

# A GENERALIZED DUAL FOR A $C^*$ -ALGEBRA<sup>(1)</sup>

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**Abstract.** Let  $\mathcal{A}$  be a  $C^*$ -algebra, let  $\mathcal{B}$  be its enveloping von Neumann algebra, and let  $\mathcal{Z}$  be the center of  $\mathcal{B}$ . Let  $\mathcal{B}_\sim$  be the set of all  $\sigma$ -weakly continuous  $\mathcal{Z}$ -module homomorphisms of the  $\mathcal{Z}$ -module  $\mathcal{B}$  into  $\mathcal{Z}$  and let  $\mathcal{A}^\sim$  be the set of all restrictions to  $\mathcal{A}$  of elements of  $\mathcal{B}_\sim$ . Then  $\mathcal{A}$  is classified as CCR, GCR, and NGCR in terms of certain naturally occurring topologies on  $\mathcal{A}^\sim$ .

**1. Introduction.** The purpose of this paper is the presentation of a set of operators on a  $C^*$ -algebra that generalizes, to a certain degree, the notion of the dual of the algebra. However, instead of making an abstract study of this set of operators at this time, we characterize CCR, GCR, and NGCR algebras in terms of the topology of this set. The motivation for the generalization is our belief that the dual is too small to characterize intrinsically the different types of algebras. As evidence, we cite the result of J. Glimm [6, Theorem 6]: If  $\mathcal{A}$  is a  $C^*$ -algebra, then the set  $\{\alpha f \mid f \text{ is a pure state of } \mathcal{A}, 0 \leq \alpha \leq 1\}$  is  $w^*$ -compact if, and only if,  $\mathcal{A}$  is CCR,  $\mathcal{A}$  has a Hausdorff structure space, and  $\mathcal{A}$  modulo the closure of the ideal of elements with continuous trace is commutative; and the result of J. Tomiyama and M. Takesaki [26]: The pure state space of  $\mathcal{A}$  is equal to the state space of  $\mathcal{A}$  if, and only if,  $\mathcal{A}$  is NGCR and the ideal (0) is prime. In both these examples it appears that the dual is not strong enough to handle the centers of the represented algebras. Now, if  $\mathcal{A}$  is a  $C^*$ -algebra, let  $\mathcal{B}$  be the enveloping von Neumann algebra of  $\mathcal{A}$ . The algebra  $\mathcal{B}$  is a module over its center  $\mathcal{Z}$ . The set  $\mathcal{B}_\sim$  of all  $\sigma$ -weakly module homomorphisms of  $\mathcal{B}$  into  $\mathcal{Z}$  is a normed  $\mathcal{Z}$ -module and the set of all continuous module homomorphisms of  $\mathcal{B}_\sim$  into  $\mathcal{Z}$  is identified with  $\mathcal{B}$ . By analogy, the set  $\mathcal{A}^\sim$  of restriction to  $\mathcal{A}$  of the elements of  $\mathcal{B}_\sim$  will be our generalized dual. We characterize CCR, GCR, and NGCR algebras in terms of the compactness of the unit sphere of  $\mathcal{A}^\sim$  and in terms of the size of the closure of the set of extreme points of the positive elements in the unit sphere of  $\mathcal{A}^\sim$  in various naturally occurring topologies. One of these might be considered as a generalization of Glimm's theorem. We use one of the characterizations to locate a CCR algebra in its enveloping von Neumann algebra.

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We hope that this application will make a more abstract study of  $\mathcal{A}^\sim$  acceptable to the mathematical public. Perhaps a generalization of the theorems of Tomiyama and Takesaki [26] will be obtained.

**2. Weakly dense subalgebras.** We first present some needed lemmas concerning weakly dense  $*$ -subalgebras of a von Neumann algebra.

**PROPOSITION 1.** *Let  $\mathcal{B}$  be a von Neumann algebra and let  $\mathcal{A}$  be a weakly dense uniformly closed  $*$ -subalgebra of  $\mathcal{B}$ . Suppose that every continuous linear functional on  $\mathcal{A}$  has an extension to a  $\sigma$ -weakly continuous functional of  $\mathcal{B}$ . If  $\varphi$  is a continuous linear function of  $\mathcal{A}$  into the center  $\mathcal{Z}$  of  $\mathcal{B}$ , then there is a unique  $\sigma$ -weakly continuous linear function  $\psi$  of  $\mathcal{B}$  into  $\mathcal{Z}$  whose restriction to  $\mathcal{A}$  is  $\varphi$ .*

**Proof.** Since  $\mathcal{A}$  is  $\sigma$ -weakly dense in  $\mathcal{B}$ , the existence of  $\psi$  implies its uniqueness. We now show  $\psi$  exists. For each  $x$  and  $y$  in the Hilbert space  $H$  of  $\mathcal{B}$ , let  $f_{x,y}$  be the unique  $\sigma$ -weakly continuous extension of  $w_{x,y} \cdot \varphi$  to  $\mathcal{B}$ . Here  $w_{x,y}(A) = (Ax, y)$ . From the Kaplansky density theorem we obtain that  $f_{x,y}$  is an isometric extension. For fixed  $A$  in  $\mathcal{B}$  we have that  $\langle x, y \rangle_A = f_{x,y}(A)$  is a bilinear form on  $H$ . It satisfies

$$|\langle x, y \rangle_A| \leq \|\varphi\| \|A\| \|x\| \|y\|;$$

hence, there is a bounded linear operator  $\psi(A)$  on  $H$  such that  $(\psi(A)x, y) = \langle x, y \rangle_A$  for every  $x$  and  $y$  in  $H$ . We have that  $A \rightarrow (\psi(A)x, y)$  is linear for each fixed  $x$  and  $y$  and so  $\psi$  is a bounded linear operator of  $\mathcal{B}$  into the algebra of all bounded linear operators on  $H$ . For each  $B$  in the commutant of  $\mathcal{Z}$  and each  $x$  and  $y$  we have that  $w_{Bx,y} \cdot \varphi = w_{x,B^*y} \cdot \varphi$  and, therefore,

$$(\psi(A)Bx, y) = f_{Bx,y}(A) = f_{x,B^*y}(A) = (B\psi(A)x, y).$$

This proves that  $B\psi(A) = \psi(A)B$  and that  $\psi(A)$  is in  $\mathcal{Z}$ . Because  $w_{x,y}\psi = f_{x,y}$  is  $\sigma$ -weakly continuous on  $\mathcal{B}$  for each  $x$  and  $y$  in  $H$ , we have that  $\psi$  is  $\sigma$ -weakly continuous on  $\mathcal{B}$ . Q.E.D.

The next lemma indicates the circumstances under which Proposition 1 will be used.

**LEMMA 2.** *Let  $\mathcal{B}$  be a von Neumann algebra and let  $\mathcal{A}$  be a weakly dense uniformly closed  $*$ -subalgebra of  $\mathcal{B}$ . Suppose every continuous linear functional of  $\mathcal{A}$  has an extension to a  $\sigma$ -weakly continuous functional on  $\mathcal{B}$ . If  $I$  is a uniformly closed two-sided ideal in  $\mathcal{A}$  whose weak closure is equal to  $\mathcal{B}P$ , where  $P$  is a central projection of  $\mathcal{B}$ , then every continuous linear functional on  $I$  (resp.  $\mathcal{A}(1-P)$ ) has an extension to a  $\sigma$ -weakly continuous functional on  $\mathcal{B}P$  (resp.  $\mathcal{B}(1-P)$ ).*

**Proof.** If  $f$  is a continuous linear functional on  $I$ , then  $f$  has an extension to a continuous linear functional on  $\mathcal{A}$  which, in turn, has an extension to a  $\sigma$ -weakly continuous linear functional  $g$  on  $\mathcal{B}$ . The restriction of  $g$  to  $\mathcal{B}P$  is the desired extension of  $f$ .

If  $f$  is a continuous linear functional on  $\mathcal{A}(1-P)$ , then  $g(A)=f(A(1-P))$  ( $A \in \mathcal{A}$ ) defines a continuous linear functional on  $\mathcal{A}$ . There is a  $\sigma$ -weakly continuous functional  $h$  on  $\mathcal{B}$  whose restriction to  $\mathcal{A}$  coincides with  $g$ . We have that  $h(A)=0$  for every  $A \in I$  and, therefore,  $h(A)=0$  for every  $A \in \mathcal{B}P$ . This means that  $f(A(1-P))=g(A)=h(A)=h(A(1-P))$  for every  $A$  in  $\mathcal{A}$ . Therefore, the functional  $h$  restricted to  $\mathcal{B}(1-P)$  is the desired  $\sigma$ -weakly continuous extension of  $f$  to  $\mathcal{B}(1-P)$ . Q.E.D.

The following, essentially due to M. Tomita [27, Theorem 6], is an approximation lemma (cf. also [21]).

**LEMMA 3.** *Let  $\mathcal{B}$  be a type I von Neumann algebra and let  $\mathcal{A}$  be a weakly dense uniformly closed  $*$ -subalgebra of  $\mathcal{B}$ . Let  $E_1, E_2, \dots, E_n$  be abelian projections in  $\mathcal{B}$  and let  $B$  be an element (resp. hermitian element) in  $\mathcal{B}$ . Then there exists a net  $\{P_i \mid i \in D\}$  of central projections of  $\mathcal{B}$  and a corresponding net  $\{A_i \mid i \in D\}$  of elements (resp. hermitian elements) in  $\mathcal{A}$  such that  $\{P_i\}$  converges strongly to 1 and such that  $A_i E_j P_i = B E_j P_i$  for all  $j=1, 2, \dots, n$  and all  $i \in D$ . Furthermore, the net  $\{A_i\}$  may be chosen so that  $\{\|A_i\|\}$  is bounded by  $8m^{1/2}\|B\|$  where  $m$  is some number such that  $\text{lub}\{E_1, E_2, \dots, E_n\}$  can be written as the sum of  $m$  mutually orthogonal abelian projections (cf. [11, proof of Theorem 4.3] and [9, Theorem 2.1]).*

Let  $\mathcal{B}$  be a type I von Neumann algebra with center  $\mathcal{Z}$  and let  $I_a$  be the closed two-sided ideal in  $\mathcal{B}$  which is generated by the abelian projections of  $\mathcal{B}$  [15]. If  $A \in I_a^+$ , then there is a monotonely decreasing sequence  $\{A_n\}$  in  $\mathcal{Z}^+$  and a sequence of mutually orthogonal abelian projections  $\{E_n\}$  such that

$$(1) \text{ closure } \{\zeta \in Z \mid A_n^\wedge(\zeta) \neq 0\} = \{\zeta \in Z \mid P_n^\wedge(\zeta) = 1\},$$

where  $Z$  is the spectrum of  $\mathcal{Z}$ ,  $B^\wedge$  is the Gelfand transform of  $B$  in  $\mathcal{Z}$ , and  $P_n$  is the central support of  $E_n$ ;

$$(2) \lim_n A_n = 0; \text{ and}$$

$$(3) A = \sum A_n E_n.$$

The sum  $\sum A_n E_n$  is called a spectral resolution or decomposition for  $A$  [9, §2].

