$, it follows that the set $\{x, c_p, c'_p\}$ is noncollinear and hence that these points determine a unique (vertical) plane which we denote by P_x . The open annulus $A = S' \cap (E^3 - (B_p \cup B'_p))$ intersects the plane P_x in the union of two disjoint open arcs. If $x \in A$ we denote the open arc in $A \cap P_x$ which contains x by A_x .

(B) If a point $x \in A$ lies in a ball B_y of radius δ such that B_y is tangent to S at $y \in S$ and $B_y - \{y\} \subset \text{Int } S$, then each point of A_x below x lies in $\text{Int } S$. Similarly if $B_y - \{y\} \subset \text{Ext } S$, then each point of A_x above x lies in $\text{Ext } S$.

Proof of (B). It follows from (A) that B_y contains c_p ; hence the disk $D = B_y \cap P_x$ contains both c_p and x . This means that D has a radius larger than $\delta/2$ because $\text{diam } D \geq d(c_p, x) > \delta$. Notice that D does not contain A_x for if so then D would intersect $B'_p - \{p\}$ which lies in $\text{Ext } S$. It follows that $\text{Bd } D$ must intersect A_x , and we have the following cases to consider.

Case 1. $(S' \cap P_x) \cap \text{Bd } D = \{x\}$. In this case the disks D and $P_x \cap B$ are tangent at x so both of their centers, p and c , lie on a line L containing x . Since $d(p, c) > \delta/2$ and V has radius less than $\delta/2$, we see that $c \in L - V$. However $L \cap B_p \subset V$, so c is forced to lie outside B_p , and the center c_p of B_p does not lie in D as required. This case cannot occur.

Case 2. $A_x \cap \text{Bd } D$ contains two points a and b . Then the center c of D and p must both lie on the line L which perpendicularly bisects the segment ab . Thus the contradiction here is the same as in Case 1.

Case 3. $A_x \cap \text{Bd } D$ consists of a single point a and D is not tangent to $P_x \cap B$. In this case, the point a cannot be below x , for otherwise D would intersect B'_p . Since the upper endpoint of A_x lies in $\text{Ext } S$, and hence is not in D , it follows that all points of A_x below a lie in $\text{Int } S$.

(C) If there exists a point w in $A \cap W$, then every point of A_w below w lies in $\text{Int } S$ and every point of A_w above w lies in $\text{Ext } S$.

Proof of (C). Since $w \in A \cap W$ we have tangent balls of radius δ on both sides of S at w , and it is clear that in this case the disks $B_w \cap P_w$ and $B'_w \cap P_w$ are tangent at w . Thus (C) follows from (B).

Since S separates the disk $H' = B'_p \cap S'$ from $H = B_p \cap S'$ on S' , the component E' of $S' - S$ which contains H' has a boundary component M' in S' that also separates H from H' on S' . Let E be the component of $S' - M'$ containing E' . Thus E is an open disk in S' such that $\text{Bd } E = M'$ and no component of $S \cap E$ separates M' from H' in S' . It follows that $S' \cap W \subset M'$, for suppose there exists a point $w \in S' \cap W$. Now the open arc A_w in P_w must intersect M' , and from (C) it follows that those points of A_w above w lie in $\text{Ext } S$ while those below w lie in $\text{Int } S$. Thus $A_w \cap S = \{w\}$ and w must belong to M' . This proves that $E \cap W = \emptyset$. In the next two paragraphs we identify an open set O containing E within which we shall adjust E and S to general position.

