

GENERALIZATIONS OF QF-3 ALGEBRAS

BY

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Abstract. This paper consists of three parts. The first is devoted to investigating the equivalence and left-right symmetry of several conditions known to characterize finite dimensional algebras which have a unique minimal faithful representation—QF-3 algebras—in the class of left perfect rings. It is shown that the following conditions are equivalent and imply their right-hand analog: R contains a faithful Σ -injective left ideal, R contains a faithful Π -projective injective left ideal; the injective hulls of projective left R -modules are projective, and the projective covers of injective left R -modules are injective. Moreover, these rings are shown to be semi-primary and to include all left perfect rings with faithful injective left and right ideals.

The second section is concerned with the endomorphism ring of a projective module over a hereditary or semihereditary ring. More specifically we consider the question of when such an endomorphism ring is hereditary or semihereditary.

In the third section we establish the equivalence of a number of conditions similar to those considered in the first section for the class of hereditary rings and obtain a structure theorem for this class of hereditary rings. The rings considered are shown to be isomorphic to finite direct sums of complete blocked triangular matrix rings each over a division ring.

Thrall [36] called an algebra A of finite rank (left) QF-3 if it has a unique minimal faithful left module, that is, a unique (up to isomorphism) module M with the property that no proper direct summand is faithful. It is not difficult to verify the equivalence of the following statements.

- (1) A is (left) QF-3.
- (2) A contains a faithful injective left ideal.
- (3) The injective hull of ${}_A A$ is projective.
- (4) The injective hull of every projective left A -module is projective.
- (5) The projective cover of every injective left A -module is injective.

Moreover, the right-hand analogue of each of these conditions is easily established by forming duals with respect to the field so that for algebras being QF-3 is a two-sided property and distinctions between left and right are unnecessary.

It is apparent from the above list that there are several possible ring theoretic generalizations of QF-3 algebras and this paper is primarily devoted to resolving certain questions which arise naturally in connection with such extensions.

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In §1 we show that if R is a left perfect ring then the assumptions that R contains a faithful Σ -injective left ideal (see Faith [12]), that R contains a faithful Π -projective injective left ideal, that the injective hulls of projective left R -modules are projective, and that the projective covers of injective left modules are injective are equivalent, and that they imply their right-hand counterparts. Furthermore, we deduce that these rings are semiprimary and show that perfect rings with faithful injective left and right ideals have the above properties. These results include and extend those of Harada [19], Fuller [14], [15], Morita [27], and Tachikawa [35].

The second section begins with some preliminary results connecting submodules of a module with one-sided ideals of its endomorphism ring which were essentially noted by Wolfson [37] and are included for completeness and clarity. We combine these results and an idea of Small [34] to prove that the endomorphism ring of any projective (respectively, finitely generated projective) left module over a left hereditary ring is left semihereditary (respectively, hereditary) and several related results. Similar techniques are applied to show that every projective left module over a right semihereditary ring is a direct sum of finitely generated modules. The corresponding result for left modules over left semihereditary rings was proved by Albrecht [1] and our result, like his, generalizes Kaplansky's treatment of the commutative case and depends heavily on Kaplansky's reduction of the problem [23].

Mochizuki [25] proved that a hereditary QF-3 algebra is a direct sum of complete blocked triangular matrix algebras each over a division algebra. Harada [18] and the present authors [7] extended this result to semiprimary and semiperfect hereditary left QF-3 rings, respectively. In §3 we show that one can drop all finiteness assumptions and still obtain this structure theorem. More precisely, a left hereditary ring has the above form if and only if one of the following hold:

- (1) The injective hull of ${}_R R$ is projective.
- (2) R contains faithful injective left and right ideals.
- (3) R contains a faithful injective left ideal and contains no infinite set of orthogonal idempotents.
- (4) R contains a faithful Σ -injective left ideal.

The results of this section complement those of [8].

1. Perfect QF-3 rings. Let R and Λ denote rings with identity, N the radical of R . All modules considered are unital.

Let ${}_R M_\Lambda$ be a left R , right Λ -bimodule, $X \subseteq M$ and $U \subseteq \Lambda$. We shall consider two type of annihilators.

$$\text{Ann}_\Lambda(X) = \{\lambda \in \Lambda \mid X\lambda = 0\}$$

is a right ideal of Λ and

$$\text{Ann}_M(U) = \{m \in M \mid mU = 0\}$$

is an R -submodule of M . We shall abbreviate

$$\text{Ann}_{\Lambda\Lambda}(U) = l(U) \quad \text{and} \quad \text{Ann}_{\Lambda\Lambda}(U) = r(U).$$

These are always left and right ideals, respectively, and we shall refer to one-sided ideals of this type as annihilator ideals. We shall use similar notation and terminology for annihilators involving R and M .

R is *left perfect* if every left R -module has a projective cover. Equivalently, R/N is Artinian and every nonzero right R -module contains a simple submodule (see Eilenberg [10] and Bass [3]).

An R -module M is Σ -*injective* if the direct sum of any number of copies of M is injective (see Faith [12]).