Let  $\mathcal{B}$  be any von Neumann algebra and let  $\mathcal{Z}$  be the spectrum of the center of  $\mathcal{B}$ . For each  $\zeta \in Z$ , let  $[\zeta]$  be the smallest closed two-sided ideal in  $\mathcal{B}$  which contains  $\zeta$ . We write the image of  $A \in \mathcal{B}$  under the canonical homomorphism of  $\mathcal{B}$  into  $\mathcal{B}([\zeta]) = \mathcal{B}/[\zeta]$  by  $A(\zeta)$ . The map  $\zeta \rightarrow \|A(\zeta)\|$  is continuous on  $Z$  for each fixed  $A \in \mathcal{B}$  [5, Lemma 10].

The following extension of Lemma 3 is needed.

**LEMMA 4.** *Let  $\mathcal{B}$  be a type I von Neumann algebra and let  $\mathcal{A}$  be a weakly dense uniformly closed  $*$ -subalgebra of  $\mathcal{B}$  that is contained in the ideal  $I_a$  generated by the abelian projections of  $\mathcal{B}$ . Then for any sequence  $\{C_n\}$  of elements of  $I_a$ , there exists a net  $\{P_i\}$  of central projections of  $\mathcal{B}$  such that  $\{P_i\}$  converges strongly to 1 and  $C_n P_i \in \mathcal{A} P_i$  for every  $n$  and  $i$ .*

**Proof.** Let  $\mathcal{Z}$  be the center of  $\mathcal{B}$ . Due to the existence of the spectral decomposition for elements of  $I_a^+$ , it is sufficient to show that, given an element  $C$  in  $\mathcal{Z}^+$ ,

an abelian projection  $E$  in  $\mathcal{B}$ , a finite set  $\{x_i\}$  of unit vectors in the Hilbert space of  $\mathcal{B}$ , and  $\varepsilon > 0$ , then there exists a projection  $P \in \mathcal{Z}$  such that  $\|(1-P)x_i\| < \varepsilon$  for every  $x_i$  and  $CEP \in \mathcal{AP}$ . Indeed, by induction we can find a projection  $P'$  in  $\mathcal{Z}$  with  $B_k E_k P' \in \mathcal{AP}'$  for every  $B_k E_k$  that appears as a summand in the spectral resolution of  $C_n + C_n^*$  or  $i(C_n - C_n^*)$  ( $n = 1, 2, \dots$ ) and with  $\|(1-P')x_i\| < \varepsilon$  for every  $x_i$ . Because  $\mathcal{AP}'$  is uniformly closed and the partial sums of the spectral decomposition converge uniformly, we have  $C_n P' \in \mathcal{AP}'$  ( $n = 1, 2, \dots$ ) as desired.

Let  $\{Q_i\}$  be a maximal set of mutually orthogonal nonzero projections in  $\mathcal{Z}$  such that for each  $Q_i$  there is an  $A_i \in \mathcal{A}^+$  with  $A_i(\zeta) \neq 0$  whenever  $Q_i$  does not lie in the maximal ideal  $\zeta$  in the spectrum  $Z$  of  $\mathcal{Z}$ . We have that  $1 = \sum Q_i$ ; otherwise, we have  $\mathcal{A}(1 - \sum Q_i) = 0$  since  $A(\zeta) \neq 0$  for some  $\zeta$  which does not contain  $1 - \sum Q_i$  implies  $A(\zeta') \neq 0$  for every  $\zeta'$  in an open and closed neighborhood of  $\zeta$  and so  $\{Q_i\}$  would not be maximal. Now there is a finite subset of  $\{Q_i\}$  with sum  $Q$  such that  $\|(1-Q)x_i\| < \varepsilon/2$  for every  $x_i$ . If  $A$  is the sum of the  $A_i$ 's corresponding to this finite subset, then we have  $A(\zeta) \neq 0$  for every  $\zeta$  not containing  $Q$ . Let  $A$  have the spectral resolution  $A = \sum B_i F_i$ ; then  $|B_1(\zeta)| = \|A(\zeta)\| \neq 0$  for every  $\zeta$  which does not contain  $Q$ . Therefore, there is a  $\delta > 0$  such that  $B_1 Q \geq \delta Q$ . Let  $f$  be the continuous real-valued function on  $[0, \infty)$  given by  $f(\alpha) = 0$  if  $\alpha \in [0, \delta/2]$ ,  $f(\alpha) = \alpha$  if  $\alpha \in [\delta, \infty)$  and  $f(\alpha)$  linear on  $[\delta/2, \delta]$ . Let  $p$  be a polynomial with  $p(0) = 0$  and  $|p(\alpha) - f(\alpha)| \leq \rho$  for every  $\alpha \in [0, \|A\| + \delta]$ . There is an  $m_0$  such that  $\|B_m\| \leq \delta/2$  whenever  $m \geq m_0$ . We have that

$$\begin{aligned} & \left\| \sum \{p(B_i)F_i \mid 1 \leq i \leq m\} - \sum \{p(B_i)F_i \mid 1 \leq i \leq m_0\} \right\| \\ & \leq \left\| \sum \{|p(B_i)|F_i \mid m_0 + 1 \leq i \leq m\} \right\| \leq \rho \end{aligned}$$

whenever  $m > m_0$ . Since  $p(A) = \lim \sum \{p(B_i)F_i \mid 1 \leq i \leq m\}$ , we have that

$$\begin{aligned} & \left\| f(A) - \sum \{f(B_i)F_i \mid 1 \leq i \leq m\} \right\| \\ & \leq \|f(A) - p(A)\| + \left\| p(A) - \sum \{p(B_i)F_i \mid 1 \leq i \leq m\} \right\| \\ & \quad + \left\| \sum \{(p(B_i) - f(B_i))F_i \mid 1 \leq i \leq m\} \right\| \leq 3\rho \end{aligned}$$

whenever  $m \geq m_0$ . Since  $m_0$  is independent of  $p$  and  $\rho$ , we have that  $f(A) = \sum \{f(B_i)F_i \mid 1 \leq i \leq m_0\}$ . Note that  $f(B_1)Q = B_1 Q$  and, therefore, there is a  $C_1 \in \mathcal{Z}^+$  with  $C_1 B_1 Q = Q$ . Because the central support of  $F_1$  majorizes  $Q$  (by the definition of the spectral resolution), there is a partial isometry  $V$  in  $\mathcal{B}$  such that  $VV^* = EQ$  and  $V^*V \leq F_1 Q$  [2, III, 3, Lemma 1]. By Lemma 3 there is a  $U \in \mathcal{A}$  and a projection  $P \in \mathcal{Z} Q$  such that  $\|(Q-P)x_i\| < \varepsilon/2$  for all  $x_i$  and  $Uf(A)P = C_1^{1/2}VC_1f(A)P$ . Therefore, the element  $CEP$  is in  $\mathcal{AP}$  and  $\|(1-P)x_i\| < \varepsilon$  for every  $x_i$ . Q.E.D.

**2. A generalization of the dual space.** Let  $\mathcal{P}$  be the set of all continuous positive functionals on a  $C^*$ -algebra  $\mathcal{A}$ . For each  $f \in \mathcal{P}$  let  $L(f)$  be the left ideal  $L(f) = \{A \in \mathcal{A} \mid f(A^*A) = 0\}$  and let  $H(f)$  be the Hilbert space obtained by completing

the prehilbert space  $\mathcal{A} - L(f)$  with inner product  $\langle A - L(f), B - L(f) \rangle = f(B^*A)$ . Let  $\pi_f$  be the representation of  $\mathcal{A}$  on  $H(f)$  which extends left multiplication on  $\mathcal{A} - L(f)$ . Let  $\pi_0$  denote the representation  $\pi_0 = \sum \bigoplus \{\pi_f \mid f \in \mathcal{P}\}$  on the Hilbert space  $H_0 = \sum \bigoplus \{H(f) \mid f \in \mathcal{P}\}$ . The representation  $\pi_0$  is faithful and the weak closure  $\mathcal{B}$  of  $\pi_0(\mathcal{A})$  in  $H_0$  is the von Neumann algebra generated by  $\pi_0(\mathcal{A})$  in  $H_0$ . The algebra  $\mathcal{B}$  is called the enveloping von Neumann algebra of  $\mathcal{A}$  and  $H_0$  is called the canonical Hilbert space of  $\mathcal{B}$ . We identify  $\mathcal{A}$  with its image  $\pi_0(\mathcal{A})$  in  $\mathcal{B}$ . The algebra  $\mathcal{B}$  possesses the following universal property: If  $\pi$  is a representation of  $\mathcal{A}$  on a Hilbert space  $H$ , then there is a unique normal representation  $\pi'$  of  $\mathcal{B}$  on  $H$  such that  $\pi'(A) = \pi(A)$  for every  $A$  in  $\mathcal{A}$  and  $\pi'(\mathcal{B})$  is the weak closure of  $\pi(\mathcal{A})$  on  $H$ . If  $\pi$  is irreducible, then there is a minimal projection  $E'$  in the commutator of  $\mathcal{B}$  on  $H_0$  and an isometric isomorphism  $U$  of  $E'(H_0)$  onto  $H$  such that  $UAx = \pi(A)Ux$  for every  $x \in E'(H_0)$ . We shall always identify  $E'(H_0)$  with  $H$  (cf. [3, 12.1.5]).

