

## $\sigma$ -FINITE INVARIANT MEASURES ON INFINITE PRODUCT SPACES<sup>(1)</sup>

BY  
DAVID G. B. HILL

**Abstract.** A necessary and sufficient condition in terms of Hellinger integrals is established for the existence of a  $\sigma$ -finite invariant measure on an infinite product space. Using this it is possible to construct a wide class of transformations on the unit interval which have no  $\sigma$ -finite invariant measure equivalent to Lebesgue measure. This class includes most of the previously known examples of such transformations.

**1.1. Introduction.** This paper deals with a special case of the following well-known problem from ergodic theory. Suppose that  $G$  is a group of measurable, invertible, nonsingular transformations of the  $\sigma$ -finite measure space  $(X, \mathcal{B}, \mu)$ . Under what conditions does there exist a  $\sigma$ -finite,  $G$ -invariant measure  $\nu$  on  $(X, \mathcal{B}, \mu)$  such that  $\nu$  is equivalent to  $\mu$ ?

For our purposes we take  $(X, \mathcal{B}, \mu)$  to be the infinite product space  $(X, \mathcal{B}, \mu) = \prod_{k=1}^{\infty} (X_k, \mathcal{B}_k, \mu_k)$ , and  $G = \sum_{k=1}^{\infty} \bigoplus G_k$  where for each positive integer  $k$ ,  $G_k$  is an ergodic group of measurable, invertible, nonsingular transformations on  $(X_k, \mathcal{B}_k, \mu_k)$ . In Theorem 3.20 we solve this problem in terms of the convergence of an infinite series of Hellinger integrals. In fact this result is derived from a more general theorem on the equivalence of certain types of  $\sigma$ -finite product measures with a finite product measure.

In §4 we apply this to the case when all the factor spaces are countable. In particular we are able to improve an existing theorem of C. C. Moore [9] by removing an unnecessary side condition from the hypotheses. Next, using our criterion, we construct a class of ergodic, measurable, invertible, nonsingular transformations on the unit interval which do not have any  $\sigma$ -finite invariant measures equivalent to Lebesgue measure. This class includes the examples of D. S. Ornstein [10], A. Brunel [2], R. V. Chacon [3], and L. K. Arnold [1]. Furthermore our method reveals the essential features these transformations have in common.

**1.2. DEFINITIONS.** Let  $(X, \mathcal{B}, \mu)$  be a measure space. A measurable transformation of the space is a map  $T: X \rightarrow X$  such that  $T^{-1}\mathcal{B} \subset \mathcal{B}$ . It is necessary only that

---

Received by the editors July 3, 1969 and, in revised form, February 10, 1970.

*AMS 1969 subject classifications.* Primary 2875, 2870; Secondary 2840.

*Key words and phrases.* Invariant measures, measures on infinite products, Hellinger integrals, Ornstein's transformation, transformations without invariant measures,  $\sigma$ -finite invariant measures.

<sup>(1)</sup> The results presented here are contained in the author's Ph.D. dissertation written at Yale University under the supervision of Professor S. Kakutani.

Copyright © 1971, American Mathematical Society

$T$  be defined  $\mu$ -almost everywhere. When the inverse function is also defined  $\mu$ -a.e. and furthermore  $T\mathcal{B} \subset \mathcal{B}$ ,  $T$  is said to be invertible. A measurable, invertible, transformation  $T$  on the space with the property that  $\mu T^{-1} = \mu$  is called non-singular. Following the notation used by A. Ionescu Tulcea in [7], measurable, invertible, nonsingular transformations are called automorphisms of  $(X, \mathcal{B}, \mu)$ .

An automorphism  $T$  is called measure preserving iff  $\mu T^{-1} = \mu$ . In this case  $\mu$  is said to be  $T$ -invariant. A  $T$ -invariant set  $B \in \mathcal{B}$  has the property  $T^{-1}B = B$  ( $\mu$ -a.e.). An automorphism  $T$  is said to be ergodic iff for any  $B \in \mathcal{B}$ ,  $B$   $T$ -invariant implies that  $\mu(B) = 0$  or  $\mu(X \setminus B) = 0$ . Similarly if  $G$  is a group of automorphisms of  $(X, \mathcal{B}, \mu)$  we say that  $\mu$  is  $G$ -invariant when for all  $g \in G$ ,  $\mu$  is  $g$ -invariant. And  $G$  is ergodic iff for all  $B \in \mathcal{B}$ ,  $B$   $G$ -invariant implies  $\mu(B) = 0$  or  $\mu(X \setminus B) = 0$ .

2. In this section we state without proof a few well-known results needed as a preliminary to the main results in §3.

2.1. PROPOSITION. *Let  $G$  be a group of automorphisms of the finite measure space  $(X, \mathcal{B}, \mu)$ .*

- (a)  *$G$  is ergodic iff the only  $G$ -invariant measurable functions are the constants.*
- (b) *Suppose that  $G$  is ergodic. Then if there exists a  $\sigma$ -finite  $G$ -invariant measure  $\nu$  which is equivalent to  $\mu$ ,  $\nu$  is unique to within a scalar multiple. In particular we cannot have both a finite and an infinite solution to the invariant measure problem when  $G$  is ergodic.*

2.2. *Hellinger integrals.* Suppose that  $\mu_1$  and  $\mu_2$  are a pair of finite measures on the measurable space  $(X, \mathcal{B})$  such that  $\mu_1(X) = 1$  and  $\mu_2(X) = 1$ . By a countable measurable partition of  $X$  we shall mean a family of subsets,  $\{B_n : n \in N\} \subset \mathcal{B}$ , such that the  $B_n$  are mutually disjoint and  $\bigcup_{n=1}^{\infty} B_n = X$ . Let  $\mathcal{D}$  be the family of countable, measurable partitions of  $X$  and direct  $\mathcal{D}$  by refinement. Given  $\Delta \in \mathcal{D}$ , define

$$S(\Delta) = \sum_{B \in \Delta} [\mu_1(B)\mu_2(B)]^{1/2}.$$

Then  $\{S(\Delta) : \Delta \in \mathcal{D}\}$  is a decreasing net and has a limit. This limit is called the Hellinger integral of  $\mu_1$  and  $\mu_2$  and we write

$$\int [d\mu_1 d\mu_2]^{1/2} = \lim_{\Delta \in \mathcal{D}} S(\Delta).$$

The next theorem collects some of the useful properties of the Hellinger integral.