1.1 PROPOSITION (FAITH [12]). *Let ${}_R M$ be injective and let $\Lambda = \text{Hom}_R(M, M)$. The following are equivalent:*

- (1) M is Σ -injective.
- (2) The direct sum of a countable family of copies of M is injective.
- (3) R satisfies the ascending chain condition on annihilators of subsets of M .
- (4) M_Λ satisfies the descending chain condition on annihilators of subsets of R .

The following result was proved for left Artinian rings by Fuller [15] and for left or right Artinian rings by Morita [27], with the assumption of Σ -injectivity replaced by injectivity.

1.2 THEOREM. *Let R be a ring. The following are equivalent:*

- (1) R is left perfect and contains a faithful Σ -injective left ideal.
- (2) R is semiprimary and contains idempotents e and f such that
 - (a) ${}_R Re$ and fR_R are faithful left and right ideals, respectively.
 - (b) Re_{eRe} and ${}_f Rf$ are finitely generated, eRe is right Artinian and fRf is left Artinian.

(c) *The functors $\text{Hom}_{fRf}(_, fRe)$ and $\text{Hom}_{eRe}(_, fRe)$ define a duality between the categories of finitely generated left fRf -modules and finitely generated right eRe -modules (see [26]).*

Proof. Assume condition (1) holds. Let Re , $e^2 = e$ be a faithful projective Σ -injective left ideal of R . Then $Re \subseteq R \subseteq \bigcap Re$ so the lattices of annihilators in R of subsets of Re and of annihilators in R of subsets of R are the same so R is semiprimary by Proposition 1 and Faith [12, Proposition 4.1]. Let e_1, \dots, e_n be a complete set of orthogonal idempotents of R . Let f be the sum of the idempotents e_i which have the property that Re_i/Ne_i is isomorphic to a simple left ideal of Re . By Fuller [15, Lemma 1.1], fR is faithful. By Tachikawa [35, Proposition 2.5] or Fuller [15, Lemma 2.2], $\text{Hom}_{fRf}(fR, fRe) = Re$, $\text{Hom}_{fRf}(fRe, fRe) = eRe$ and ${}_f Rf$ is an injective cogenerator. Thus, applying Morita [26, Lemma 2.1], if X is any submodule of ${}_f Rf$, then

$$\begin{aligned} X &= \text{Ann}_{fR}(\text{Ann}_{\text{Hom}(fR, fRe)}(X)) \\ &= \text{Ann}_{fR}(\text{Ann}_{eRe}(X)) = f \text{Ann}_R(\text{Ann}_{eRe}(X)). \end{aligned}$$

Hence since R satisfies the a.c.c. on annihilators of subsets of Re , ${}_f Rf$ satisfies the

a.c.c. on submodules so fRf is finitely generated and fRf is left perfect and left Noetherian, hence left Artinian. Thus 2(c) holds by Morita [26, Theorem 6.3 VI] and since Re_{eRe} is reflexive, Re_{eRe} is finitely generated so 2 holds.

Conversely, Re is injective as in Fuller [15, Theorem 3.1] and is Σ -injective by Proposition 1.1(4).

A module ${}_R P$ is Π -projective if $\prod_{\alpha \in A} P$ is projective for any index set A . We denote the injective hull of a module ${}_R M$ by $E({}_R M)$ (see [9]). The equivalence of conditions (4), (5) and (6) of the following theorem has been proved by Fuller for left Artinian rings [15].

1.3 THEOREM. *Let R be a ring. The following are equivalent:*

- (1) *R is left perfect and contains a faithful Σ -injective left ideal.*
- (2) *R is left perfect and contains a faithful Π -projective injective left ideal.*
- (3) *R is left perfect and contains an idempotent e such that*
 - (a) *${}_R Re$ is a faithful injective left ideal.*
 - (b) *eRe is right Artinian and Re_{eRe} is finitely generated.*
- (4) *R is left perfect and the injective hull of every projective left module is projective.*
- (5) *R is left perfect and the projective cover of every injective left R -module is injective.*
- (6) *R is left and right perfect and contains faithful injective left and right ideals, respectively.*

Proof. It is clear from Theorem 2 that conditions (1) and (3) are equivalent. We shall prove that (1) \rightarrow (2) \rightarrow (4) \rightarrow (1), (1) \leftrightarrow (6), and (1) \leftrightarrow (5).

(1) *implies* (2). As we saw in Theorem 2, R is semiprimary. Hence $\prod Re$ has an essential socle $T = \bigoplus \sum S_\alpha$, S_α simple. Since each S_α is isomorphic to a submodule of Re , $E(S_\alpha)$ is a summand of Re so is Σ -injective by assumption. Since there are only finitely many isomorphism types of simple modules $\bigoplus \sum E(S_\alpha)$ is the injective hull of T . Since T is essential in $\prod Re$,

$$\prod Re = \bigoplus \sum E(S_\alpha)$$

so $\prod Re$ is projective since each $E(S_\alpha)$ is.

(2) *implies* (4). Let P be a faithful Π -projective injective left ideal of R . (4) follows since any projective left R -module can be embedded in a direct product of copies of P .