Let  $\mathcal{B}$  be a von Neumann algebra and let  $\mathcal{A}$  be a weakly dense uniformly closed  $*$ -subalgebra of  $\mathcal{B}$ . In particular let  $\mathcal{B}$  be the enveloping von Neumann algebra of  $\mathcal{A}$ . Let  $\mathcal{Z}$  be the center of  $\mathcal{B}$  and let  $\mathcal{B}_\sim$  (resp.  $\mathcal{B}^\sim$ ) be the space of  $\sigma$ -weakly continuous (resp. continuous)  $\mathcal{Z}$ -module homomorphisms of  $\mathcal{B}$  into  $\mathcal{Z}$  (cf. [9], [10], and [11]). We call the elements of  $\mathcal{B}_\sim$  (resp.  $\mathcal{B}^\sim$ )  $\sigma$ -weakly continuous (resp. continuous) functionals of the module  $\mathcal{B}$ . Let  $R = R(\mathcal{B}_\sim, \mathcal{A})$  be the set of all restrictions to  $\mathcal{A}$  of elements in  $\mathcal{B}_\sim$ . If  $\mathcal{B}$  is the enveloping algebra of  $\mathcal{A}$ , we denote  $R$  by  $\mathcal{A}^\sim$ . Since each functional in  $\mathcal{B}_\sim$  is uniquely determined by its restriction to  $\mathcal{A}$ , no notational distinction will be made between an element in  $\mathcal{B}_\sim$  and its restriction to  $\mathcal{A}$ . Notice that  $R$  is a  $\mathcal{Z}$ -module under pointwise operations and that the operator bound

$$\text{lub } \{\|\varphi(A)\| \mid A \in \mathcal{A}, \|A\| \leq 1\}$$

for  $\varphi$  in  $R$  is equal to the operator bound

$$\text{lub } \{\|\varphi(A)\| \mid A \in \mathcal{B}, \|A\| \leq 1\}$$

for  $\varphi$  in  $\mathcal{B}_\sim$  by the Kaplansky density theorem [13]. If  $X$  is a subspace of  $\mathcal{B}^\sim$  and  $Y$  is a subspace of  $\mathcal{B}$ , then the topology on  $X$  given by pointwise convergence on  $Y$  where  $\mathcal{Z}$  is taken with its uniform (resp.  $\sigma$ -weak) topology will be called the  $\sigma_u(X, Y)$ -topology (resp.  $\sigma_w(X, Y)$ -topology) on  $X$ .

Let  $\mathcal{S}(\mathcal{B}^\sim)$  denote the positive elements in  $\mathcal{B}^\sim$  of norm not exceeding one. Here  $\varphi \in \mathcal{B}^\sim$  is positive if and only if  $\varphi(A) \geq 0$  for every  $A \in \mathcal{B}^+$ . The set  $\mathcal{S}(\mathcal{B}^\sim)$  is a convex set. Also  $\mathcal{S}(\mathcal{B}^\sim)$  is  $\mathcal{Z}$ -convex in the sense that  $A\varphi + (1-A)\varphi' \in \mathcal{S}(\mathcal{B}^\sim)$  whenever  $\varphi$  and  $\varphi'$  are in  $\mathcal{S}(\mathcal{B}^\sim)$  and  $A \in \mathcal{Z}$  with  $0 \leq A \leq 1$ . The functional  $\varphi \in \mathcal{S}(\mathcal{B}^\sim)$  is said to be  $\mathcal{Z}$ -irreducible if given  $\psi \in \mathcal{S}(\mathcal{B}^\sim)$  with  $\varphi - \psi \in \mathcal{S}(\mathcal{B}^\sim)$ , then there is a  $C \in \mathcal{Z}^+$  with  $C\varphi = \psi$ . Also  $\varphi$  is said to be a  $\mathcal{Z}$ -extreme point of  $\mathcal{S}(\mathcal{B}^\sim)$  if  $\varphi'$  and  $\varphi''$  in  $\mathcal{S}(\mathcal{B}^\sim)$ ,  $A$  in  $\mathcal{Z}^+$  with  $0 < A < 1$  (i.e. the spectrum of  $A$  is contained in the open interval  $(0, 1)$ ), and  $A\varphi' + (1-A)\varphi'' = \varphi$  imply  $\varphi' = \varphi'' = \varphi$ .

The same terminology is employed if  $\mathcal{B}^\sim$  is replaced by either  $\mathcal{B}_\sim$  or  $R$  and if  $\mathcal{S}(\mathcal{B}^\sim)$  is replaced by either  $\mathcal{S}(\mathcal{B}_\sim)$  or  $\mathcal{S}(R)$  respectively. The functional  $\varphi \in \mathcal{S}(\mathcal{B}_\sim)$  is a  $\mathcal{Z}$ -extreme point (resp.  $\mathcal{Z}$ -irreducible) in  $\mathcal{S}(\mathcal{B}_\sim)$  if and only if  $\varphi$  considered as an element of  $\mathcal{S}(\mathcal{B}^\sim)$  is a  $\mathcal{Z}$ -extreme point (resp.  $\mathcal{Z}$ -irreducible) in  $\mathcal{S}(\mathcal{B}^\sim)$ . Indeed, if  $\varphi \in \mathcal{S}(\mathcal{B}_\sim)$ ,  $\psi \in \mathcal{S}(\mathcal{B}^\sim)$  and  $\varphi - \psi \in \mathcal{S}(\mathcal{B}^\sim)$ , then  $\psi \in \mathcal{S}(\mathcal{B}_\sim)$ . Also  $\varphi$  is a  $\mathcal{Z}$ -extreme point of  $\mathcal{S}(\mathcal{B}^\sim)$  if and only if  $\varphi$  is  $\mathcal{Z}$ -irreducible and  $\varphi(1)$  is a projection [10, Theorem 8]. But given  $\varphi \in \mathcal{S}(\mathcal{B}^\sim)$  (resp.  $\mathcal{S}(\mathcal{B}_\sim)$ ), there is a  $\psi \in \mathcal{S}(\mathcal{B}^\sim)$  (resp.  $\mathcal{S}(\mathcal{B}_\sim)$ ) and a  $C$  in  $\mathcal{Z}^+$  such that  $\psi(1)$  is a projection and  $C\psi = \varphi$ . Now a result of E. Størmer [24, Proposition 3.2] states that an extreme point of  $\mathcal{S}(\mathcal{B}^\sim)$  is  $\mathcal{Z}$ -irreducible. Therefore, an extreme point of  $\mathcal{S}(\mathcal{B}^\sim)$  is equal to the product of a  $\mathcal{Z}$ -extreme point of  $\mathcal{S}(\mathcal{B}^\sim)$  and an element of  $\mathcal{Z}^+$ .

Let  $\varphi$  be any  $\mathcal{Z}$ -irreducible functional in  $\mathcal{S}(\mathcal{B}_\sim)$ . There is an element  $A_0$  in the ideal  $I_a^+$  of the commutant  $\mathcal{Z}'$  of  $\mathcal{Z}$  with a spectral resolution  $A_0 = \sum A_i E_i$  in  $\mathcal{Z}'$  so that

$$\left\{ \sum \{A_i \mid 1 \leq i \leq n\} \right\}_n$$

is bounded such that  $\varphi(A) = \sum A_i \tau_{E_i}(A)$  for every  $A$  in  $\mathcal{B}$ . If  $P$  is the central support of abelian projection  $E \in \mathcal{Z}'$ , then  $\tau_E(A)$  is the unique element  $B$  in  $\mathcal{Z}P$  with  $BE = EAE$  [9, §4] and [10, Theorem 2]. But the relation  $\varphi(A) \geq A_1 \tau_{E_1}(A)$  for every  $A \in \mathcal{B}^+$  implies the existence of  $B \in \mathcal{Z}^+$  with  $B\varphi = A_1 \tau_{E_1}$ . For every central projection  $P$  with  $PB = 0$  we have that  $A_1 E_1 P = 0$  and so by definition of the spectral resolution  $A_0 P = 0$  and  $P \cdot \varphi = 0$ . Thus there is a  $C \in \mathcal{Z}^+$  with  $C \tau_{E_1} = \varphi$ . Therefore we have a specific form for  $\mathcal{Z}$ -irreducible elements of  $\mathcal{S}(\mathcal{B}_\sim)$  and for extreme points of  $\mathcal{S}(\mathcal{B}_\sim)$ .

A positive functional on a  $C^*$ -algebra  $\mathcal{A}$  with identity is said to be abelian if the commutant  $\pi(\mathcal{A})'$  of the image of  $\mathcal{A}$  under the canonical representation  $\pi$  induced by  $f$  is an abelian algebra. These functionals were studied by M. Tomita for separable algebras. In order that a positive functional  $f$  on a  $C^*$ -algebra  $\mathcal{A}$  be abelian, a necessary and sufficient condition is that, for every positive functional  $g$  on  $\mathcal{A}$  with  $g \leq f$ , there is a positive element  $C$  in the center of  $\mathcal{A}$  such that  $g(A) = f(CA)$  for every  $A$  in  $\mathcal{A}$  [8, Lemma 2.2]. Now let  $\mathcal{A}$  be a von Neumann algebra on the Hilbert space  $H$  and let  $f$  be a normal functional on  $\mathcal{A}$ ; then in order that  $f$  be abelian, a necessary and sufficient condition is that there exist a vector  $x$  in  $H$  such that  $f = w_x$  and such that the projection in  $\mathcal{A}$  corresponding to the closure of the linear manifold  $\{A'x \mid A' \text{ in } \mathcal{A}'\}$  generated by the commutant  $\mathcal{A}'$  of  $\mathcal{A}$  is an abelian projection in  $\mathcal{A}$  [8, Theorem 2.3].

We prove the following proposition about  $\mathcal{Z}$ -extreme points for  $\mathcal{S}(\mathcal{B}_\sim)$  when  $\mathcal{B}$  is continuous.

**PROPOSITION 5.** *If  $\mathcal{B}$  is a continuous von Neumann algebra, then the set of positive elements  $\mathcal{S} = \mathcal{S}(\mathcal{B}_\sim)$  in the unit sphere of  $\mathcal{B}_\sim$  contains no  $\mathcal{Z}$ -extreme points save 0.*

**Proof.** We argue by contradiction. Let  $\varphi$  be a nonzero  $\mathcal{Z}$ -extreme point of  $\mathcal{S}$ . There is a unit vector  $x$  in the Hilbert space of  $\mathcal{B}$  such that  $\varphi(1)x = x$  since  $\varphi(1)$  is a projection. Let  $f$  be a positive functional on  $\mathcal{B}$  such that  $f \leq w_x \cdot \varphi$ . Then there is a vector  $y$  and a positive functional  $\psi$  in  $\mathcal{B}_\sim$  such that  $w_y \cdot \psi = f$  and  $\psi(1)y = y$  [10, Theorem 1]. Since  $w_y \leq w_x$  on the center  $\mathcal{Z}$  of  $\mathcal{B}$ , there is an element  $C$  in  $\mathcal{Z}^+$  with  $w_y \cdot \psi = w_x \cdot C\psi$  (Radon-Nikodym theorem). Thus  $PC\psi \leq P\varphi$  where  $P$  is the support of  $w_x$  restricted to  $\mathcal{Z}$ . This means that there is  $D$  in  $\mathcal{Z}^+$  such that  $f = w_x \cdot D\varphi$ . Therefore, the functional  $w_x \cdot \varphi$  is a nonzero normal abelian functional on  $\mathcal{B}$ . By the remarks preceding this proposition, we see that  $\mathcal{B}$  contains a nonzero abelian projection. This is impossible since  $\mathcal{B}$  is a continuous algebra. Therefore  $\mathcal{S}$  contains no nonzero extreme points. Q.E.D.