2.3. THEOREM. (a) *If  $\mu$  is a  $\sigma$ -finite measure on  $(X, \mathcal{B})$  such that  $\mu_1 \ll \mu$  and  $\mu_2 \ll \mu$  (e.g.  $\mu = \mu_1 + \mu_2$ ), then*

$$\int [d\mu_1 d\mu_2]^{1/2} = \int [d\mu_1/d\mu d\mu_2/d\mu]^{1/2} d\mu,$$

where  $d\mu_1/d\mu$  and  $d\mu_2/d\mu$  are Radon-Nikodym derivatives.

(b) *In particular when  $\mu_1 \ll \mu_2$ , then*

$$\int [d\mu_1 d\mu_2]^{1/2} = \int [d\mu_1/d\mu_2]^{1/2} d\mu_2.$$

(c)  $0 \leq \int [d\mu_1 d\mu_2]^{1/2} \leq 1$ .

(d)  $\int [d\mu_1 d\mu_2]^{1/2} = 1$  iff  $\mu_1 = \mu_2$ .

(e)  $\int [d\mu_1 d\mu_2]^{1/2} = 0$  iff  $\mu_1 \perp \mu_2$ .

As long as  $\mu_1$  and  $\mu_2$  are finite measures we can still define the Hellinger integral as above and in place of (c) we will have  $0 \leq \int [d\mu_1 d\mu_2]^{1/2} \leq [\mu_1(X)\mu_2(X)]^{1/2}$ . A similar equation will hold when one of the measures is infinite provided we replace  $X$  by some set of finite measure. Also, note that Theorem 2.3(e) gives a criterion for the singularity of a pair of finite measures. As long as we knew that the measures had to be either equivalent or singular, this would be also a criterion for equivalence. There are two important cases of this situation, one of which occurs in Kakutani's theorem (2.6 below) and the other in Proposition 2.4.

**2.4. PROPOSITION.** *Let  $\mu_1$  and  $\mu_2$  be  $\sigma$ -finite measures on  $(X, \mathcal{B})$  and let  $G$  be an ergodic group of automorphisms with respect to both  $\mu_1$  and  $\mu_2$ . Then either  $\mu_1 \perp \mu_2$  or  $\mu_1 \sim \mu_2$ .*

**2.5. The Zero One Law.** This well-known result from probability theory is the principle upon which are based all the theorems on infinite product spaces in this paper. The version given below is taken from Halmos' book on Measure Theory [5, p. 201].

Let  $(X, \mathcal{B}, \mu) = \prod_{k=1}^{\infty} (X_k, \mathcal{B}_k, \mu_k)$  where for each index  $k$ ,  $\mu_k(X_k) = 1$ . Then if  $A \in \mathcal{B}$  has the property that for any finite  $F \subset N$  there is a measurable subset  $A_F^*$  of  $\prod_{k \notin F} X_k$  such that  $A = (\prod_{k \in F} X_k) \times A_F^*$ , either  $\mu(A) = 0$  or  $\mu(A) = 1$ .

The following theorem due to S. Kakutani [8, pp. 214–224] is an interesting consequence of the Zero One Law.

**2.6. THEOREM.** *For each  $k \in N$  let  $\mu_k$  and  $\mu'_k$  be a pair of finite measures on the measurable space  $(X_k, \mathcal{B}_k)$  with  $\mu_k(X_k) = 1 = \mu'_k(X_k)$ . Let  $(X, \mathcal{B}) = \prod_{k=1}^{\infty} (X_k, \mathcal{B}_k)$  and  $\mu = \prod_{k=1}^{\infty} \mu_k$ ,  $\mu' = \prod_{k=1}^{\infty} \mu'_k$ . Then the following statements are equivalent:*

(a)  $\mu \sim \mu'$ .

(b) *For each index  $k$ ,  $\mu_k \sim \mu'_k$  and  $\prod_{k=1}^{\infty} \int [d\mu_k d\mu'_k]^{1/2} > 0$ .*

(c) *For each index  $k$ ,  $\mu_k \sim \mu'_k$  and*

$$\sum_{k=1}^{\infty} \left[ 1 - \int [d\mu_k d\mu'_k]^{1/2} \right] < \infty.$$

(d) *For each index  $k$ ,  $\mu_k \sim \mu'_k$  and*

$$\sum_{k=1}^{\infty} \|1 - \sqrt{f_k}\|_2^2 < \infty,$$

where  $f_k$  is the Radon-Nikodym  $d\mu'_k/d\mu_k$  in  $L_2(X_k, \mathcal{B}_k, \mu_k)$ . On the other hand, if any one of the conditions in (b), (c), or (d) fails to hold, then  $\mu \perp \mu'$ .

Both 2.4 and 2.6 are proved by looking at the largest subset of  $X$  upon which the two measures are equivalent. In the first case this is a  $G$ -invariant set and in the second, a set of the sort to which the Zero One Law can be applied.

3. Throughout this section, for each positive integer  $k$ ,  $(X_k, \mathcal{B}_k, \mu_k)$  will be a finite measure space with  $\mu_k(X_k)=1$  and  $\nu_k$  will be a  $\sigma$ -finite measure on  $(X_k, \mathcal{B}_k)$ . We denote the product measure space  $\prod_{k=1}^{\infty} (X_k, \mathcal{B}_k, \mu_k)$  by  $(X, \mathcal{B}, \mu)$ . It follows that  $\mu(X)=1$ . By  $\mathcal{F}$  we mean the collection of finite subsets of the positive integers directed by inclusion.  $F$  is used as a subscript in the following way:

$$\begin{aligned} X_F &= \prod_{k \in F} X_k, & \mu_F &= \prod_{k \in F} \mu_k, & \mathcal{B}_F &= \prod_{k \in F} \mathcal{B}_k, \\ X_F^* &= \prod_{k \notin F} X_k, & \mu_F^* &= \prod_{k \notin F} \mu_k, & \mathcal{B}_F^* &= \prod_{k \notin F} \mathcal{B}_k \end{aligned}$$

and so on.