(4) *implies* (1). Since $E({}_R R)$ is isomorphic to a direct sum of modules of the form Re_i , we obtain a faithful injective left ideal by letting Re be the sum of one of each isomorphism type of these Re_i . If M is any direct sum of copies of Re then since R is left perfect, $E(M)$ is the direct sum of indecomposable projective injective modules, and since M is also such a sum, M is injective by Faith and Walker [13, Theorem 6.4].

(1) *implies* (6). This follows immediately from the left-right symmetry of Theorem 2.

(6) *implies* (1). Since ${}_f R f Re$ and $f Re_{eRe}$ are injective cogenerators and $\text{Hom}_{{}_f R f}(f Re, f Re) = eRe$, and $\text{Hom}_{eRe}(f Re, f Re) = f R f$ (see Tachikawa [35, Proposition 2.5] and since $f R f$ and eRe are perfect, we can apply a result of Osofsky [30, Theorem 3.3] to obtain that $f R f$ is left artinian and eRe is right artinian. Since Re_{eRe} is reflexive with respect to $f Re$, the socle of Re_{eRe} is also reflexive so the socle of Re_{eRe} must be a finite sum of simple modules. It follows that Re_{eRe} can be imbedded in a finite direct sum of copies of $f Re$. By Morita [26, Theorem 6.3] $f Re_{eRe}$ is finitely generated. Hence Re_{eRe} is finitely generated and (3), hence (1), holds.

(1) *implies* (5). Let Re , $e^2 = e$, be a faithful Σ -injective left ideal. By the above, Re is also Π -projective. R is semiprimary by Theorem 2 and, as in Fuller [15, Theorem 2.6] the projective cover of every indecomposable injective module is injective. Let ${}_R E$ be injective, and $\bigoplus \sum S_\alpha$, S_α simple, be the socle of ${}_R E$. Since ${}_R E$ is a direct summand of $\prod E(S_\alpha)$ it suffices to show that the projective cover of $\prod E(S_\alpha)$ is injective. Let P_α be a projective cover of $E(S_\alpha)$. By Fuller's result mentioned above P_α is injective.

Since each P_α is a direct sum of summands of Re , $\prod P_\alpha$ is contained in, hence is a direct summand of, a direct product of copies of Re so $\prod P_\alpha$ is projective since Re is Π -projective. Let Q be a projective cover of $\prod E(S_\alpha)$. Then $\prod E(S_\alpha)$ is an epimorph of both Q and $\prod P_\alpha$, and, since $\prod P_\alpha$ is projective, Q is a direct summand of $\prod P_\alpha$ so is injective.

(5) *implies* (1). Let S_1, S_2, \dots, S_n be a complete set of nonisomorphic simple left R -modules. Then

$$M = E(S_1) \oplus \dots \oplus E(S_n)$$

is a faithful, injective left R -module. Let P be a projective cover of M . Since M is faithful, P is faithful, and since R is left perfect, P is a direct sum of indecomposable projective injective direct summands of R .

$$P = \bigoplus \sum Re_{j_i}, \quad 1 \leq j_i \leq n.$$

Letting e be the sum of the distinct e_{j_i} which occur in the above sum, we obtain a faithful projective injective left ideal Re . It suffices to show that Re is countably Σ -injective. Since $\prod_{i=1}^\infty Re$ is injective, its projective cover is injective. Since R/N is semisimple Artinian,

$$T = \prod Re/N(\prod Re)$$

is an R/N -module, and $N(\prod Re) \subseteq \prod (Ne)$, T contains a direct summand isomorphic to $\prod Re/\prod Ne \cong \prod (Re/Ne)$ and hence also a summand isomorphic to $\sum_{i=1}^\infty (Re/Ne)$. It follows that $\bigoplus \sum_{i=1}^\infty Re$ is isomorphic to a direct summand of the projective cover of $\prod Re$ so is injective.

1.4 REMARKS. (1) Tachikawa [35] has shown that if R satisfies condition (6) of Theorem 3, then $E({}_R R)$ and $E(R_R)$ are both projective.

(2) Müller [28] proved that if R is perfect and $E({}_R R)$ is projective, then $E(R_R)$ is projective iff a duality slightly different than the one considered above holds.

(3) A ring satisfying the conditions of Theorem 3 is semiprimary but need not be Artinian.

For example, let B be the matrix ring

$$B = \left\{ \begin{pmatrix} r_1 & 0 & 0 \\ r_2 & q & 0 \\ r_3 & r_4 & r_5 \end{pmatrix} \mid r_i \text{ real, } q \text{ rational} \right\}.$$

(4) A semiprimary ring with $E(R_R)$ projective need not have $E({}_R R)$ projective. (See Müller [28, p. 178, Example 3] with B as in (3) above.)

2. Endomorphism rings of projective modules. As in the first section, let R and Λ denote rings with identity and ${}_R M_\Lambda$ a left R , right Λ -bimodule. If $X \subseteq M$ and $U \subseteq \Lambda$, we define

$$(M : X) = \{\lambda \in \Lambda \mid M\lambda \subseteq X\}$$

and

$$MU = \left\{ \sum_{i=1}^k m_i u_i \mid m_i \in M, u_i \in U \text{ and } k \text{ is a positive integer} \right\}.$$

They are, respectively, a left ideal of Λ and an R -submodule of M .