We are now ready to state the main theorem.

**THEOREM 6.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra and let  $\mathcal{B}$  be the enveloping von Neumann algebra of  $\mathcal{A}$ . Let  $\mathcal{Z}$  be the center of  $\mathcal{B}$ . The following are equivalent:*

- (a1)  $\mathcal{A}$  is CCR;
- (a2) if  $\psi \in \mathcal{B}_\sim$ , then there is a unique  $\varphi \in \mathcal{B}_\sim$  such that  $\psi(A) = \varphi(A)$  for every  $A \in \mathcal{A}$ ;
- (a3) the unit sphere of  $\mathcal{A}_\sim$  is compact in the  $\sigma_w(\mathcal{A}_\sim, \mathcal{A})$ -topology;
- (a4) if  $\{\varphi_n\}$  is a net of  $\mathcal{Z}$ -irreducible functionals in  $\mathcal{S}(\mathcal{B}_\sim)$  such that  $\{\varphi_n(A)\}$  converges in the uniform topology for every  $A$  in  $\mathcal{A}$ , then there is a  $\mathcal{Z}$ -irreducible functional  $\varphi$  in  $\mathcal{S}(\mathcal{B}_\sim)$  such that  $\varphi(A) = \lim \varphi_n(A)$  for every  $A \in \mathcal{A}$ ; and
- (a5) if  $\{\varphi_n\}$  is a net of  $\mathcal{Z}$ -extreme points of  $\mathcal{S}(\mathcal{A}_\sim)$  such that  $\{\varphi_n(A)\}$  converges uniformly for each  $A$  in  $\mathcal{A}$ , then there is a  $\mathcal{Z}$ -irreducible functional  $\varphi$  in  $\mathcal{S}(\mathcal{A}_\sim)$  such that  $\{\varphi_n\}$  converges in the  $\sigma_u(\mathcal{A}_\sim, \mathcal{A})$ -topology to  $\varphi$ .

**REMARK.** It is the equivalence of (a1) and (a5) that we consider to be the generalization of Glimm's theorem quoted in the introduction.

The method of proof will be (a1)  $\Rightarrow$  (a2)  $\Rightarrow$  (a3)  $\Rightarrow$  (a1) and (a3)  $\Rightarrow$  (a4)  $\Rightarrow$  (a5)  $\Rightarrow$  (a1). This scheme is used because our proof of (a3)  $\Rightarrow$  (a4) requires the hypothesis that  $\mathcal{A}$  is CCR. Note (a4) implies (a5) is obvious.

**Proof of (a1) implies (a2).** If  $\psi \in \mathcal{B}_\sim$ , then the restriction  $\varphi$  of  $\psi$  to  $\mathcal{A}$  can be extended uniquely to a  $\sigma$ -weakly continuous linear function  $\varphi$  to  $\mathcal{B}$  to  $\mathcal{Z}$  (Proposition 1). The main difficulty lies in proving that  $\varphi$  is a module homomorphism. In order to prove  $\varphi$  is a module homomorphism, we have divided the proof into two parts: in the first we study the restrictions to a special ideal of  $\mathcal{A}$  of the linear maps of  $\mathcal{A}$  into the center of a von Neumann algebra which is more general than the enveloping algebra of  $\mathcal{A}$ . We show that the unique extensions of these restrictions to  $\sigma$ -weakly continuous functions into the center are actually module homomorphisms on the weak closure of the ideal. In the second part of the proof we construct a composition series for  $\mathcal{A}$  so that for the first part of the proof can be applied to each succeeding term in the series.

Let us assume until further notice that  $\mathcal{A}$  is a weakly dense uniformly closed

\*-subalgebra of a von Neumann algebra  $\mathcal{B}$  with center  $\mathcal{Z}$ ; also let us assume that  $\mathcal{A}$  is a CCR algebra and that every continuous linear functional on  $\mathcal{A}$  has a (necessarily unique) extension to a  $\sigma$ -weakly continuous functional on  $\mathcal{B}$ . Note that  $\mathcal{B}$  is a type I algebra. Let  $\psi$  be an element in the unit sphere of  $\mathcal{B}^*$  and let  $\varphi$  be the restriction of  $\psi$  to  $\mathcal{A}$ . Then  $\varphi$  is a continuous linear function of  $\mathcal{A}$  into  $\mathcal{Z}$  and therefore  $\varphi$  has a unique extension to a  $\sigma$ -weakly continuous function of  $\mathcal{B}$  into  $\mathcal{Z}$  (Proposition 1). We again call this extension  $\varphi$ . We now construct an ideal of  $\mathcal{A}$  on whose weak closure  $\varphi$  will be a module homomorphism.

Now  $\mathcal{A}$  contains a nonzero closed two-sided ideal  $I$  so that there exists a nonzero  $A$  in  $I^+$  such that the range of  $\pi(A)$  has dimension not greater than one for every irreducible representation  $\pi$  of  $I$  [1] and [4]. This means that for each  $B$  and  $C$  in  $I$  the element  $ABA^2CA - ACA^2BA$  is in the intersection of the kernels of the irreducible representations of  $I$ . Since this intersection is 0, we find that  $ABA^2CA - ACA^2BA$  vanishes for every  $B$  and  $C$  in  $I$ ; and, therefore, working with the approximate identity for  $I$ , we find that  $ABA^2CA - ACA^2BA$  vanishes for every  $B$  and  $C$  in  $\mathcal{A}$ . By the strong continuity of multiplication on bounded spheres of  $\mathcal{B}$  and by the Kaplansky density theorem, we have that  $ABA^2CA = ACA^2BA$  for every  $B$  and  $C$  in  $\mathcal{B}$ . We claim that  $A$  is in the closed two-sided ideal  $I_a$  generated by abelian projections of  $\mathcal{B}$ . Indeed, for each  $\zeta$  in the spectrum  $Z$  of  $\mathcal{Z}$  let  $\pi_\zeta$  be an irreducible representation of the type I algebra  $\mathcal{B}$  on the Hilbert space  $H(\zeta)$  such that the kernel of  $\pi_\zeta$  is equal to  $[\zeta]$  [7, p. 118]. Then it is easily seen that the range projection of  $\pi_\zeta(A)$  has dimension no greater than one. There is an abelian projection  $E_\zeta$  in  $\mathcal{B}$  and a scalar  $\alpha_\zeta \geq 0$  such that  $\pi_\zeta(\alpha_\zeta E_\zeta) = \pi_\zeta(A)$  [5, §4]. Because  $\zeta \rightarrow \|\pi_\zeta(B)\|$  is continuous on  $Z$  for each fixed  $B$  in  $\mathcal{B}$  and because  $Z$  is extremely disconnected, a compactness argument allows us to show that given any  $\varepsilon > 0$  there is an abelian projection  $F$  in  $\mathcal{B}$  and a positive central element  $C$  of  $\mathcal{B}$  such that  $\|\pi_\zeta(CF) - \pi_\zeta(A)\| < \varepsilon$  for every  $\zeta \in Z$ . This means that  $\|CF - A\| < \varepsilon$  and therefore that  $A$  is in the closed two-sided ideal  $I_\varepsilon$  in  $\mathcal{B}$  generated by the abelian projections. Because every ideal in  $\mathcal{A}$  is a CCR algebra and because every nonzero CCR algebra contains a nonzero closed two-sided ideal with Hausdorff structure, we may find a nonzero closed two-sided ideal  $I$  in  $\mathcal{A}$  with Hausdorff structure space which is contained in  $I_a$ .

Let  $P$  be the central projection in  $\mathcal{B}$  such that  $\mathcal{B}P$  is the weak closure of  $I$ . We show that  $\varphi(B_0A_0) = B_0\varphi(A_0)$  for every  $B_0$  in  $\mathcal{Z}$  and  $A_0$  in  $\mathcal{B}P$ . Because  $I$  is weakly dense in  $\mathcal{B}P$  and because  $\varphi$  is weakly continuous, it is sufficient to assume that  $A_0$  is in  $I^+$ . We may also assume that  $B_0 \in \mathcal{Z}^+$ . Let  $B_0A_0$  have the spectral resolution  $B_0A_0 = \sum A_n E_n$ . There is a net  $\{P_i\}$  of projections in  $\mathcal{Z}P$  that converges strongly to  $P$  and there is a net  $\{B_i\}$  in  $I$  such that  $B_iP_i = B_0A_0P_i$  and  $\|B_i\| \leq \|B_0A_0\| + 1$  for every  $i$  (Lemma 4). Thus, given a normal functional  $g$  on  $\mathcal{Z}$  and  $\varepsilon > 0$ , there is a  $j$  such that  $i > j$  implies that

$$|g(\varphi(B_i) - B_0\varphi(A_0))| = |g(\psi(B_i - B_0A_0))| \leq (g(P - P_i)\|g\|)^{1/2} \|B_i - B_0A_0\| \leq \varepsilon.$$

However, the net  $\{B_i\}$  converges strongly to  $B_0A_0$ . This implies that

$$|g(\varphi(B_0A_0) - B_0\varphi(A_0))| \leq \varepsilon$$

because the composite function  $g \cdot \varphi$  is weakly continuous on  $\mathcal{B}$ . Since  $g$  and  $\varepsilon > 0$  are arbitrary, we have that  $\varphi(B_0A_0) = B_0\varphi(A_0)$ .