The usual definition of infinite product measure requires that at most a finite number of the factor measures be infinite. However, by adopting a slightly different procedure it is possible to construct a sort of product of  $\sigma$ -finite measures. This procedure is taken from C. C. Moore's paper [9].

3.1. *Restricted product measures.* For each index  $k$ , let  $Y_k$  be chosen in  $\mathcal{B}_k$  such that  $0 < \nu_k(Y_k) < \infty$ . Put  $Y = \prod_{k=1}^{\infty} Y_k$ ,  $Y_F = \prod_{k \in F} Y_k$  and  $Y_F^* = \prod_{k \notin F} Y_k$ . Also, for each  $F \in \mathcal{F}$  define mixed product sets  $S^{(Y,F)} = X_F \times Y_F^*$  and put  $S^{(Y)} = \bigcup_{F \in \mathcal{F}} S^{(Y,F)}$ . This is measurable since  $\mathcal{F}$  is countable. Next, define normalized measures  $\bar{\nu}_k = \nu_k/\nu_k(Y_k)$  and let  $\lambda_k$  be the restriction of  $\bar{\nu}_k$  to  $Y_k$  (i.e.  $\lambda_k(B_k) = \nu_k(B_k \cap Y_k)/\nu_k(Y_k)$  for any  $B_k \in \mathcal{B}_k$ ). Note that  $\bar{\nu}_k$  is a  $\sigma$ -finite measure on  $(X_k, \mathcal{B}_k)$  and  $\lambda_k$  a finite measure with  $\bar{\nu}_k(Y_k) = \lambda_k(X_k)$ .

Now let  $F \in \mathcal{F}$  and define the infinite product measure  $\nu^{(Y,F)} = (\prod_{k \in F} \bar{\nu}_k) \times (\prod_{k \notin F} \lambda_k) = \bar{\nu}_F \times \lambda_F^*$ . This is well defined since all but a finite number of the factors are unit measures. Notice that  $\nu^{(Y,F)}$  is supported by  $S^{(Y,F)}$  and  $\nu^{(Y,F)}(Y) = 1$ . Finally, let  $\mathcal{C}$  be the subfamily of  $\mathcal{B}$  consisting of finite unions of sets of the form  $R = \prod_{k=1}^{\infty} R_k$  where for each  $k$ ,  $R_k \in \mathcal{B}_k$ .

3.2. THEOREM. (a) For each  $B \in \mathcal{B}$ ,  $\{\nu^{(Y,F)}(B) : F \in \mathcal{F}\}$  defines an increasing net so  $\lim_F \nu^{(Y,F)}(B)$  exists (though possibly infinite). This limit is called  $\nu^{(Y)}(B)$ .

(b)  $\nu^{(Y)}$  is a countably additive,  $\sigma$ -finite measure on  $(X, \mathcal{B})$  supported by  $S^{(Y)}$  and  $\nu^{(Y)}(Y) = 1$ . This measure is called the restricted product of  $\{\nu_k : k \in \mathbb{N}\}$  with respect to  $Y$ .

(c) For any  $F \in \mathcal{F}$ ,  $\nu^{(Y)} = \bar{\nu}_F \times \nu_F^*$  where  $\bar{\nu}_F = \prod_{k \in F} \bar{\nu}_k$  and  $\nu_F^*$  is the restricted product of  $\{\nu_k : k \notin F\}$  with respect to  $Y_F^*$ .

(d)  $\mathcal{C}$  is dense in  $\mathcal{B}$  with respect to  $\nu^{(Y)}$ , i.e. if  $\nu^{(Y)}(B) < \infty$  and  $\varepsilon > 0$ , there is an  $E \in \mathcal{C}$  such that  $\nu^{(Y)}(E \Delta B) < \varepsilon$ .

**Proof.** (a) and (b) follow from standard calculations.

(c) Let  $K \in \mathcal{F}$ . Then,

$$\begin{aligned}\nu^{(Y)} &= \lim_F \nu^{(Y, F)} \\ &= \bar{\nu}_K \times \left( \lim_{F \supset K} \bar{\nu}_{F \setminus K} \times \lambda_F^* \right) \\ &= \bar{\nu}_K \times \nu_K^*,\end{aligned}$$

where  $\nu_K^*$  is the restricted product of  $\{\nu_k : k \notin K\}$  with respect to  $Y_K^*$ .

(d) Let  $B \in \mathcal{B}$  such that  $\nu^{(Y)}(B) < \infty$  and let  $\varepsilon > 0$ . From the definition of restricted product measure it follows that there exists  $K \in \mathcal{F}$  such that if  $F \supset K$  then  $\nu^{(Y)}(B \setminus S^{(Y, F)}) < \frac{1}{2}\varepsilon$ . Now  $\nu^{(Y, K)} = \bar{\nu}_K \times \lambda_K^*$  and  $\lambda_K^*(X_K^*) = 1$ . Therefore there is an elementary set  $E \in \mathcal{B}$  such that  $\nu^{(Y, K)}(E \Delta B) < \frac{1}{2}\varepsilon$ . But clearly,  $E \cap S^{(Y, K)} \in \mathcal{C}$ . And,

$$(E \cap S^{(Y, K)}) \Delta B \subset [(E \Delta B) \cap S^{(Y, K)}] \cup [B \setminus S^{(Y, K)}].$$

So

$$\nu^{(Y)}((E \cap S^{(Y, K)}) \Delta B) \leq \nu^{(Y, K)}(E \Delta B) + \nu^{(Y)}(B \setminus S^{(Y, K)}) \leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon.$$

**3.3. REMARK.** In the particular case that each  $\nu_k$  is actually a finite measure we may construct  $\nu^{(X)}$ . But clearly  $\nu^{(X)} = \prod_{k \in N} \bar{\nu}_k$  where in this instance  $\bar{\nu}_k = \nu_k / \nu_k(X_k)$  for each index  $k$ . So the restricted product concept is an extension of the normal product.

We now investigate equivalence of restricted products of the same measures  $\{\nu_k : k \in N\}$  with respect to different subsets.