The proofs of the next two lemmas are straightforward and will be omitted.

2.1 LEMMA. *If M_Λ is faithful, then*

- (i) $(M : \text{Ann}_M(U)) = l(U)$.
- (ii) $\text{Ann}_\Lambda(MU) = r(U)$.

We say that T is a *closed* submodule of ${}_R M$ if $T = \text{Ann}_M(U)$ for some $U \subseteq \Lambda$.

2.2 LEMMA. *If M_Λ is faithful and for each R -submodule T of M , $T = M(M : T)$, then*

- (i) $\text{Ann}_\Lambda(T) = r((M : T))$.
- (ii) *If T is a closed submodule of ${}_R M$, $(M : T) = l(\text{Ann}_\Lambda(T))$.*
- (iii) *If J is an annihilator right ideal of Λ , $J = \text{Ann}_\Lambda(\text{Ann}_M(J))$.*
- (iv) *If L is an annihilator left ideal of Λ , $(M : ML) = L$.*

If F is a free left R -module and $\Lambda = \text{Hom}_R(F, F)$, then ${}_R F_\Lambda$ satisfies the hypotheses of these lemmas.

Let ${}_R M_\Lambda$ be as in Lemma 2, $C_R(M)$ denote the lattice of closed R -submodules of M and $A_l(\Lambda)$ and $A_r(\Lambda)$ denote, respectively, the lattices of annihilator left ideals and annihilator right ideals of Λ . The content of the two preceding lemmas may be expressed as follows:

(1) The correspondences

$$T \rightarrow \text{Ann}_\Lambda(T) \quad \text{and} \quad J \rightarrow \text{Ann}_M(J)$$

are inverse lattice anti-isomorphisms between $C_R(M)$ and $A_r(\Lambda)$.

(2) The correspondences

$$T \rightarrow (M:T) \quad \text{and} \quad L \rightarrow ML$$

are inverse lattice isomorphisms between $C_R(M)$ and $A_l(\Lambda)$.

(3) The following diagram commutes in both directions

$$\begin{array}{ccc} & C_R(M) & \\ \swarrow & & \searrow \\ A_l(\Lambda) & \longleftrightarrow & A_r(\Lambda) \end{array}$$

where the arrows represent the correspondences of (1) and (2) above and the usual correspondences between $A_l(\Lambda)$ and $A_r(\Lambda)$.

A ring R is *left hereditary* if every left ideal of R is projective and *left semi-hereditary* if every finitely generated left ideal is projective. Recall that a principal left ideal Ra is projective if and only if $l(a) = Re$ with $e^2 = e \in R$.

2.3 THEOREM. *R is a left hereditary ring if and only if for every free left R -module F , the ring of R -endomorphisms of F , $\Lambda = \text{Hom}_R(F, F)$, has principal left ideals projective.*

Proof. Assume R is left hereditary and let $\alpha \in \Lambda$. We must show $l(\alpha) = \Lambda e$ with $e^2 = e$. But $\ker(\alpha) = \text{Ann}_F(\alpha) = Fl(\alpha)$ by (i) of Lemma 1 and by the fact that since ${}_R F$ is free $T = F(F:T)$ for all submodules T of ${}_R F$. Furthermore, since R is left hereditary every submodule of F is projective (see [4, Theorem 5.3, p. 13]) and hence in particular so is the image of α . Thus the kernel of α is a direct summand of F and so $\ker \alpha = Fe = F\Lambda e$ with $e^2 = e \in \Lambda$. Since $l(\alpha)$ and Λe are both annihilator left ideals of Λ , it follows from the discussion after Lemma 2 that $\Lambda e = l(\alpha)$.

To prove the converse let I be any left ideal of R . Then there exists a free module F and an epimorphism of F onto I . Since F is free, it contains a submodule T isomorphic to I and there exists $\alpha \in \Lambda$ such that $F\alpha = T$. Then the sequence

$$0 \longrightarrow \ker \alpha \longrightarrow F \xrightarrow{\alpha} T \longrightarrow 0$$

is exact and just as in the first half of the proof of this proposition $\ker(\alpha) = Fl(\alpha)$. But $Fl(\alpha) = F\Lambda e = Fe$, $e = e^2$ by hypothesis and hence the above exact sequence splits and so T and hence also I is projective.

With minor modifications the same argument also serves to establish the next proposition which was proved originally by Small [34] using a different technique which does not seem to extend to the situation just considered.

If R is a ring and n is a positive integer, R_n denotes the ring of $n \times n$ matrices over R .

2.4 THEOREM. *R is left semihereditary if and only if, for all n , R_n has principal left ideals projective.*

2.5 LEMMA. *If R has principal left ideals projective and $e^2 = e \in R$, then eRe has principal left ideals projective.*

Proof. Let $x \in eRe$. Then Rex is R -projective and so the exact sequence

$$Re \rightarrow Rx \rightarrow (0)$$

obtained by sending $re \rightarrow rex$ splits. Thus $Re = Rf \oplus Rf'$ with f and f' idempotents, $Rf \cong Rx$ and $fx = x$. Hence $eRex = eRx \cong eRf = eRe efe$, which is a direct summand of eRe . Therefore $eRex$ is eRe -projective.