We define Y to be the union of all the sets $B_w \cup B'_w$ where w ranges over $W \cap V$. We shall prove that Y is a closed set that does not intersect $E \cap S$. Suppose that y is a limit point of Y that is not contained in Y . As in the proof that each X_i is closed, we see that some sequence $\{w_i\}$ of points of $W \cap V$ exists such that the sequence $\{B_{w_i} \cup B'_{w_i}\}$ of sets converges to the union of two balls B_1 and B_2 , each of radius δ , such that $y \in B_1 \cup B_2$, $\text{Int } B_1 \subset \text{Int } S$, $\text{Int } B_2 \subset \text{Ext } S$, and B_1 and B_2 are tangent to each other at some point x of S . Since $W \cap V$ is closed and $\{w_i\}$ converges to x , we see that $x \in W \cap V$. Let δ' be a number such that $\delta < \delta' < 1/t$. It follows that, for each w_i , there exist balls $B_{w_i}(\delta')$ and $B'_{w_i}(\delta')$ containing B_{w_i} and B'_{w_i} , respectively. The sequence whose elements are these larger ball pairs must converge to the union of two balls F_x and F'_x , each of radius δ' , such that $B_1 - \{x\} \subset \text{Int } F_x \subset \text{Int } S$ and $B_2 - \{x\} \subset \text{Int } F'_x \subset \text{Ext } S$. Thus we know that $y \in B_1 \cup B_2 \subset Y$, and from this it follows that Y is closed. Since $Y \cap S = W \cap V$ and $E \cap W = \emptyset$, it is clear that $Y \cap (E \cap S) = \emptyset$.

Since the closed set Y does not intersect $E \cap S$, we may choose an open set O in V such that $O \cap Y = \emptyset$, $O \cap S' \subset G = E - H'$, and $S \cap E \subset O$. Now S is locally tame at each point of $O \cap S$ so with a small space homeomorphism fixed outside O we may adjust G and $S \cap O$ so that they are locally polyhedral [1] and in general position. This forces each component of $S \cap G$ to be either a simple closed curve in G or an open arc in G with its endpoints in M' . Next we apply another homeomorphism fixed outside O to return G to its former round shape. We still use S to refer to the adjusted S since the important property that S has tangent balls of radius δ at each point of $W \cap V$ is possessed by both S and its adjustment. This is the reason for insisting that $Y \cap O = \emptyset$.

Let C be the component of $E - S$ containing H' . Since the closed set $\text{Bd } C$ separates H from H' in S' , so does some continuum in $\text{Bd } C$, and we let M be the component of $\text{Bd } C$ containing this continuum. Note that M is a tame continuum

since it lies on the round 2-sphere S' . The choice of C and the fact that S and E are in general position forces each component of $\text{Bd } C$ different from M to be a simple closed curve in $E-H'$. We denote the collection of these disjoint simple closed curves in $E-H'$ by $\{J_i\}$ and, for each i , we let E_i and D_i be the disks in E and $S \cap V$, respectively, which are bounded by J_i . Although the collection $\{E_i\}$ is pairwise disjoint, the same may not be true of $\{D_i\}$. However, since no D_i intersects M , no D_i can contain infinitely many other D_j . This permits us to assume that the subscripts have been assigned in such a manner that if $D_i \subset D_j$, then $i < j$.

Suppose that, for some i , D_i contains a wild point w of S . Since $E \cap W = \emptyset$ we may assume that $w \in \text{Int } D_i$. Let I be the line segment joining c_w to c'_w and note that I pierces S at w . Since the length of I is 2δ , the diameter of V is less than δ , and the midpoint w of I lies in V , it follows that a simple closed curve J exists such that $I \subset J$ and $J - I \subset E^3 - V$. Consequently J links J_i and there must exist a point x in $I \cap \text{Int } E_i$. It follows that $w \neq p$. Since J_i bounds E_i in $C-H'$, we see that there exist two points a and b of J_i in the intersection of the plane P_x with A_x such that a is above x and b is below. Now it follows from (C) that $x \neq w$, so either $x \in \text{Int } B'_w$ or $x \in \text{Int } B_w$. But in either case it follows from (B) that either a lies in $\text{Ext } S$ or b lies in $\text{Int } S$. This is a contradiction since $\{a, b\} \subset J_i \subset S$. Thus S is locally tame at each point of each D_i .