**3.4. THEOREM.** Let  $Y = \prod_{k \in N} Y_k$  and  $Z = \prod_{k \in N} Z_k$  where for each  $k \in N$ ,  $Y_k, Z_k \in \mathcal{B}_k$  and  $0 < \nu_k(Y_k), \nu_k(Z_k) < \infty$ . Then the following conditions are equivalent:

- (a)  $0 < \nu^{(Y)}(Z) < \infty$ .
- (b)  $0 < \nu^{(Z)}(Y) < \infty$ .
- (c)  $\sum_{k=1}^{\infty} \nu_k(Y_k \Delta Z_k) / \nu_k(Y_k) < \infty$ .
- (d)  $\sum_{k=1}^{\infty} \nu_k(Y_k \Delta Z_k) / \nu_k(Z_k) < \infty$ .

And furthermore, if one of these conditions holds, then not only do we have  $\nu^{(Y)} \sim \nu^{(Z)}$  but also  $\nu^{(Y)} = \nu^{(Y)}(Z) \nu^{(Z)}$  and

$$\nu^{(Y)}(Z) = \prod_{k \in N} \nu_k(Z_k) / \nu_k(Y_k) = 1 / \nu^{(Z)}(Y).$$

On the other hand, if any one of the conditions fails, the measures are singular and either  $\nu^{(Y)}(Z) = 0$  or  $\nu^{(Z)}(Y) = 0$ .

**Proof.** First we will show that (a) and (c) are equivalent. As

$$\nu^{(Y)}(Z) = \lim_F \left[ \prod_{k \in F} \nu_k(Z_k) / \nu_k(Y_k) \right] \left[ \prod_{k \notin F} \nu_k(Y_k \cap Z_k) / \nu_k(Y_k) \right],$$

the l.h.s. is positive and finite iff both

$$0 < \prod_{k \in N} \nu_k(Z_k) / \nu_k(Y_k) < \infty$$

and

$$0 < \prod_{k \notin K} \nu_k(Y_k \cap Z_k) / \nu_k(Y_k) < \infty,$$

for some  $K \in \mathcal{F}$ .

From the second relation,

$$\sum_{k=1}^{\infty} [1 - \nu_k(Y_k \cap Z_k) / \nu_k(Y_k)] < \infty,$$

and exchanging  $Y_k$  for  $Z_k$  in the denominators by means of the first product,  $\sum_{k=1}^{\infty} [1 - \nu_k(Y_k \cap Z_k) / \nu_k(Z_k)] < \infty$  also, i.e.

$$\sum_{k=1}^{\infty} \nu_k(Y_k \setminus Z_k) / \nu_k(Y_k) < \infty$$

and

$$\sum_{k=1}^{\infty} \nu_k(Z_k \setminus Y_k) / \nu_k(Z_k) < \infty.$$

Finally,  $\nu_k(Y_k) / \nu_k(Z_k) \rightarrow 1$ , so we can replace the denominators in the latter series with the terms  $\nu_k(Y_k)$ . A similar reverse calculation yields the converse implication.

In the same way, (b) and (d) are equivalent. The remaining equivalence comes from the fact that both (a) and (b) imply that  $\nu_k(Y_k) / \nu_k(Z_k) \rightarrow 1$ .

Notice that if one of the four conditions holds then  $\lim_F \prod_{k \notin F} \nu_k(Z_k \cap Y_k) / \nu_k(Y_k) = 1$  as this is the tail of a convergent infinite product. Thus,

$$\nu^{(Y)}(Z) = \prod_{k \in N} \nu_k(Z_k) / \nu_k(Y_k) = 1 / \nu^{(Z)}(Y).$$

We want to show that in these circumstances  $\nu^{(Y)} = \nu^{(Y)}(Z) \nu^{(Z)}$ . From 3.2(d) it is sufficient to show that if  $R = \prod_{k \in N} R_k$  where each  $R_k \in \mathcal{B}_k$  and  $0 < \nu^{(Y)}(R) < \infty$ , then  $\nu^{(Y)}(R) = \nu^{(Y)}(Z) \nu^{(Z)}(R)$ . But this follows immediately from the infinite product formula for  $\nu^{(Y)}(Z)$  above and the similar product for  $\nu^{(Y)}(R)$  obtained by replacing  $Z$  with  $R$ .

Finally, if any one of the four conditions does not hold, then either  $\nu^{(Y)}(Z) = 0$  or  $\nu^{(Z)}(Y) = 0$ . If  $\nu^{(Y)}(Z) = 0$ , then  $\nu_F^*(Z_F^*) = 0$ , for any  $F \in \mathcal{F}$ , so

$$\nu^{(Y)}(S^{(Z)}) = \lim_F \bar{\nu}_F(X_F) \nu_F^*(Z_F^*) = 0.$$

As  $\nu^{(Z)}$  is supported by  $S^{(Z)}$ ,  $\nu^{(Y)} \perp \nu^{(Z)}$ .

**3.5. COROLLARY.** *With the same hypotheses as Theorem 3.4 the following statements are equivalent:*

- (a)  $\nu^{(Y)}$  is a finite measure.
- (b) For each index  $k$ ,  $\nu_k$  is finite and  $\nu^{(X)}(Y) > 0$ .
- (c) For each index  $k$ ,  $\nu_k$  is finite and

$$\sum_{k=1}^{\infty} [1 - \nu_k(Y_k) / \nu_k(X_k)] < \infty.$$

- (d) For each index  $k$ ,  $\nu_k$  is finite and  $\nu^{(Y)} \sim \nu^{(X)}$ .

**Proof.** Note that if  $\nu^{(Y)}$  is finite then  $1 = \nu^{(Y)}(Y) \leq \nu^{(Y)}(X) < \infty$ .

Now recalling Kakutani's criterion for the equivalence of a pair of unit product measures (Theorem 2.6) it seems likely that there should be a similar result for restricted product measures. Indeed there is as the next theorem demonstrates.