Continuing to work with the ideal  $I$ , we show that

$$\varphi(A_0(1-P)) = (1-P)\varphi(A_0(1-P))$$

for every  $A_0$  in  $\mathcal{A}$ . Let  $A_0 \in \mathcal{A}^+$  and let  $F$  be the spectral projection of  $A_0$  in  $\mathcal{B}$  which corresponds to the interval  $[\varepsilon, +\infty)$  where  $\varepsilon > 0$ . We claim that there is no nonzero central projection  $Q$  majorized by  $P$  such that  $FQ$  majorizes an infinite set  $\{F_n \mid 1 \leq n < \infty\}$  of mutually orthogonal abelian projections each with central support  $Q$ . We argue this point by contradiction. Indeed suppose such a  $Q$  exists. By Lemma 4 there is no loss of generality in the assumption that  $F_n \in IQ$  for every  $n = 1, 2, \dots$ . If  $\zeta$  is an element in the spectrum  $Z$  of  $\mathcal{Z}$  which does not contain  $Q$ , then  $I(\zeta) = I/[\zeta]$  is nonzero and hence isomorphic to the set  $C(H)$  of all completely continuous operators on the Hilbert space  $H$  [5, proof of Lemma 12]. We identify  $I(\zeta)$  with  $C(H)$ . Since  $I(\zeta)$  is a subalgebra of  $\mathcal{B}(\zeta)$ , there is an irreducible representation  $\pi$  of  $\mathcal{B}(\zeta)$  on a Hilbert space  $K$  which contains  $H$  as a subspace invariant under  $\pi(I(\zeta))$  such that  $\pi(A)x = Ax$  for every  $A \in I(\zeta)$  and  $x \in H$  [19, 4.7.20]. Let  $E'$  be the projection of  $K$  on  $H$ . Since  $I$  is an ideal in  $\mathcal{A}$  and since  $I(\zeta)$  is irreducible on  $H$ , the space  $H$  is invariant under  $\pi(\mathcal{A}(\zeta))$ . This means that  $\pi(\mathcal{A}(\zeta))E'$  is irreducible on  $H$ ; and therefore,  $\pi(\mathcal{A}(\zeta))E' = C(H)$  because  $\mathcal{A}$  is CCR. In particular, we have that  $\pi(A_0(\zeta))E' \in C(H)$ . On the other hand, no  $F_n(\zeta)$  is zero and hence  $\{\pi(F_n(\zeta))E'\}$  is an infinite sequence of nonzero projections on  $H$ . Note that these projections are mutually orthogonal. But for each  $n = 1, 2, \dots$  we have

$$\pi(F_n(\zeta))\pi(A_0(\zeta))\pi(F_n(\zeta))E' = \pi((F_nA_0F_n)(\zeta))E' \geq \varepsilon\pi(F_n(\zeta))E'.$$

This contradicts the fact that  $\pi(A_0(\zeta))E' \in C(H)$ . Hence no such  $Q$  exists. So we may find a net  $\{P_i\}$  of projections in  $\mathcal{Z}P$  which converge strongly to  $P$  such that  $FP_i \in I_a$  for every  $P_i$ . There is no loss of generality in the assumption that there is a  $B_i \in I$  such that  $\|B_i\| \leq \|A_0F\|$  and  $B_iP_i = A_0FP_i$  (Lemma 4). For unit vectors  $x$  and  $y$  in the range of  $P_i$  we have that

$$|(\varphi(A_0 - B_i)x, y)| = |(\psi(A_0 - B_i)x, y)| \leq \|\psi((A_0F - B_i)P_i)\| + \|\psi(A_0(1-F))\| \leq \varepsilon.$$

Therefore, we have that  $\|P_i\varphi(A_0 - B_i)\| \leq \varepsilon$  and

$$(1) \quad \|(1-P_i)\varphi(A_0 - B_i) - \varphi(A_0 - B_i)\| \leq \varepsilon.$$

Also we obtain that

$$(2) \quad (1-P_i)\varphi(A_0 - B_i) = (P-P_i)\varphi((A_0 - B_i)(P-P_i)) + (1-P_i)\varphi((A_0 - B_i)(1-P)).$$

However, the nets  $\{P-P_i\}$  and  $\{B_i\}$  converge strongly to 0 and  $A_0FP$  respectively.

The latter is true because

$$\begin{aligned}\|(A_0FP - B_i)x\| &\leq \|(A_0FP - B_i)(1 - P_i)x\| \\ &\leq \|(A_0FP - B_i)\| \|(P - P_i)x\|\end{aligned}$$

for every vector  $x$  in the Hilbert space. Since  $\varphi$  is  $\sigma$ -weakly continuous and  $\{B_i\}$  is bounded, we have by (2) that

$$(3) \quad \begin{aligned}\lim (((1 - P_i)\varphi(A_0 - B_i) - \varphi(A_0 - B_i))x, y) \\ = (((1 - P)\varphi(A_0(1 - P)) - \varphi(A_0 - A_0FP))x, y)\end{aligned}$$

for every  $x$  and  $y$  in the Hilbert space. But this implies that

$$\begin{aligned}(4) \quad &|(((1 - P)\varphi(A_0(1 - P)) - \varphi(A_0(1 - P)))x, y)| \\ &\leq |(((1 - P)\varphi(A_0(1 - P)) - \varphi(A_0(1 - PF)))x, y)| \\ &\quad + |(\varphi(A_0P(1 - F))x, y)| \\ &\leq \limsup (\|(1 - P_i)\varphi(A_0 - B_i) - \varphi(A_0 - B_i)\| \|x\| \|y\|) \\ &\quad + \|A_0(1 - F)\| \|x\| \|y\| \leq 2\varepsilon \|x\| \|y\|\end{aligned}$$

by (1) and (3). Since  $x$ ,  $y$  and  $\varepsilon > 0$  are arbitrary, we obtain  $\varphi(A_0(1 - P)) = (1 - P)\varphi(A_0(1 - P))$  from (4).

We have completed the first step of the proof and we now construct the required composition series. Let  $\mathcal{A}$  be a CCR algebra, let  $\mathcal{B}$  be the enveloping von Neumann algebra of  $\mathcal{A}$ , and let  $\varphi$  be the unique extension to a  $\sigma$ -weakly continuous linear function of  $\mathcal{B}$  into its center  $\mathcal{Z}$  of the restriction to  $\mathcal{A}$  of the function  $\psi$  in the unit sphere of  $\mathcal{B}^\sim$ . Let  $\{P_i \mid 0 \leq i \leq i_0\}$  be a set of projections in  $\mathcal{Z}$  indexed by the ordinals such that

- (i)  $P_0 = 0$ ;
- (ii)  $P_i < P_{i+1}$  ( $i < i_0$ );
- (iii)  $\text{lub } \{P_i \mid i < j\} = P_j$  if  $j$  is a limit ordinal with  $j \leq i_0$ ; and
- (iv)  $\mathcal{B}(P_{i+1} - P_i)$  is the weak closure of a closed two-sided ideal in  $\mathcal{A}(1 - P_i)$  which is generated by elements in  $\mathcal{A}(1 - P_i)$  which are contained in the ideal  $I_a(1 - P_i)$  and which has Hausdorff structure space. Here  $I_a$  is the ideal of  $\mathcal{B}$  generated by the abelian projections of  $\mathcal{B}$ .

Let  $I_i$  be the closed two-sided ideal given by  $I_i = \{A \in \mathcal{A} \mid A(1 - P_i) = 0\}$ . We claim that the weak closure of  $I_i$  is  $\mathcal{B}P_i$ . Otherwise, let  $i$  be the smallest ordinal such that  $I_i$  is not weakly dense in  $\mathcal{B}P_i$ . By (iv) we conclude that  $i > 1$ . If  $i$  is a limit ordinal, then the weak closure of  $I_i$  contains the weak closure of  $\bigcup \{I_k \mid k < i\}$  which is equal to  $\mathcal{B}P_i$  and hence the weak closure of  $I_i$  is  $\mathcal{B}P_i$ . If  $i$  is the successor of the ordinal  $j$ , then we show that  $A \in \mathcal{B}P_i^+$  is in the weak closure of  $I_i$ . Let  $\{A_n\}$  be a net of hermitian elements in the sphere of  $\mathcal{A}(1 - P_j)$  of radius  $\|A\|$  about 0 such that  $A_nP_i = A_n$  for every  $n$  and  $\lim A_n = A(1 - P_j)$  strongly ((iv) and [13]). Let  $\{B_n\}$  be a bounded net of hermitian elements in  $\mathcal{A}$  with  $B_n(1 - P_j) = A_n$  for every  $n$ . Note that every  $B_n$  is in  $I_i$ . Passing to a subnet if necessary, we may assume

that  $\{B_n\}$  converges weakly to an element  $B$  in  $\mathcal{B}P_i$ . We have that  $(A-B)$  is the weak limit of elements in  $I_j$  since  $(A-B)(1-P_j)=0$ . Thus  $A$  is the weak limit of a net in  $I_i$ . This is a contradiction. So the weak closure of  $I_i$  is  $\mathcal{B}P_i$  for all  $i$ . By Lemma 2 every continuous linear functional on  $\mathcal{A}(1-P_i)$  has a unique extension to a  $\sigma$ -weakly continuous functional on  $\mathcal{B}(1-P_i)$ . Because  $\mathcal{A}(1-P_i)$  is isomorphic to  $\mathcal{A}/I_i$ , the algebra  $\mathcal{A}(1-P_i)$  is CCR. We may therefore assume that  $P_i=1$  by the principle of transfinite induction.

Now we show that  $\varphi(B_0A_0P_i)=B_0\varphi(A_0P_i)$  and  $\varphi(A_0(1-P_i))=(1-P_i)\varphi(A_0(1-P_i))$  for every  $A_0 \in \mathcal{B}$ ,  $B_0 \in \mathcal{L}$ , and every  $i$ . On the contrary, let  $i$  be the smallest ordinal for which these inequalities fail for some  $A_0$  or  $B_0$ . By relation (iv) and the weak continuity of  $\varphi$  it is obvious that  $i$  must be the successor of an ordinal  $j$ . Let  $P=1-P_j$  and let  $P \cdot \varphi$  be the  $\sigma$ -weakly continuous linear function of  $\mathcal{B}P$  into  $\mathcal{L}P$  given by  $P \cdot \varphi(B)=P\varphi(B)$  for every  $B \in \mathcal{B}P$ . The restriction of  $P \cdot \varphi$  to  $\mathcal{A}P$  is equal to the restriction of  $\psi$  to  $\mathcal{A}P$  because  $\psi(AP)=P\psi(A)=P\varphi(A)=P\varphi(AP)+P\varphi(AP_j)=(P \cdot \varphi)(AP)$  for every  $A$  in  $\mathcal{A}$ . We notice that  $\psi$  restricted to  $\mathcal{B}P$  is a functional in  $\mathcal{B}P^\sim$ . By the analysis contained in the first part of the proof we have that  $P \cdot \varphi(BAP_i)=B\varphi(AP_i)$  and  $P \cdot \varphi(A(1-P_i))=(1-P_i)\varphi(A(1-P_i))$  for every  $A \in \mathcal{B}P$  and  $B \in \mathcal{L}P$ . Then we have that

$$\begin{aligned}\varphi(B_0A_0P_i) &= \varphi(B_0A_0P_j) + \varphi(B_0A_0(P_i-P_j)) \\ &= B_0\varphi(A_0P_j) + P\varphi(B_0A_0(P_i-P_j)) \\ &= B_0\varphi(A_0P_j) + B_0P\varphi(A_0(P_i-P_j)) \\ &= B_0\varphi(A_0P_j) + B_0\varphi(A_0(P_i-P_j)) \\ &= B_0\varphi(A_0P_i)\end{aligned}$$

and

$$\begin{aligned}\varphi(A_0(1-P_i)) &= \varphi(A_0(1-P_i)(1-P_j)) \\ &= P \cdot \varphi(A_0(1-P_i)P) = (1-P_i)\varphi(A_0(1-P_i)).\end{aligned}$$

This contradicts the choice of  $i$ . So we have that  $\varphi(B_0A_0P_i)=B_0\varphi(A_0P_i)$  for all  $A_0 \in \mathcal{B}$ ,  $B_0 \in \mathcal{L}$ , and all  $i$ . Because  $P_{i_0}=1$ , we have that  $\varphi(B_0A_0)=B_0\varphi(A_0)$  for all  $A_0 \in \mathcal{B}$  and  $B_0$  in  $\mathcal{L}$ . Q.E.D.