We indicate briefly the first stages of an inductive procedure for filling the holes in C to obtain an open disk D in $\text{Ext } S$ having M as its boundary. In this outline we assume that there are infinitely many D_i since this is the most difficult case. Since $D_1 \cap W = \emptyset$ we may assume that S is locally polyhedral at each point of D_1 . We first add D_1 to C , then we push D_1 and a small neighborhood of D_1 in $D_1 \cup C$ into $\text{Ext } S$. We call the resulting 2-manifold C_1 and we note that it does not intersect D_1 . In the second step we use the same procedure relative to C_1 and D_2 to construct another 2-manifold C_2 which does not intersect $D_1 \cup D_2$. This procedure is continued inductively, and we define the open disk D to be $\bigcup_{i=1}^{\infty} C_i$. Since the disks D_i all lie on S , the limit of their diameters must be zero. This important fact must be used in the construction of D to insure that it has M as its boundary.

We now check the open disk D and its boundary, the tame continuum M , against the conditions in the definition of S being locally capped on tame continua. Clearly $p \notin M$, so we may choose a disk R in $S-M$ with p in its interior. Let K be the component of $(\text{Ext } S) - D$ such that R lies in its boundary. We need to show that the diameter of K is less than $\xi = \delta$; for this it is sufficient that $K \subset V$. If this is not the case, there must be an arc F_1 from p to a point q in $\text{Bd } V$ such that $F_1 \cap D = \emptyset$. We assume that F_1 does not intersect the vertical segment F_2 from p to the point $h \in H' \subset D$ directly above p , for this situation can always be accomplished with small adjustments of F_1 in $E^3 - D$. Since $D \subset V$ and $D \cap B'_p = H'$, there is an arc F_3 from q to h such that $F_3 \cap D = \{h\}$; we define J to be the simple closed curve $F_1 \cup F_2 \cup F_3$. It follows that $J \cap D = \{h\}$ and J pierces D at h . Since the vertical line containing F_2 pierces S at p , we see from [7] that p must lie in some tame arc

A on S . Let S_1 be a 2-sphere containing $A \cup M$ such that S_1 is locally polyhedral modulo $A \cup M$, $S_1 \cap J = \{p\}$, and $J - p \subset \text{Ext } S_1$ [1]. Since A and M are each tame and nondegenerate, it follows that S_1 is tame [6]. Thus we may push S_1 slightly into its exterior at points of M and into its interior at p to obtain a 2-sphere S_2 with M in its interior and J in its exterior. We choose a simple closed curve J' in D such that J' separates H' from M in \bar{D} and such that J' is close enough to M to insure that it lies in $\text{Int } S_2$. Clearly J links J' since $J \cap D = \{h\}$ and the disk in D bounded by J' is pierced at h by J . This is impossible as J' bounds a disk in the interior of S_2 which does not intersect J . This means that $K \subset V$ and it follows that K has diameter less than ξ .

We have proven that S is locally capped on tame continua at p from $\text{Ext } S$, and a similar argument would establish the same result from $\text{Int } S$. Since p was an arbitrary wild point of U it follows from Theorem 2 that S is locally tame at each point of U . This is a contradiction since U was chosen to intersect W ; hence $W = \emptyset$. Thus S must be tame.

The proof of Theorem 2 below is essentially the same as the proof given for Theorem 11 of [9]; however, for completeness and because a recent result by Cannon [6] is needed here, we give an outline. Property $(*, F, S)$, where F is a closed subset of S , roughly means that Bing's Side Approximation Theorem can be applied where the intersection of the approximating sphere with S is covered by a finite collection of disjoint small disks in $S - F$. A precise definition can be found in [7] or [10].