**3.6. THEOREM.** *If  $\nu^{(Y)}$  is a restricted product measure and  $\mu$  the unit product measure  $\mu = \prod_{k \in N} \mu_k$ , then the following conditions are equivalent:*

- (a)  $\nu^{(Y)} \sim \mu$ .
- (b) *For each index  $k$ ,  $\mu_k \sim \nu_k$  and the restrictions of  $\mu$  and  $\nu^{(Y)}$  to  $Y$  are equivalent.*
- (c) *For each index  $k$ ,  $\mu_k \sim \nu_k$  and*

$$\sum_{k=1}^{\infty} \left[ 1 - \int_{Y_k} [d\mu_k d\nu_k]^{1/2} / [\mu_k(Y_k) \nu_k(Y_k)]^{1/2} \right] < \infty.$$

- (d) *For each index  $k$ ,  $\mu_k \sim \nu_k$  and*

$$\sum_{k=1}^{\infty} \left[ 1 - \int_{Y_k} [d\mu_k d\nu_k]^{1/2} / [\nu_k(Y_k)]^{1/2} \right] < \infty.$$

*Conversely, if  $\mu_k \sim \nu_k$  for each  $k$ , but one of the conditions above fails to hold, then  $\mu \perp \nu^{(Y)}$ .*

**Proof.** First we shall show that (b), (c) and (d) are equivalent. Notice that the restrictions of  $\nu^{(Y)}$  and  $\mu/\mu(Y)$  to  $Y$  are both unit measures. Moreover, when  $\mu(Y) > 0$  we can express each of them in product form.  $\nu^{(Y)}|_Y = \prod_{k \in N} (\nu_k|_{Y_k})/\nu_k(Y_k)$  and  $(\mu|_Y)/\mu(Y) = \prod_{k \in N} (\mu_k|_{Y_k})/\mu_k(Y_k)$ .

Now suppose that for each index  $k$ ,  $\mu_k \sim \nu_k$ . Then their restrictions to  $Y_k$  are also equivalent and since  $\nu_k(Y_k) > 0$  each  $\mu_k(Y_k) > 0$  too and the product on the right above is well defined even if  $\mu(Y) = 0$ . We now apply Theorem 2.6 to these products and conclude that (b) and (c) are equivalent statements.

To obtain (d) look at the equation

$$\begin{aligned} 1 - \int_{Y_k} [d\mu_k d\nu_k]^{1/2} / [\mu_k(Y_k) \nu_k(Y_k)]^{1/2} \\ = \left[ 1 - \int_{Y_k} [d\mu_k d\nu_k]^{1/2} / [\nu_k(Y_k)]^{1/2} \right] / [\mu_k(Y_k)]^{1/2} + [1 - 1/[\mu_k(Y_k)]^{1/2}]. \end{aligned}$$

Given (b) we know that  $\mu(Y) > 0$  since  $\nu^{(Y)}(Y) = 1$ . Therefore,  $\sum_{k=1}^{\infty} [1 - \mu_k(Y_k)] < \infty$ , and  $\mu_k(Y_k) \rightarrow 1$ . So,  $\sum_{k=1}^{\infty} [1 - 1/[\mu_k(Y_k)]^{1/2}] < \infty$ . It follows that

$$\sum_{k=1}^{\infty} \left[ 1 - \int_{Y_k} [d\mu_k d\nu_k]^{1/2} / [\nu_k(Y_k)]^{1/2} \right] < \infty.$$

Conversely, if this series does converge we consider the equation

$$1 - \int_{Y_k} [d\mu_k d\nu_k]^{1/2} / [\nu_k(Y_k)]^{1/2} \\ = \left[ 1 - \int_{Y_k} [d\mu_k d\nu_k]^{1/2} / [\mu_k(Y_k)\nu_k(Y_k)]^{1/2} \right] [\mu_k(Y_k)]^{1/2} + [1 - [\mu_k(Y_k)]^{1/2}].$$

It follows that  $\sum_{k=1}^{\infty} [1 - \mu_k(Y_k)] < \infty$ , and

$$\sum_{k=1}^{\infty} \left[ 1 - \int_{Y_k} [d\mu_k d\nu_k]^{1/2} / [\mu_k(Y_k)\nu_k(Y_k)]^{1/2} \right] < \infty.$$

Finally we show that (a) and (b) are equivalent. Suppose that  $\mu \sim \nu^{(Y)}$ . Then for each index  $k$ ,  $\mu = \mu_k \times \mu_k^*$  and  $\nu^{(Y)} = \bar{\nu}_k \times \nu_k^*$  so  $\mu_k \sim \bar{\nu}_k \sim \nu_k$ . Also,  $\nu^{(Y)}|Y \sim \mu|Y$  trivially.

Conversely, let  $\nu^{(Y)}|Y \sim \mu|Y$  and  $\nu_k \sim \mu_k$  for each index  $k$ . Take  $F \in \mathcal{F}$ . Then,  $\mu = \mu_F \times \mu_F^*$  and  $\nu^{(Y,F)} = \bar{\nu}_F \times \lambda_F^*$ . But  $\mu_F \sim \bar{\nu}_F$  and  $\lambda_F^* \sim \mu_F^*|Y_F^*$  so it follows that  $\nu^{(Y,F)} \sim \mu_F \times \mu_F^*|Y_F^*$ . Now if  $B \in \mathcal{B}$ ,

$$\begin{aligned} \nu^{(Y)}(B) = 0 & \text{ iff } \nu^{(Y,F)}(B) = 0 \text{ for all } F \in \mathcal{F}; \\ & \text{ iff } (\mu_F \times \mu_F^*|Y_F^*)(B) = 0 \text{ for all } F \in \mathcal{F}; \\ & \text{ iff } \mu(B) = 0. \end{aligned}$$

The very last statement of Theorem 3.6 is a consequence of the Zero One Law, which says that either  $\mu(S^{(Y)}) = 0$  or  $\mu(S^{(Y)}) = 1$ . The latter can not hold if  $\mu \sim \nu^{(Y)}$ .

So far we have constructed restricted product measures directly. However, there is another way to show that certain  $\sigma$ -finite measures on  $(X, \mathcal{B})$  are actually restricted products.

**3.7. DEFINITION.** Let  $\nu_k$  be a  $\sigma$ -finite measure on  $(X_k, \mathcal{B}_k)$  for each index  $k$ . Then a  $\sigma$ -finite measure  $\nu$  on  $(X, \mathcal{B})$  is said to be a product of  $\{\nu_k : k \in N\}$  iff for each  $F \in \mathcal{F}$ , there exists a measure  $\nu_F^*$ ,  $\sigma$ -finite on  $(X_F^*, \mathcal{B}_F^*)$  such that  $\nu = \nu_F \times \nu_F^*$  where  $\nu_F = \prod_{k \in F} \nu_k$  as usual.