**Proof of (a2) implies (a3).** By hypothesis the restriction map of elements of  $\mathcal{B}^\sim$  to  $\mathcal{A}$  is a continuous map of  $\mathcal{B}^\sim$  with the  $\sigma_w(\mathcal{B}^\sim, \mathcal{B})$ -topology onto  $\mathcal{A}^\sim$  with the  $\sigma_w(\mathcal{A}^\sim, \mathcal{A})$ -topology. This map takes  $\mathcal{S}(\mathcal{B}^\sim)$  onto  $\mathcal{S}(\mathcal{A}^\sim)$ . Since  $\mathcal{S}(\mathcal{B}^\sim)$  is compact, so is  $\mathcal{S}(\mathcal{A}^\sim)$ . Q.E.D.

The following lemma is used in the next step of the proof.

**LEMMA 7.** *If  $\mathcal{A}$  is a GCR algebra that is not a CCR algebra, then there is a net  $\{\varphi_n\}$  of extreme points of  $\mathcal{S}(\mathcal{B}^\sim)$  that converges in the  $\sigma_u(\mathcal{B}^\sim, \mathcal{B})$ -topology to a functional  $\varphi \in \mathcal{S}(\mathcal{B}^\sim)$  whose restriction to  $\mathcal{A}$  is not in  $\mathcal{A}^\sim$ .*

**Proof.** Because  $\mathcal{A}$  is a GCR algebra which is not CCR, there is an  $A_0$  in  $\mathcal{A}^+$  and an irreducible representation  $\pi$  of  $\mathcal{A}$  on a Hilbert space  $H$  such that  $\pi(A_0)$  is not in the set of completely continuous operators  $C(H)$  of  $H$ . Setting  $H_0$  equal to

the canonical Hilbert space of  $\mathcal{B}$ , we may find a minimal projection  $E'$  in the commutant of  $\mathcal{B}$  on  $H_0$  such that  $E'(H_0) = H$ , the algebra  $\mathcal{B}E'$  is the algebra of all bounded linear operators on  $H$ , and  $\pi(A)x = Ax$  for every  $A$  in  $\mathcal{A}$  (cf. remarks, beginning of §2). If  $P$  is the central support of  $E'$ , then  $P$  is a minimal projection of the center of  $\mathcal{B}$  and  $\Phi(B) = BE'$  is a  $\sigma$ -weakly continuous isomorphism of  $\mathcal{B}P$  onto  $\mathcal{B}E'$  [2, I, §4, Theorem 2, Corollary 1]. There is a state  $f$  of  $\mathcal{B}E'$  such that  $f(A_0E') \neq 0$  and  $f(A) = 0$  for all  $A \in C(H)$ . There is a net  $\{x_n\}$  of unit vectors of  $H$  such that  $\{w_{x_n}\}$  converges to  $f$  in the  $w^*$ -topology of  $\mathcal{B}^*$  [5, Theorem 2]. Then let  $\varphi_n(A) = Pw_{x_n}(A)$  for all  $A \in \mathcal{B}$  and all  $n$  and let  $\varphi(A) = Pf(A)$ . It is easy to see that each module homomorphism  $\varphi_n$  is  $\sigma$ -weakly continuous and  $\mathcal{L}$ -irreducible, and takes 1 into  $P$ , therefore, each  $\varphi_n$  is an extreme point of  $\mathcal{S}(\mathcal{B}_\sim)$  and  $\{\varphi_n\}$  converges to  $\varphi$  in the  $\sigma_u(\mathcal{B}_\sim, \mathcal{B})$ -topology. There is, however, no  $\psi$  in  $\mathcal{B}_\sim$  such that  $\psi(A) = \varphi(A)$  for every  $A$  in  $\mathcal{A}$ . Otherwise, we have  $P\psi(A) = P\varphi(A) = \varphi(A) = \psi(A)$  for every  $A$  in  $\mathcal{A}$  and thus  $P\psi = \psi$ . Because  $\mathcal{A}$  is GCR, there is an ideal  $I$  in  $\mathcal{A}$  such that  $IE' = C(H)$ . This means that  $\psi(I) = (0)$  and that  $Pf(A_0) = \psi(A_0) = \psi(A_0P) = 0$  since  $C(H)$  is  $\sigma$ -weakly dense in  $\mathcal{B}E'$  and since  $\Phi^{-1}$  is  $\sigma$ -weakly continuous [2, I, §4, Theorem 2, Corollary 1]. This is a contradiction. Therefore, there is no  $\psi \in \mathcal{B}_\sim$  such that  $\psi(A) = \varphi(A)$  for every  $A$  in  $\mathcal{A}$ . Q.E.D.

**Proof of (a3) implies (a1).** We first assume that  $\mathcal{A}$  is not a GCR algebra and obtain a contradiction. If  $\mathcal{A}$  is not GCR, then there is a nonzero projection  $P$  in the center of  $\mathcal{B}$  such that  $\mathcal{B}P$  is a continuous von Neumann algebra. It is clear that  $P\mathcal{S}(\mathcal{A}_\sim) = \{\varphi \in \mathcal{S}(\mathcal{A}_\sim) \mid (1-P)\varphi = 0\}$  is a nonzero compact convex subset of  $\mathcal{S}(\mathcal{A}_\sim)$  in the  $\sigma_w(\mathcal{A}_\sim, \mathcal{A})$ -topology. Let  $\varphi$  be a nonzero extreme point of  $P\mathcal{S}(\mathcal{A}_\sim)$ . Considering  $\varphi$  a functional in  $(P\mathcal{B})_\sim$ , we see that  $\varphi$  is, by the result of E. Størmer [24] cited in the preliminaries, a  $\mathcal{L}$ -irreducible functional in  $(\mathcal{B}P)_\sim$ . This contradicts the conclusion of Proposition 5. Therefore,  $\mathcal{A}$  must be a GCR algebra. Now we may apply Lemma 7 to prove that  $\mathcal{A}$  is CCR. Q.E.D.

**Proof of (a3) implies (a4).** We may assume that  $\mathcal{A}$  is a CCR algebra because (a3) is equivalent to (a1). Let  $\{B_n\}$  be a net of positive elements in the unit sphere of the center  $\mathcal{Z}$  of  $\mathcal{B}$  and let  $\{F_n\}$  be a net of abelian projections in  $\mathcal{B}$  such that the net  $\{B_n\tau_{F_n}(A)\}$  (cf. remarks preceding Proposition 5) converges uniformly to  $\varphi(A)$  for each  $A$  in  $\mathcal{A}$ . By hypothesis the function  $\varphi$  has a unique extension to a positive function in the unit sphere of  $\mathcal{B}_\sim$ . We call this extension  $\varphi$ . We show that  $\varphi$  is a  $\mathcal{L}$ -irreducible functional. Let  $A_0$  be a positive element in the ideal  $I_a$  generated by the abelian projections of the type I algebra  $\mathcal{B}$  such that  $A_0$  has the spectral resolution  $A_0 = \sum A_i E_i$  with  $\{\sum \{A_i \mid 1 \leq i \leq m\}\}$  bounded and  $\varphi(A) = \sum A_i \tau_{E_i}(A)$  for every  $A$  in  $\mathcal{B}$ . It is sufficient to show that  $A_2 = 0$ . Indeed, we then have that  $\varphi(A) = A_1 \tau_{E_1}(A)$  and the proof of Theorem 9 [10] is applicable. We may find a set  $\{P_i \mid 0 \leq i \leq i_0\}$  of projections in  $\mathcal{Z}$  indexed by the ordinals such that properties (i)–(iv) of the proof of (a1) implies (a2) are satisfied. Let  $i$  be the smallest ordinal such that  $A_2 P_i \neq 0$ . Then it is obvious that  $i$  must be the successor of an ordinal  $j$ . Now there is a nonzero central projection  $P$  majorized by  $P_i - P_j$  such that

(1)  $A_2 P \geq \varepsilon P$  for some  $\varepsilon > 0$  and

(2) there are elements  $U_i$  in  $\mathcal{A}$  such that  $U_i^* U_i P = E_i P$  and  $U_i U_i^* P \leq E_i P$  for  $i = 1, 2, \dots$  (Lemma 4 and [2, III, §3, Lemma 1]).

Let  $\zeta$  be an element of the spectrum of  $\mathcal{Z}$  such that  $P^\wedge(\zeta) = 1$  and such that  $\{\sum \{A_n^\wedge(\zeta) \mid 1 \leq n \leq m\}\}$  converges to  $(\sum A_n)^\wedge(\zeta)$ . Since  $\{\sum \{A_n P \mid 1 \leq n \leq m\}\}$  converges strongly to  $\sum A_n P$ , such a choice is possible. There is an irreducible representation  $\pi$  of  $\mathcal{B}$  on a Hilbert space  $H(\pi)$  such that the kernel of  $\pi$  is  $[\zeta]$  [7, p. 118]. Since every abelian projection of  $\mathcal{B}$  whose central support is not in  $\zeta$  is carried into a 1-dimensional projection on  $H(\pi)$ , there is a net  $\{y_n\}$  of vectors in  $H(\pi)$  such that  $\tau_{F_n}(A)^\wedge(\zeta) = w_{y_n}(\pi(A))$  for every  $A$  in  $\mathcal{B}$ . There is also a sequence of vectors  $\{x_n\}$  in the unit sphere of  $H(\pi)$  such that  $\tau_{E_n}(A)^\wedge(\zeta) = w_{x_n}(\pi(A))$  for every  $A$  in  $\mathcal{B}$ . Now if  $E_0$  is the sum of the sequence  $\{\pi(E_n)\}$  of orthogonal projections on  $H(\pi)$ , we must have that the set  $C(E_0(H(\pi)))$  of completely continuous operators on  $E_0(H(\pi))$  is contained in the image of  $\mathcal{A}$ . The existence of the matrix units  $\pi(U_j^* U_i)$  ( $i, j = 1, 2, \dots$ ) for an orthonormal basis of  $E_0(H(\pi))$  in the image of  $\mathcal{A}$  insures this. But we have that