THEOREM 2. *If G is an open subset of a 2-sphere S in E^3 , V is a component of $E^3 - S$, and S is locally capped on tame continua at each point of G from V , then S is locally tame from V at each point of G .*

Outline of proof. Let $p \in G$ and let N be an open subset of E^3 containing p . We shall prove the existence of an open set U containing p having the property that each unknotted simple closed curve in $U \cap V$ bounds a disk in $N \cap V$. Since this property would be true of each $p \in G$, the local tameness of S from V at points of G would follow from the techniques of Bing in [2] or from results in [10]. Let K be a disk in $G \cap N$ such that $p \in \text{Int } K$, and let O be an open subset of E^3 such that $p \in O \cap S \subset \text{Int } K$ and $O \subset N$. Theorem 12 of [10] guarantees the existence of an open subset U of O such that $p \in U$ and if J is an unknotted simple closed curve in $U \cap V$ then J bounds a disk in $O - F$ where F is any closed subset of S satisfying Property $(*, F, S)$. Let J be an unknotted simple closed curve in $U \cap V$. For each $q \in K$ we use the hypothesis to find an open disk D_q in V and a component R_q of $S - \bar{D}_q$ such that $q \in R_q$, $\bar{D}_q - D_q = M_q$ is a tame continuum on S , \bar{D}_q separates R_q from J in \bar{V} , and $D_q \subset N$. Since K is compact, a finite collection, say R_1, R_2, \dots, R_n , of the R_q suffices to cover K . We denote the open disk in V corresponding to R_i by D_i and the continuum $\bar{D}_i - D_i$ by M_i . Since each M_i is a tame nondegenerate continuum on S , Property $(*, M_i, S)$ can be established as in Gillman's proof of

Theorem 2 in [7], provided we first invoke a result by Cannon [6] to see that each 2-sphere which contains M_i and which is locally tame modulo M_i must be tame. Property $(*, \bigcup_{i=1}^n M_i, S)$ then follows from Theorem 21 of [10].

By the definition of U, J bounds a tame disk H in $O - \bigcup M_i$. Each component of $H \cap S$ lies in some R_i and is separated from J on H by D_i . We assume D_1 and H are in general position and we construct a set H_1 by filling the holes in the component of $H - D_1$ which contains J . These holes are filled with disks near D_1 . Now we do the same thing with H_1 and D_2 to obtain H_2 . This can be done so that H_2 does not intersect $R_1 \cup R_2$. Continuing inductively we finally obtain a disk H_n bounded by J such that $H_n \cap S = \emptyset$ and $H_n \subset N$.

The following consequence of Theorem 2 also follows directly from a combination of Theorem 11 of [9] with [6].

COROLLARY. *If G is an open subset of a 2-sphere S in E^3 such that for each point p of G and for each $\xi > 0$ there exists a 2-sphere S' of diameter less than ξ such that $p \in \text{Int } S'$ and $S' \cap S$ is a tame continuum, then S is locally tame at each point of G .*

Questions. 1. Is a 2-sphere S in E^3 tame from its exterior if, for each $p \in S$, there exists a round ball B_p such that $B_p \cap S = \{p\}$ and $B_p - \{p\} \subset \text{Int } S$?

2. Is S tame if there exists a positive number δ such that, for each p in S , there exists a round ball B_p of radius δ such that $B_p \cap S = \{p\}$ and $B_p - \{p\} \subset \text{Int } S$?

3. Is S tame if, for each point p of S , there exist two right circular cones C_1 and C_2 such that $C_1 \cap C_2 = \{p\}$ and C_1 and C_2 lie, except for p , on opposite sides of S ; that is, is S tame if it can be touched with a pencil from both sides at each of its points (see [3] and [4])?

4. In the characterization of tame 2-spheres given in Theorem 2, can the tameness condition of the continuum M be removed (see [11, p. 172])?

J. W. Cannon has recently generalized our main theorem and has obtained affirmative answers to Questions 1 and 3 above. Apparently H. G. Bothe, working independently, has also established our main result.

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