Clearly the ordinary product of a finite number of  $\sigma$ -finite measures with an infinite number of unit measures fits in with this definition, as does the restricted product. Notice as well that if  $\nu$  is a product in the sense of Definition 3.7 then the measures  $\nu_F^*$  above are themselves products of  $\{\nu_k : k \notin F\}$  on  $X_F^*$ .

**3.8. PROPOSITION.** Let  $\nu_k$  be a  $\sigma$ -finite measure on  $(X_k, \mathcal{B}_k)$  and  $Y_k \in \mathcal{B}_k$  such that  $0 < \nu_k(Y_k) < \infty$ , for each  $k \in N$ . Put  $Y = \prod_{k \in N} Y_k$ . Then, if  $\nu$  is a product of  $\{\nu_k : k \in N\}$ ,  $\nu \sim \nu^{(Y)}$  iff there is an  $a > 0$  such that  $a\nu = \nu^{(Y)}$ .

**Proof.** Obviously if  $a\nu = \nu^{(Y)}$ , then  $\nu \sim \nu^{(Y)}$ . To get the converse we first construct a finite measure  $\alpha = \prod_{k \in N} \alpha_k$  such that  $\alpha \sim \nu \sim \nu^{(Y)}$ .

For each index  $k$ , choose a sequence of disjoint sets  $\{Y_{k,j} : j \in N\}$  such that  $0 < \nu_k(Y_{k,j}) < \infty$  for every  $j$  and  $\bigcup_{j \in N} Y_{k,j} = X_k \setminus Y_k$ . Define  $\alpha_k$  by

$$\alpha_k(B_k) = [(2^k - 1)/2^k][\nu_k(Y_k \cap B_k)/\nu_k(Y_k)] + \sum_{j=1}^{\infty} \nu_k(Y_{k,j} \cap B_k)/[\nu_k(Y_{k,j})2^{k+j}],$$



for each  $B_k \in \mathcal{B}_k$ . Clearly,  $\alpha_k \sim \nu_k$  and  $\alpha_k(X_k) = 1$ . Then,

$$\sum_{k=1}^{\infty} \left[ 1 - \int_{Y_k} [d\alpha_k d\nu_k]^{1/2} / [v_k(Y_k)]^{1/2} \right] = \sum_{k=1}^{\infty} [1 - [(2^k - 1)/2^k]^{1/2}] < \infty.$$

So from Theorem 3.6,  $\alpha \sim \nu^{(Y)} \sim \nu$ .

To complete the proof let  $h(x)$  be the Radon-Nikodym derivative  $d\nu^{(Y)}/d\nu$ , and  $h_F = d\bar{\nu}_F/d\nu_F$ ,  $h_F^* = d\lambda_F^*/d\nu_F^*$  where for any  $F \in \mathcal{F}$  we can factorize the measures thus:

$$\nu = \nu_F \times \nu_F^* \quad \text{and} \quad \nu^{(Y)} = \bar{\nu}_F \times \lambda_F^*.$$

Then, given  $x \in X$ ,  $h(x) = h_F(x_F)h_F^*(x_F^*)$ , where  $x = (x_F; x_F^*)$ . But,  $h_F(x_F) = 1/\nu_F(Y_F)$  and is independent of  $x_F$ . So for any  $F \in \mathcal{F}$ ,  $h(x)$  depends only on  $x_F^*$ . In this situation it follows from the Zero One Law that  $h(x)$  is constant  $\alpha$ -almost everywhere. But  $\alpha \sim \nu \sim \nu^{(Y)}$  and so there exists an  $a > 0$  such that  $\nu^{(Y)} = a\nu$ .

However, it can also be demonstrated that any product measure equivalent to a finite product must be a multiple of a restricted product measure. This is the main result of this paper.

**3.9. THEOREM.** *Suppose that  $\nu$  is a product of the  $\sigma$ -finite measures  $\{\nu_k : k \in N\}$ . Then there exists a finite product measure  $\mu = \prod_{k \in N} \mu_k$  such that  $\nu \sim \mu$  iff  $\nu$  is a multiple of a restricted product  $\nu^{(Y)}$  of  $\{\nu_k : k \in N\}$  with respect to some  $Y = \prod_{k \in N} Y_k$ .*

**Proof.** Going one way is easy. If  $\nu$  is a multiple of some  $\nu^{(Y)}$  then using the construction in Proposition 3.8 we produce an equivalent finite product measure.

The converse result is much more difficult and for convenience the proof has been split up into four propositions. The method of proof is based on that used by C. C. Moore in [9] though this proof does differ from his in many respects other than just being in a more general setting. First we establish some notation.

The assumption is that  $\nu$  is a product of  $\{\nu_k : k \in N\}$  and there exists  $\mu \sim \nu$  such that  $\mu = \prod_{k \in N} \mu_k$  where each  $\mu_k$  is a unit measure on  $(X_k, \mathcal{B}_k)$ . Define the Radon-Nikodym derivatives,  $f = d\mu/d\nu$ ,  $f_F^* = d\mu_F^*/d\nu_F^*$ ,  $f_F = d\mu_F/d\nu_F$  and so on. For any  $F \in \mathcal{F}$  we can factorize  $f = f_F \times f_F^*$ . Without loss of generality we can regard all functions as having domain  $X$  by making them constant in variables upon which they do not depend. However, the functions above may not be integrable with respect to  $\mu$  and it is necessary to deal with the related functions:

$$\begin{aligned} g^t(x) &= \exp(-2\pi i t \log f(x)), \\ g_F^t(x) &= \exp(-2\pi i t \log f_F(x)), \\ g_F^{t*}(x) &= \exp(-2\pi i t \log f_F^*(x)) \quad \text{where } t \in \mathbf{R}. \end{aligned}$$

These functions all have absolute value 1 and hence are all in  $L_2(X, \mathcal{B}, \mu)$ . Furthermore, the product formula is preserved, i.e. for each  $F \in \mathcal{F}$  and for each  $t \in \mathbf{R}$ ,  $g^t(x) = g_F(x)g_F^{t*}(x)$ .