$$\begin{aligned} \lim B_n^\wedge(\zeta) w_{E_0 y_n}(\pi(A)) &= \lim B_n^\wedge(\zeta) w_{y_n}(\pi(A)) \\ &= \lim (B_n \tau_{F_n}(A))^\wedge(\zeta) = \left( \sum A_n \tau_{E_n}(A) \right)^\wedge(\zeta) \\ &= \sum A_n^\wedge(\zeta) \tau_{E_n}(A)^\wedge(\zeta) = \sum A_n^\wedge(\zeta) w_{x_n}(\pi(A)) \end{aligned}$$

for every  $A$  in  $\mathcal{A}$  such that  $\pi(A) \in C(E_0(H(\pi)))$ . There is a vector  $x$  in  $E_0(H)$  such that  $w_x(A) = \sum A_n^\wedge(\zeta) w_{x_n}(A)$  for every  $A \in C(E_0(H(\pi)))$  since every  $w^*$ -limit of vector states on  $C(E_0(H(\pi)))$  is proportional to a vector state [5, Theorem 2]. This means that  $A_1^\wedge(\zeta) w_{x_1}$  is proportional to  $A_2^\wedge(\zeta) w_{x_2}$  on  $C(E_0(H(\pi)))$ . Since  $A_2^\wedge(\zeta) \neq 0$ , we have that  $E_2(\zeta) \neq 0$  by the definition of the spectral resolution for  $A_0$  and hence  $x_2 \neq 0$ . This is impossible since  $A_1 \geq A_2$ ,  $E_1 \succ E_2$ , and  $E_1 E_2 = 0$ . Therefore, we must conclude that  $A_2 = 0$ . Q.E.D.

The last step of the proof requires this lemma.

**LEMMA 8.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra and let  $\mathcal{B}$  be the enveloping von Neumann algebra of  $\mathcal{A}$ . If  $\mathcal{A}$  is not a GCR algebra, then there exists a net  $\{\varphi_n\}$  of  $\mathcal{Z}$ -extreme points in  $\mathcal{S}(\mathcal{B}_\sim)$  that converge in the  $\sigma_u(\mathcal{B}_\sim, \mathcal{A})$ -topology to a functional  $\varphi \in \mathcal{S}(\mathcal{B}_\sim)$  which is not  $\mathcal{Z}$ -irreducible.*

**Proof.** Because  $\mathcal{A}$  is not a GCR algebra, there is a nonzero irreducible representation  $\pi$  of  $\mathcal{A}$  on the Hilbert space  $H$  such that the intersection of  $\pi(\mathcal{A})$  with the set  $C(H)$  of completely continuous operators on  $H$  is  $(0)$  [20]. There is a minimal projection  $E'$  in the commutant of  $\mathcal{B}$  on its canonical Hilbert space  $H_0$  such that  $H = E'(H_0)$  and  $\pi(A) = A E'$  for every  $A \in \mathcal{A}$ . Because  $E'$  is minimal, the central support  $P$  of  $E'$  is a minimal projection in the center of  $\mathcal{B}$ . Let  $x_1$  and  $x_2$  be orthogonal unit vectors in  $H$ . The functionals  $\psi_i(A) = w_{x_i}(A)P$  ( $i = 1, 2$ ;  $A \in \mathcal{B}$ )

are in  $\mathcal{S}(\mathcal{B}_\sim)$ ; and hence,  $\varphi = (\psi_1 + \psi_2)/2$  is in  $\mathcal{S}(\mathcal{B}_\sim)$ . It is clear that  $\varphi$  is not  $\mathcal{Z}$ -irreducible. Using Glimm's theorem [5, Theorem 2] again, we can find a net  $\{y_n\}$  of unit vectors in  $H$  such that  $\{w_{y_n}(A)\}$  converges to  $(w_{x_1}(A) + w_{x_2}(A))/2$  for every  $A \in \mathcal{A}$ . Setting  $\varphi_n(A) = w_{y_n}(A)P$ , we obtain a net  $\{\varphi_n\}$  of  $\mathcal{Z}$ -extreme points in  $\mathcal{S}(\mathcal{B}_\sim)$  which converges to  $\varphi$  in the  $\sigma_u(\mathcal{B}_\sim, \mathcal{A})$ -topology. Q.E.D.

**Proof of (a5) implies (a1).** From Lemma 8 we see that  $\mathcal{A}$  must be a GCR algebra. Now by Lemma 7, if  $\mathcal{A}$  is not CCR, then there is a net  $\{\varphi_n\}$  of  $\mathcal{Z}$ -extreme points of  $\mathcal{S}(\mathcal{B}_\sim)$  which converge in the  $\sigma_u(\mathcal{B}_\sim, \mathcal{B})$ -topology to an element  $\varphi \in \mathcal{S}(\mathcal{B}_\sim)$  such that the restriction of  $\varphi$  to  $\mathcal{A}$  is not in  $\mathcal{A}^\sim$ . Hence  $\mathcal{A}$  is a CCR algebra. Q.E.D.

Let  $\mathcal{B}$  be a von Neumann algebra with center  $\mathcal{Z}$  and let  $\mathcal{A}$  be a subset of  $\mathcal{B}$ . Let  $\langle \mathcal{A} \rangle$  denote the set of all elements of  $\mathcal{B}$  of the form  $\sum A_i R_i$ , where  $\{A_i\}$  is a bounded subset of  $\mathcal{A}$  and  $\{R_i\}$  is a set of mutually orthogonal projections in  $\mathcal{Z}$  of sum 1. Notice that  $\langle \mathcal{A} \rangle$  is closed if  $\mathcal{A}$  is a closed  $\mathcal{Z}$ -submodule of  $\mathcal{B}$ . Indeed, let  $\{A_n\}$  be a sequence in  $\langle \mathcal{A} \rangle$  which converges uniformly to  $\mathcal{A}$ . For every  $\zeta$  in the complement of a nowhere dense set  $N_n$  of the spectrum  $Z$  of  $\mathcal{Z}$  we have that  $A_n(\zeta) \in \mathcal{A}(\zeta)$ . Thus  $A(\zeta) \in \mathcal{A}(\zeta)$  for every  $\zeta \notin \bigcup N_n$  which is nowhere dense. So there is a net  $\{R_i\}$  of mutually orthogonal projections in  $\mathcal{Z}$  of sum 1 such that  $A(\zeta) \in \mathcal{A}(\zeta)$  whenever  $R_i \wedge(\zeta) = 1$  for at least one  $R_i$ . Thus  $AR_i$  is the uniform limit of elements of  $\mathcal{A}$  since  $\mathcal{A}$  is a closed  $\mathcal{Z}$ -submodule. This proves  $\langle \mathcal{A} \rangle$  is closed. Also notice that closure  $\langle \mathcal{A} \rangle$  is a submodule of  $\mathcal{B}$  provided  $\mathcal{A}$  is a linear subspace of  $\mathcal{B}$ . Indeed, let  $A \in \mathcal{A}$ , let  $B \in \mathcal{Z}$ , and let  $\varepsilon > 0$ . There is a set  $\{P_i\}$  of mutually orthogonal projections  $\{P_i\}$  of sum 1 in  $\mathcal{Z}$  such that for each  $P_i$  there is a scalar  $\alpha_i$  with  $\|(\alpha_i - B)P_i\| < \varepsilon$ . Then  $\sum \alpha_i A P_i \in \langle \mathcal{A} \rangle$  and  $\|BA - \sum \alpha_i A P_i\| < \varepsilon \|A\|$ . So  $BA \in \text{closure } \langle \mathcal{A} \rangle$ . Then it is clear that closure  $\langle \mathcal{A} \rangle$  is a  $\mathcal{Z}$ -module.

**COROLLARY.** *Let  $\mathcal{A}$  be a CCR algebra; let  $\mathcal{B}$  be the enveloping von Neumann algebra of  $\mathcal{A}$ ; and let  $I_a$  be the ideal generated by the abelian projections of  $\mathcal{B}$ ; then  $\langle I_a \rangle = \text{closure } \langle \mathcal{A} \rangle$ .*

**Proof.** Let  $Z$  be the spectrum of the center  $\mathcal{Z}$  of  $\mathcal{B}$ . We first show that for every  $A \in \mathcal{B}$ , the function

$$\zeta \rightarrow p_\zeta(A) = \text{glb } \{\|A + B + C\| \mid B \in \langle I_a \rangle, C \in [\zeta]\}$$

is continuous on  $Z$ . Given  $\varepsilon \geq 0$  it is sufficient to show that set  $X = \{\zeta \in Z \mid p_\zeta(A) < \varepsilon\}$  is open and the set  $Y = \{\zeta \in Z \mid p_\zeta(A) \leq \varepsilon\}$  is closed. In the former we may assume  $\varepsilon > 0$ ; then the continuity of  $\zeta \rightarrow \|B(\zeta)\|$  on  $Z$  for fixed  $B \in \mathcal{B}$  permits us to conclude that  $X$  is open. In the latter let  $\{\zeta_n\}$  be a net in  $Y$  which converges to  $\zeta_0$ . Let  $\{Q_i\}$  be a maximal set of nonzero orthogonal projections in  $\mathcal{Z}$  such that for each  $i$  there is at least one  $\zeta_{n_i}$  with  $Q_i \wedge(\zeta_{n_i}) = 1$ . If  $\sum Q_i = Q_0$ , then we have that  $Q_0 \wedge(\zeta_0) = 1$ ; otherwise, we would find that  $\{Q_i\}$  is not maximal. Given a natural number  $m$  there is an  $A_i \in \langle I_a \rangle$  such that

$$\|(A + A_i)(\zeta_{n_i})\| < \varepsilon + m^{-1}.$$

This means there is a nonzero central projection  $Q'_{n_i}$  majorized by  $Q_{n_i}$  with  $Q'_{n_i}(\zeta_{n_i})=1$  and

$$\|(A - A_i)Q'_{n_i}\| < \varepsilon + m^{-1}.$$

So  $\sum A_i Q'_{n_i} \in \langle I_a \rangle$  and thus

$$\begin{aligned} \|(A - \sum A_i Q'_{n_i})(\zeta_0)\| &\leq \limsup \|(A - \sum A_i Q'_{n_i})(\zeta_{n_i})\| \\ &\leq \varepsilon + m^{-1}. \end{aligned}$$

Since  $m$  is arbitrary, we have  $\zeta_0 \in Y$ . Thus  $Y$  is closed. This proves that  $\zeta \rightarrow p_\zeta(A)$  is continuous on  $Z$  for every  $A \in \mathcal{B}$ . Let  $p(A)$  be the unique element in  $\mathcal{Z}^+$  such that  $p(A)\zeta = p_\zeta(A)$  for every  $\zeta \in Z$ . We then have that  $p(A+B) \leq p(A) + p(B)$  and  $p(CB) = |C|p(B)$  for every  $A, B \in \mathcal{B}$  and  $C \in \mathcal{Z}$ . Indeed, we have that  $p_\zeta(A+B) \leq p_\zeta(A) + p_\zeta(B)$  and  $p_\zeta(CB) = |C(\zeta)|p_\zeta(B)$  for every  $\zeta \in Z$ .