2.6 THEOREM. *If R is left hereditary and P is a finitely generated projective left R -module, then $\Lambda = \text{Hom}_R(P, P)$ is left hereditary.*

Proof. There exists a projective module Q such that $P \oplus Q = F$ is a free module of finite rank. Since $\text{Hom}_R(F, F)$ is left hereditary (see [24]) and $\Lambda \cong f \text{Hom}_R(F, F)f$, where f is the projection of F on P orthogonal to Q , it suffices to show that eSe is left hereditary for any left hereditary ring S and idempotent $e \in S$.

Let G be a free left eSe module. Then $G \cong \bigoplus \sum_{\alpha \in A} X_\alpha$ with $X_\alpha = eSe$ for all α in some index set A . Consider the free S -module $\bigoplus \sum_{\alpha \in A} S_\alpha$ with $S_\alpha = S$ for all $\alpha \in A$ and let

$$\bar{e}: \bigoplus \sum_{\alpha \in A} S_\alpha \rightarrow \bigoplus \sum_{\alpha \in A} S_\alpha$$

be defined by $\bar{e}: 1_\alpha \rightarrow e_\alpha$ for all $\alpha \in A$. Then

$$\begin{aligned} \text{Hom}_{eSe}(G, G) &\cong \text{Hom}_{eSe} \left(\bigoplus \sum_{\alpha \in A} X_\alpha, \bigoplus \sum_{\alpha \in A} X_\alpha \right) \\ &\cong \bar{e} \text{Hom}_S \left(\bigoplus \sum_{\alpha \in A} S_\alpha, \bigoplus \sum_{\alpha \in A} S_\alpha \right) \bar{e}. \end{aligned}$$

It, therefore, follows from Theorem 3 and Lemma 5 that $\text{Hom}_{eSe}(G, G)$ has principal left ideals projective. Thus Theorem 3 implies that eSe is left hereditary.

2.7 REMARK. Since R is left hereditary if and only if the left global projective dimension of R (l. gl. pd. (R)) is ≤ 1 (see [22]), Theorem 6 says that if l. gl. pd. (R) ≤ 1 , then l. gl. pd. (Λ) \leq l. gl. pd. (R) where $\Lambda = \text{Hom}_R(P, P)$ and ${}_R P$ is finitely generated and projective. There is no more general inequality of this type even for algebras of finite rank over a field. To see this let K be a field,

$$R = \left\{ \begin{pmatrix} u & 0 & 0 \\ v & x & 0 \\ w & y & u \end{pmatrix} \mid u, v, w, x, y \in K \right\}$$

and $P = Re$ where

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\Lambda \cong eRe$ and applying [2] or [22, p. 56] one shows easily that l. gl. pd. $(R) = 2$ but l. gl. pd. $(eRe) = \infty$ since eRe is a local ring with nonzero socle.

2.8 THEOREM. *If R is left hereditary and P is a projective left R -module, then $\Lambda = \text{Hom}_R(P, P)$ is left semihereditary.*

Proof. Let ${}_R F$ be a free R -module, $\Gamma = \text{Hom}_R(F, F)$ and n be any positive integer. Then $F' = \bigoplus_{i=1}^n F_i$ with $F_i = F$, for $i = 1, 2, \dots, n$, is a free R -module and hence it follows from Theorem 3 that $\Gamma_n \cong \text{Hom}_R(F', F')$ has principal left ideals projective. Thus Γ is left semihereditary by Theorem 4. The proof can be completed by applying the same argument used to prove Theorem 6 except that in this instance one applies Theorem 4 in place of Theorem 3.

2.9 REMARK. The endomorphism ring of a free R -module with an infinite basis contains an infinite direct product of copies of R (the diagonal) and is, therefore, *never* hereditary (see [31, p. 1384]).

An argument similar to those used to prove the two preceding theorems can be used to prove the next theorem.

2.10 THEOREM. *If ${}_R P$ is a finitely generated projective module over a left semihereditary ring, then $\text{Hom}_R(P, P)$ is left semihereditary.*

In a remarkable paper [23], Kaplansky proved that every projective module is the direct sum of countably generated modules. He noted, further, that if every element of a countably generated module X can be embedded in a finitely generated direct summand, then X is a direct sum of finitely generated modules. Kaplansky used these results to deduce, among other things, that if R is a commutative semihereditary ring then every projective R -module is a direct sum of finitely generated modules. This result was generalized by Albrecht [1] who used Kaplansky's reduction to give a very easy proof that every projective right R -module over a right semihereditary ring is the direct sum of finitely generated projective modules. We close this section with a different generalization of Kaplansky's results.