- 3.10. PROPOSITION A. (a) Let  $\varphi(t) = \int g^t d\mu$ . Then  $\varphi$  is continuous and  $\varphi(0) = 1$ .  
 (b) There exists  $t_0 > 0$  such that if  $|t| \leq t_0$

$$\prod_{k \in N} |\langle g_k^t, 1 \rangle| = |\varphi(t)| \geq 1/e.$$

- (c) If  $|t| \leq t_0$ , then  $\sum_{k=1}^{\infty} [1 - |\langle g_k^t, 1 \rangle|] \leq 1$ .

**Proof.** (a) follows immediately from the definition of  $\varphi$ .

(b) Using the continuity of  $\varphi$  at zero, there exists  $t_0 > 0$  such that  $|t| \leq t_0$  implies  $|\varphi(t)| \geq 1/e$ .

Now  $\varphi(t) = \int g^t d\mu = \int g_F^t g_F^{t_F} d\mu = \langle g_F^t, 1 \rangle \langle g_F^{t_F}, 1 \rangle$  for any  $F \in \mathcal{F}$ . Let  $P_F: L_2(X) \rightarrow L_2(X_F)$  be the orthogonal projection. By the Mean Fubini-Jessen Theorem (see [4, p. 207])  $P_F(g^t)$  converges to  $g^t$  in  $L_2(X)$ . Therefore  $|P_F(g^t)| \xrightarrow{F} |g^t| = 1$ . But  $P_F(g^t) = g_F^t \langle g_F^{t_F}, 1 \rangle$  and  $|g_F^t| = 1$ , so  $|\langle g_F^t, 1 \rangle| \xrightarrow{F} 1$ . Hence  $|\langle g_k^t, 1 \rangle| > 0$  for all but a finite number of values of  $k$  and  $|\varphi(t)| = \prod_{k \in N} |\langle g_k^t, 1 \rangle|$ .

(c) From the inequality  $1 - u \leq -\log u$  for  $0 < u \leq 1$ , we deduce that if  $|t| \leq t_0$ , then

$$\begin{aligned} \sum_{k=1}^{\infty} [1 - |\langle g_k^t, 1 \rangle|] &\leq \sum_{k=1}^{\infty} [-\log |\langle g_k^t, 1 \rangle|] \\ &= -\log |\varphi(t)| \leq 1. \end{aligned}$$

3.11. REMARK. The next step is to define the sets  $Y_k$  mentioned in the statement of Theorem 3.9. We take  $c > 0$  and let  $Y_k = \{x_k \in X_k : |\log(f_k(x_k)/q_k)| \leq c\}$ , where the numbers  $\{q_k : k \in N\}$  are to be chosen. In my original proof of the theorem the  $q_k$  were defined in terms of the means of the distributions of the functions  $\log f_k$  on suitable subsets. However, the proof contained an error and was not true in the general case. I am indebted to Neil Rickert for noticing the mistake and suggesting an alternate approach which is also much neater. The idea is to use the median rather than the mean as a measure of the centre of the distribution of  $\log f_k$ . First we need the following lemma which estimates the spread of a probability measure on the line in terms of its Fourier transform.

3.12. LEMMA. Let  $\gamma$  be a probability measure on the line. Then there exists  $s_0 \in \mathbb{R}$  such that for all  $c > 0$ ,

$$1 - \gamma([s_0 - c, s_0 + c]) \leq \left[ (1/t_0) \int_{-t_0}^{t_0} (1 - |\hat{\gamma}(t)|) dt \right] / (1 - A_c),$$

where  $t_0 > 0$  and  $A_c = \sup \{(\sin u)/u : |u| \geq 2\pi c t_0\}$ .

**Proof.**  $\hat{\gamma}(t) = \int \exp[-2\pi i t u] \gamma(du)$ . Therefore,

$$|\hat{\gamma}(t)|^2 = \hat{\gamma}(t) \overline{\hat{\gamma}(t)} = \int \exp[-2\pi i t u] \gamma(du) \int \exp[2\pi i t s] \gamma(ds).$$

So,

$$\begin{aligned} \int_{-t_0}^{t_0} |\hat{\gamma}(t)|^2 dt &= \int_{-t_0}^{t_0} \left[ \iint \exp [-2\pi it(u-s)] \gamma(du) \gamma(ds) \right] dt \\ &= \iint \left[ \int_{-t_0}^{t_0} \exp [-2\pi it(u-s)] dt \right] \gamma(du) \gamma(ds) \\ &= 2t_0 \iint \psi(u-s) \gamma(du) \gamma(ds), \end{aligned}$$

where  $\psi$  is defined by  $\psi(s) = \sin(2\pi t_0 s)/(2\pi t_0 s)$  if  $s \neq 0$  and  $\psi(0) = 1$ .

Now since  $\gamma$  is a probability measure, there exists  $s_0 \in \mathbf{R}$  such that

$$\begin{aligned} (1/2t_0) \int_{-t_0}^{t_0} |\hat{\gamma}(t)|^2 dt &\leq \int \psi(u-s_0) \gamma(du) \\ &\leq \int_I \gamma(du) + \int_{\mathbf{R} \setminus I} A_c \gamma(du), \end{aligned}$$

where  $I = [s_0 - c, s_0 + c]$ , and  $A_c$  is defined as above, i.e.

$$(1/2t_0) \int_{-t_0}^{t_0} |\hat{\gamma}(t)|^2 dt \leq \gamma(I) + A_c[1 - \gamma(I)].$$

Therefore,

$$\begin{aligned} 1 - \gamma(I) &\leq \left[ 1 - (1/2t_0) \int_{-t_0}^{t_0} |\hat{\gamma}(t)|^2 dt \right] / (1 - A_c) \\ &= \left[ (1/2t_0) \int_{-t_0}^{t_0} (1 - |\hat{\gamma}(t)|^2) dt \right] / (1 - A_c). \end{aligned}$$

But,

$$\begin{aligned} 1 - |\hat{\gamma}(t)|^2 &= (1 - |\hat{\gamma}(t)|)(1 + |\hat{\gamma}(t)|) \\ &\leq 2(1 - |\hat{\gamma}(t)|). \end{aligned}$$

So,

$$1 - \gamma(I) \leq \left[ (1/t_0) \int_{-t_0}^{t_0} (1 - |\hat{\gamma}(t)|) dt \right] / (1 - A_c).$$

**3.13. PROPOSITION B.** For each index  $k$ , there exists  $q_k > 0$  such that if  $c > 0$  and  $Y_k = \{x_k \in X_k : |\log(f_k(x_k)/q_k)| \leq c\}$ , then  $\sum_{k=1}^{\infty} [1 - u_k(Y_k)] < \infty$ .