We have that every  $A$  in  $I_a$  is contained in closure  $\langle \mathcal{A} \rangle$ . Let  $\{P_i \mid 1 \leq i \leq i_0\}$  be the projections satisfying (i)–(iv) of (a1) implies (a2). Then it is enough to prove that  $A(P_{i+1} - P_i) \in \text{closure } \langle \mathcal{A} \rangle$  for every  $0 \leq i \leq i_0$ . But  $B(P_{i+1} - P_i)$  is the weak closure of an ideal in  $\mathcal{A}(1 - P_i)$  which is contained in  $I_a(1 - P_i)$  and thus by Lemma 4 there is a net  $\{Q_j\}$  of projections in  $\mathcal{Z}(P_{i+1} - P_i)$  such that  $\lim Q_j = P_{i+1} - P_i$  strongly and  $AQ_j \in \mathcal{A}Q_j$ . But if  $\{R_k\}$  is a maximal net of mutually orthogonal nonzero projections in  $\mathcal{Z}(P_{i+1} - P_i)$  with  $AR_k \in \mathcal{A}R_k$ , then it is clear that  $\sum R_k = P_{i+1} - P_i$ . Thus  $A(P_{i+1} - P_i) \in \text{closure } \langle \mathcal{A} \rangle$  and  $I_a \subset \text{closure } \langle \mathcal{A} \rangle$ .

Now suppose there is an  $A_0 \in \mathcal{A} - \langle I_a \rangle$ . We have that  $p(A_0) \neq 0$ . Indeed, if  $p(A_0) = 0$ , then given  $\varepsilon > 0$  there is a set  $R_1, \dots, R_n$  of orthogonal central projections and a set  $A_1, \dots, A_n$  of elements of  $\langle I_a \rangle$  such that  $\|\sum R_i A_i - A_0\| < \varepsilon$  by a compactness argument and so  $A_0 \in \text{closure } \langle I_a \rangle = \langle I_a \rangle$ . Now let  $H(\mathcal{Z})$  be the hermitian elements in  $\mathcal{Z}$  and let  $M$  be the  $H(\mathcal{Z})$ -submodule of  $\mathcal{B}$  generated by  $\langle I_a \rangle$  and  $A_0$ . The relation  $\varphi(B + CA_0) = Cp(A_0)$  for  $B \in \langle I_a \rangle, C \in H(\mathcal{Z})$  defines an  $H(\mathcal{Z})$ -module homomorphism of  $M$  into  $H(\mathcal{Z})$  with  $\varphi(B) \leq p(B)$  for all  $B \in M$ . There is an  $H(\mathcal{Z})$ -module homomorphism  $\psi$  of  $\mathcal{B}$  into  $H(\mathcal{Z})$  such that  $\psi(B) \leq p(B)$  for all  $B \in \mathcal{B}$  and  $\psi(B) = \varphi(B)$  for all  $B \in M$  [18], [25], [28]. Then  $\theta(B) = \psi(B) - i\psi(iB)$  defines a  $\mathcal{Z}$ -module homomorphism of  $\mathcal{B}$  into  $\mathcal{Z}$  such that  $|\theta(B)| \leq p(B)$  for all  $B \in \mathcal{B}$  and  $\theta(A_0) \neq 0$ . But now it is clear that  $\theta \in \mathcal{B}^\sim$ . By Theorem 6 there is a  $\theta' \in \mathcal{B}^\sim$  such that  $\theta'(A) = \theta(A)$  for every  $A \in \mathcal{A}$ . Let  $i$  be the smallest ordinal such that  $P_i \theta'(A) \neq 0$  for some  $A$  in  $\mathcal{A}$ . Then  $i$  must be the successor of an ordinal  $j$  due to (iii). By (iv) and the Kaplansky density theorem [13] there is a bounded net  $\{A_n\}$  in  $\mathcal{A}$  such that  $A_n(1 - P_j) \in I_a$  and  $\lim A_n(1 - P_j) = A(P_i - P_j)$ . Therefore, we have

$$P_i \theta'(A) = \lim \theta'((P_i - P_j)A_n) = \lim (P_i - P_j)\theta(A_n) = 0$$

since  $\theta$  vanishes on  $I_a$ . Thus we have that  $\theta' = 0$ . This contradicts the fact that  $A_0 \in \mathcal{A}$  and  $\theta'(A_0) \neq 0$ . Thus, we have proved that  $\langle I_a \rangle \supset \mathcal{A}$ . Hence we have that  $\langle I_a \rangle = \text{closure } \langle \mathcal{A} \rangle$ . Q.E.D.

From Theorem 6 we obtain the following characterization of a GCR algebra.

**THEOREM 9.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra. The following are equivalent:*

- (b1)  $\mathcal{A}$  is a GCR algebra; and
- (b2) the set of  $\mathcal{L}$ -irreducible functionals in  $\mathcal{S}(\mathcal{A}^\sim)$  is closed in the  $\sigma_u(\mathcal{A}^\sim, \mathcal{A})$ -topology on  $\mathcal{S}(\mathcal{A}^\sim)$ .

**Proof.** The proof of (b1) implies (b2) is virtually the same as the proof of (a3) implies (a4) while the proof of (b2) implies (b1) follows from Lemma 8. Q.E.D.

Let  $\mathcal{A}$  be a  $C^*$ -algebra and let  $I$  be a closed two-sided ideal in  $\mathcal{A}^\sim$ . A functional  $\varphi \in \mathcal{S}(\mathcal{A}^\sim)$  is said to be  $\mathcal{L}$ -irreducible on the ideal  $I$  if there is a  $\mathcal{L}$ -irreducible functional  $\psi \in \mathcal{S}(\mathcal{A}^\sim)$  such that  $\varphi(A) = \psi(A)$  for every  $A \in I$ .

A characterization of an NGCR algebra is now obtainable.

**THEOREM 10.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra. The following are equivalent:*

- (1)  $\mathcal{A}$  is an NGCR algebra;
- (2)  $\mathcal{A}$  contains no nonzero closed two-sided ideal  $I$  such that every net  $\{\varphi_n\}$  in  $\mathcal{S}(\mathcal{A}^\sim)$  has a limit point in  $\mathcal{S}(\mathcal{A}^\sim)$  in the  $\sigma_w(\mathcal{A}^\sim, I)$ -topology; and
- (3)  $\mathcal{A}$  contains no nonzero closed two-sided ideal such that the set of  $\mathcal{L}$ -irreducible functionals on  $I$  in  $\mathcal{S}(\mathcal{A}^\sim)$  is closed in the  $\sigma_u(\mathcal{A}^\sim, I)$ -topology.

**Proof.** Let  $I$  be a nonzero closed two-sided ideal in  $\mathcal{A}$  and let  $P$  be the projection in the center of the enveloping von Neumann algebra  $\mathcal{B}$  of  $\mathcal{A}$  such that  $\mathcal{B}P$  is the weak closure of  $I$ . We claim that  $\mathcal{B}P$  may be identified with the enveloping von Neumann algebra of  $I$ . Indeed, there is a projection  $Q$  in the center of the enveloping von Neumann algebra  $\mathcal{C}$  of  $I$  such that  $\mathcal{C}Q$  is isomorphic to  $\mathcal{B}P$  under a map  $\Phi$  such that  $\Phi(AQ) = A$  for every  $A \in I$ . Now let  $\{A_i\}$  be a net in  $I$  such that  $\{A_iQ\}$  converges to 0 in the  $\sigma$ -weak topology of  $\mathcal{C}$ . Let  $f$  be a  $\sigma$ -weakly continuous functional in  $\mathcal{C}$ . The restriction  $g$  of  $f$  to  $I$  is continuous on  $I$  and so there is a unique  $\sigma$ -weakly continuous functional  $h$  on  $\mathcal{B}P$  such that  $h(A) = g(A)$  for every  $A \in \mathcal{A}$ . Thus we have that

$$\lim f(A_i) = \lim g(A_i) = \lim h(A_i) = \lim h \cdot \Phi(A_iQ) = 0$$

since  $\Phi$  is  $\sigma$ -weakly continuous on  $\mathcal{C}Q$  [2, I, §4, Theorem 2, Corollary 1]. Therefore, we must have that  $Q=1$  and that  $\Phi$  is an isomorphism of  $\mathcal{C}$  onto  $\mathcal{B}P$ . Now let  $P \cdot \mathcal{S}(\mathcal{A}^\sim) = \{P \cdot \varphi \mid \varphi \in \mathcal{S}(\mathcal{A}^\sim)\}$ . Then the map  $\Psi(\varphi) = \varphi \upharpoonright I$  is a one-one map of  $P \cdot \mathcal{S}(\mathcal{A}^\sim)$  onto  $\mathcal{S}(I^\sim)$ , which is bicontinuous in the  $\sigma_w(\mathcal{A}^\sim, I)$ -topology (resp.  $\varphi_u(\mathcal{A}^\sim, I)$ -topology) of  $P \cdot \mathcal{S}(\mathcal{A}^\sim)$  and the  $\sigma_w(I^\sim, I)$ -topology (resp.  $\sigma_u(I^\sim, I)$ -topology) of  $I^\sim$ . But every net  $\{\varphi_n\}$  in  $\mathcal{S}(\mathcal{A}^\sim)$  has a limit point in  $\mathcal{S}(\mathcal{A}^\sim)$  in the  $\sigma_w(\mathcal{A}^\sim, I)$ -topology if and only if  $P \cdot \mathcal{S}(\mathcal{A}^\sim)$  is compact in the  $\sigma_w(\mathcal{A}^\sim, I)$ -topology; and the set of  $\mathcal{L}$ -irreducible functionals on  $I$  in  $\mathcal{S}(\mathcal{A}^\sim)$  is closed in the  $\sigma_u(\mathcal{A}^\sim, I)$ -topology if and only if the set of  $\mathcal{L}$ -irreducible functionals in  $P \cdot \mathcal{S}(\mathcal{A}^\sim)$  is closed in the  $\sigma_u(\mathcal{A}^\sim, I)$ -topology. The theorem now follows from Theorems 6 and 9. Q.E.D.

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