2.11 THEOREM. *Let R be a right semihereditary ring. Then every projective left R -module is the direct sum of finitely generated projective modules.*

In view of the above remarks the following lemma will complete the proof of the theorem:

2.12 LEMMA. *Let R be a right semihereditary ring and P be a projective left R -module. Then any element of P can be embedded in a finitely generated direct summand of P .*

Proof. Let $x \in P$. There exists a module Q such that $F = P \oplus Q$ is free. Let $\Gamma = \text{Hom}_R(F, F)$. Then as usual F is a left R , right Γ -bimodule. Let $x = a_1 u_1 + \cdots + a_n u_n$ with $a_i \in R$ be a representation of x in terms of basis elements u_1, \dots, u_n for F and set $G = Ru_1 + \cdots + Ru_n$. Since P and G are direct summands of F there exist idempotents $e_1, e_2 \in \Gamma$ such that $P = Fe_1$ and $G = Fe_2$. Since $Rx \subseteq P \cap G$ and

$$\text{Ann}_F(\text{Ann}_\Gamma(Fe)) = Fe$$

for any idempotent $e \in \Gamma$, it follows that

$$Rx \subseteq S = \text{Ann}_F(\text{Ann}_\Gamma(Rx)) \subseteq P \cap G.$$

Thus it suffices to show that S is a direct summand of G for this implies S is finitely generated and a direct summand of F and hence also a direct summand of P .

Since S is a closed submodule of F , it is clear that

$$S = \text{Ann}_F(\text{Ann}_\Gamma(S)).$$

This is clearly equivalent to F/S being isomorphic to a submodule of a direct product of copies of F . Hence since both ${}_R F$ and ${}_R G$ are free and G/S is a submodule of F/S , it follows that G/S is isomorphic to a submodule of a direct product of copies of G . Hence setting $\Lambda = \text{Hom}_R(G, G)$, we conclude that

$$S = \text{Ann}_G(\text{Ann}_\Lambda(S)).$$

We also note that

$$\text{Ann}_\Lambda(Rx) = \text{Ann}_\Lambda(\text{Ann}_G(\text{Ann}_\Lambda(Rx))) = \text{Ann}_\Lambda(S).$$

Since ${}_R G$ is free, there is an $\alpha \in \Lambda$ such that $Rx = G\alpha$. Thus by (ii) of Lemma 1, $\text{Ann}_\Lambda(Rx) = r(\alpha)$ and since Λ is right semihereditary by (the right-hand analogue) of Theorem 10, we conclude that $r(\alpha) = f\Lambda$ with $f^2 = f \in \Lambda$. We, therefore, have $S = \text{Ann}_G(\text{Ann}_\Lambda(Rx)) = \text{Ann}_G(f\Lambda) = G(1-f)$ and so S is a direct summand of G as required. This completes the proof.

Note added in proof. Several people have kindly pointed out to us that a result essentially equivalent to Theorem 2.11 was obtained earlier by H. Bass. See Theorem 3 of *Projective modules over free groups are free*, J. Algebra **1** (1964), 367–373.

3. Hereditary rings. This section is devoted to determining the structure of all (left) hereditary rings containing faithful injective left and right ideals or whose injective hulls are projective. These turn out to be precisely direct sums of complete blocked triangular matrix rings over division rings just as did the hereditary QF-3 algebras of finite rank studied by Mochizuki [25]. We say that Γ is a *complete blocked triangular matrix ring* over a division ring D if there exists a finite-dimensional D -space V and a chain of subspaces

$$V \supseteq V_1 \supseteq \cdots \supseteq V_k = (0)$$

such that Γ consists of all linear transformations γ of V such that $V_i \gamma \subseteq V_i$ for

$i=1, \dots, k$. The name arises from the form of the matrix representation of Γ obtained by choosing a basis for V in the natural manner.

We begin by introducing some definitions and preliminary results used in the proof of the theorem. As before R denotes a ring with identity, N the Jacobson radical of R and ${}_R M$ a (unital) left R -module. A submodule L of M is an *essential* submodule if it has nonzero intersection with every nonzero submodule of M . We say that M is *finite dimensional* if every independent family of nonzero submodules of M has only finitely many members and M is *uniform* if every nonzero submodule of M is an essential submodule. M is finite dimensional if and only if M contains an essential submodule which is a finite direct sum of uniform submodules (see Goldie [16]). The *singular submodule* of M , $Z(M)$, is the set of all elements of M whose annihilators in R are essential left ideals. If R is left hereditary then every projective R -module I has zero singular submodule since if $0 \neq x \in I$, $\text{Ann}_R(x)$ is not an essential left ideal as it is the kernel of the map of R onto Rx defined by $r \rightarrow rx$ and so is a direct summand of R as Rx is projective.

3.1 LEMMA. *If P is a finitely generated projective injective left module over a left hereditary ring R , then $\Lambda = \text{Hom}_R(P, P)$ is a semisimple ring with minimum condition on one-sided ideals.*

Proof. It follows from Theorem 2.6 that Λ is left hereditary and from Theorem 5.1 of Faith [11, p. 44] that Λ is left self injective. Thus the conclusion follows from Osofsky [29, p. 650].

3.2 THEOREM. *The following conditions are equivalent.*

(1) *R is left hereditary and the injective hull of every projective left R -module is projective.*

(2) *R is left hereditary and R contains a faithful injective left ideal which is Π -projective.*

(3) *R is left hereditary and the injective hull of ${}_R R$ is projective.*

(4) *R is left hereditary and R contains a faithful injective left ideal and a faithful injective right ideal.*

(5) *R is left hereditary and R contains a faithful Σ -injective left ideal.*

(6) *R is left hereditary, R contains a faithful injective left ideal, and R does not contain an infinite set of orthogonal idempotents.*

(7) *R can be written as a direct sum of two-sided ideals each of which is isomorphic to a complete blocked triangular matrix ring over a division ring.*

Proof. It is clear that (1) and (2) each imply (3). We shall first prove that (3), (4), and (5) each imply (6) and then show that (7) follows from (6). It is known that (7) implies all the others (see [25] or [18]).

(3) *implies* (6). It follows from [13, Proposition 2.4] that since $E({}_R R)$ is projective, it is finitely generated. Hence $E({}_R R)$ is a finitely generated projective injective left

R -module and so Lemma 1 implies that $\Lambda = \text{Hom}_R(E(R), E(R))$ is a semisimple ring with minimum condition. Thus

$$E({}_R R) \cong I_1 \oplus \cdots \oplus I_t,$$

where the I_j are finitely generated indecomposable injective left ideals of R (see [13, Corollary 2.5]). Thus, since each I_j is uniform, $E({}_R R)$ is finite dimensional and hence so is R . Thus R contains no infinite set of orthogonal idempotents. Furthermore, we may assume that these ideals are indexed in such a way that each left ideal which appears in the above decomposition is isomorphic to exactly one of I_1, \dots, I_s for some $1 \leq s \leq t$. Then

$$I = I_1 \oplus \cdots \oplus I_s$$

is injective and faithful since $E({}_R R)$ is faithful. The proof of this implication will be completed by showing that R contains a left ideal isomorphic to I . Now R contains a left ideal isomorphic to I_1 so let r be the largest integer $\leq s$ such that R contains a left ideal isomorphic to $I_1 \oplus \cdots \oplus I_r$. If $r \neq s$, write

$$R = I_1 \oplus \cdots \oplus I_r \oplus L$$

and let Π_k denote the projection of R onto I_k with respect to the above decomposition for each $k = 1, \dots, r$. R also contains a copy of I_{r+1} and for each $k = 1, \dots, r$ the restriction of Π_k to I_{r+1} is either zero or a monomorphism since R is hereditary and I_{r+1} is indecomposable. Hence it is zero since I_{r+1} is injective, I_k is indecomposable and I_k and I_{r+1} are not isomorphic. Thus $I_{r+1} \subseteq L$. This contradiction completes the proof of this implication.

(4) implies (6). Let Re with $e^2 = e$ be the faithful injective left ideal of R . Then by Lemma 1 eRe is semisimple with minimum condition so we can write $e = e_1 + \cdots + e_t$ with the e_i primitive idempotents of eRe for $i = 1, \dots, t$, and hence each $e_i Re_i$ is a division ring. Furthermore, $eNe = (0)$ since it is the radical of eRe , and so $eN = (0)$ since Re is faithful. Thus $e_i R$ contains no nilpotent right ideals and so $e_i R$ is a simple module as in Jacobson [20, Proposition 1, p. 65]. Hence ReR is contained in the right socle of R . Now if H is any right ideal of R , $ReR \cap H \supseteq HReR \neq (0)$ since Re is faithful. Thus it is clear that the right socle of R is an essential right ideal, R contains only a finite number of isomorphism types of simple right R -modules and $Z(R_R) = (0)$.

Since R contains a faithful injective right ideal, it follows from the above observations that R contains a copy of the injective hull of the direct sum of one of each isomorphism type of simple right ideal of R and that this right ideal is finite dimensional and faithful. Moreover, it is of the form fR with $f^2 = f$ and since $Z(fR_R) = 0$ it is immediate from Theorem 5.1 of Faith [11, p. 44] that fRf is a semisimple ring with minimum condition on one-sided ideals.

Since Re and fR are both faithful R -modules, the bimodule fRe is clearly faithful both as a left fRf and a right eRe module. Thus since eRe and fRf are semisimple it

is an injective cogenerator over both rings. Moreover, it follows from Proposition 3 of [8] that $\text{Hom}_{fRf}(fRe, fRe) = eRe$ and $\text{Hom}_{eRe}(fRe, fRe) = fRf$. So fRe induces a Morita duality between the categories of reflexive left fRf and right eRe modules (see [30]). It is immediate from Proposition 3 of [8] that Re_{eRe} is reflexive and hence it is finitely generated over eRe since it is completely reducible and cannot be an infinite direct sum of eRe modules by Lemma 3.13 of Osofsky [30]. We conclude from this that the double centralizer $\hat{R} = \text{Hom}_{eRe}(Re, Re)$ is a semisimple ring with minimum condition on one-sided ideals. Since Re is faithful we may identify R with a subring of \hat{R} in the usual manner and this implies that R contains no infinite set of orthogonal idempotents since \hat{R} does not. Thus R satisfies condition (6).

(5) *implies* (6). Let Re be a faithful Σ -injective left ideal. Then $Re \subseteq R \subseteq \bigcup Re$ and so the lattices of annihilators in R of subsets of R and Re are the same. Thus it follows from Proposition 1.1 that R satisfies the ascending chain condition on annihilator left ideals and so R contains no infinite set of orthogonal idempotents. This completes the proof of this implication.

(6) *implies* (7). Let $Re, e^2 = e$, be a faithful injective left ideal of R . Then since R contains no infinite set of orthogonal idempotents we can write

$$e = e_1 + \cdots + e_r \quad \text{and} \quad 1 - e = e_{r+1} + \cdots + e_s$$

such that the e_i for $i = 1, \dots, s$ are a set of primitive orthogonal idempotents of R . Since the Re_i for $i = 1, \dots, r$ are indecomposable injective modules, they are uniform. Furthermore, since Re is faithful for each e_j with $r+1 \leq j \leq s$, there exists e_i with $1 \leq i \leq r$ such that $e_j Re_i \neq (0)$. Hence there exists a nonzero homomorphism of Re_j into Re_i and, since R is hereditary and Re_j is indecomposable, it must be a monomorphism. Thus Re_i is uniform and its injective hull is isomorphic to Re_j . We conclude from this that ${}_R R$ is finite dimensional and that $E({}_R R)$ is a finitely generated projective R -module.

Since $Z({}_R R) = (0)$ and ${}_R R$ is finite dimensional, it follows from Theorem 1.6 of Sandomierski [32] that the maximal left quotient ring Q of R is a semisimple ring with minimum condition on one-sided ideals and it is well known that ${}_R Q$ is isomorphic to $E({}_R R)$ (see [11]). Furthermore, since R is left hereditary and ${}_R R$ is finite dimensional it follows from Corollary 2 of Sandomierski [33] that R is left Noetherian.

Since ${}_R Qe$ is an essential extension of ${}_R Re$, $Qe = Re$ and hence $eQe = eRe$. Moreover, since Q is semisimple, eRe is semisimple and Re is finitely generated as a right eRe module and so satisfies the descending chain condition on eRe submodules. Thus the chain

$$Ne \supseteq N^2e \supseteq \cdots \supseteq N^k e \supseteq \cdots$$

becomes stationary so that there exists a t such that $N^t e = N^{t+1} e$. But this equality together with Nakayama's lemma implies that $N^t e = (0)$ and hence $N^t = (0)$ since Re is faithful.

Since Re_i is uniform for all $i=1, \dots, s$ and ${}_R Qe_i$ is essential over Re_i , it is also uniform as an R -module and hence is an indecomposable left ideal of Q . Thus $e_i Qe_i$ is a division ring. If $0 \neq x \in e_i Re_i$, it has an inverse $a \in e_i Qe_i$, i.e. $xa = ax = e_i$. Then $y = (1 - e_i) + x \in R$ has an inverse $y^{-1} = (1 - e_i) + a \in Q$. Since

$$R \subseteq Ry^{-1} \subseteq \dots \subseteq Ry^{-k} \subseteq \dots$$

is an ascending chain of R -submodules of Q it becomes stationary after a finite number of steps and so $y^{-(q+1)} = ry^{-q}$ for some integer q and $r \in R$. So $y^{-1} \in R$ and hence $e_i y^{-1} = e_i [1 - e_i + a] = a \in R$. Thus $e_i Re_i$ is a division ring. Hence $R/N(e_i + N)$ is a simple R/N module by Jacobson [20, Proposition 1, p. 65]. We conclude from this that R/N is a semisimple ring with minimum condition.

The results of the preceding two paragraphs imply that R is a semiprimary ring and as we noted previously R is left Noetherian. Thus R is also right hereditary (see [2]) and, in view of the equivalence of conditions (1) and (6) of Theorem 1.3, contains a faithful injective right ideal so that everything which we know about R is now symmetric. We shall complete the proof by showing that R is generalized uniserial, i.e. each principal indecomposable left and right ideal of R has a unique composition series, and by applying a result of Goldie [17]. Because of the left right symmetry we consider only one side. It suffices to show that if $g^2 = g$ is any primitive idempotent of R and k any positive integer such that $N^k g \neq (0)$, then $N^k g / N^{k+1} g$ is simple. However, by the same reasoning used in the first paragraph of this part of the proof one concludes that Rg is uniform and hence, since R is hereditary, $N^k g$ is an indecomposable projective R -module. Since R is semiprimary this implies $N^k g$ is isomorphic to a principal indecomposable left ideal of R and so $N(N^k g)$ is the unique maximal left ideal of $N^k g$ (see Eilenberg [10] or Bass [3]).

We have now shown that R satisfies the hypothesis of Theorem 8.11 of Goldie [17] and so the desired conclusion follows from that result. This completes the proof of the theorem.

We could have shortened the proof slightly by omitting the penultimate paragraph and appealing instead to Theorem 2 of Harada [18, p. 363] for the final conclusion. We did not do this because Goldie's paper is more self contained and so may be more accessible to the interested reader.

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