**Proof.** Given  $k \in N$ , define a probability measure  $\gamma_k$  on the real line by setting  $\gamma_k(E) = \mu_k(\{x_k \in X_k : \log f_k(x_k) \in E\})$ , for every Borel set  $E$ . Then,

$$\begin{aligned} \hat{\gamma}_k(t) &= \int_{\mathbf{R}} \exp [-2\pi it u] \gamma_k(du) \\ &= \int_{X_k} \exp [-2\pi it \log f_k(x_k)] \mu_k(dx_k) = \langle g_k^t, 1 \rangle. \end{aligned}$$

Applying Lemma 3.12 to  $\gamma_k$ , there exists  $s_k \in \mathbf{R}$  such that for all  $c > 0$ ,

$$1 - \gamma_k([s_k - c, s_k + c]) \leq \left[ (1/t_0) \int_{-t_0}^{t_0} (1 - |\langle g_k^t, 1 \rangle|) dt \right] / (1 - A_c).$$

But if  $q_k = \exp(s_k)$ ,

$$\begin{aligned}\gamma_k([s_k - c, s_k + c]) &= \mu_k(\{x_k \in X_k : |\log(f_k(x_k)/q_k)| \leq c\}) \\ &= \mu_k(Y_k).\end{aligned}$$

And  $\sum_{k=1}^{\infty} [1 - \langle g_k^t, 1 \rangle] \leq 1$ , for  $|t| \leq t_0$ , from Proposition A.

This is a positive term series so we can integrate from  $-t_0$  to  $t_0$  and interchange the order of integration and summation to obtain

$$(1/t_0) \sum_{k=1}^{\infty} \int_{-t_0}^{t_0} (1 - \langle g_k^t, 1 \rangle) dt \leq 2.$$

Therefore,  $\sum_{k=1}^{\infty} [1 - \mu_k(Y_k)] \leq 2/(1 - A_c) < \infty$ .

3.14. REMARK. Choosing the sets  $Y_k$  as above, we can assume that  $\mu_k(Y_k) > 0$ , for all but a finite number of values of  $k$ . But, on  $Y_k$ ,  $e^{-c}q_k \leq f_k(x_k) \leq e^c q_k$ . Integrating this with respect to  $\nu_k$ , we get,  $\nu_k(Y_k)e^{-c}q_k \leq \mu_k(Y_k) \leq \nu_k(Y_k)e^c q_k$ , for any  $k$ . We change arbitrarily those  $Y_k$  for which  $\nu_k(Y_k) = 0$  (i.e.  $\mu_k(Y_k) = 0$ ) so that for all  $k$ ,  $0 < \nu_k(Y_k) < \infty$ . There is no confusion in using the same notation for this modified set of  $Y_k$ . Later on in Proposition D we shall show that the  $q_k$  can actually be determined in terms of the medians of the functions  $\log f_k$ .

At this point we need a couple of elementary estimates to be used in the proof of Proposition C.

3.15. LEMMA. (a)  $u^2 \leq 12(1 - (\sin u)/u)$  if  $|u| \leq \pi/3$ .

(b) If  $|\log t| \leq 2c$ , then

$$[(\sqrt{t} - 1)c/(e^c - 1)]^2 \leq (\log t)^2.$$

**Proof.** Using Taylor's theorem, we get  $\sin u = u - u^3/(3!) \cos \theta$  where  $|\theta| \leq |u| \leq \pi/3$ . So,

$$u^2 = 6(1 - (\sin u)/u)/\cos \theta \leq 12(1 - (\sin u)/u).$$

(b)  $|\log t| \leq 2c$  iff  $e^{-2c} \leq t \leq e^{2c}$ .

Now if  $e^{-2c} \leq t \leq 1$ ,  $|\sqrt{t} - 1| = 1 - \sqrt{t} \leq -\log t$ . So since  $c/(e^c - 1) < 1$ ,  $(\log t)^2 \geq [(\sqrt{t} - 1)c/(e^c - 1)]^2$ .

On the other hand, if  $1 \leq t \leq e^{2c}$ , then putting

$$W(t) = \log t - (\sqrt{t} - 1)c/(e^c - 1),$$

$$W'(t) = 1/t - [\frac{1}{2}c/(e^c - 1)][1/\sqrt{t}].$$

Now  $W'(1) > 0$ ,  $W(1) = 0$  and  $W'(t)$  has only one zero and is eventually negative. But  $W(e^{2c}) = c$ , so clearly  $W(t) > 0$ , for  $1 \leq t \leq e^{2c}$ .

The next proposition essentially completes the proof of Theorem 3.9.

3.16. PROPOSITION C. With the modified set of  $Y_k$  chosen as in Remark 3.15, and  $c \leq 1/(6t_0)$ ,

$$\sum_{k=1}^{\infty} \left[ 1 - \int_{Y_k} [d\mu_k d\nu_k]^{1/2} / [\nu_k(Y_k)]^{1/2} \right] < \infty.$$

Hence,  $\mu \sim \nu^{(Y)}$  and  $\nu$  is a multiple of  $\nu^{(Y)}$ .

**Proof.** If  $\sum_{k=1}^{\infty} [1 - \int_{Y_k} [d\mu_k dv_k]^{1/2} / [v_k(Y_k)]^{1/2}] < \infty$ , then  $\nu^{(Y)} \sim \mu$  by Theorem 3.6 and  $\nu$  is a multiple of  $\nu^{(Y)}$  by Proposition 3.8. So we have to show this series converges.

Choose  $c > 0$  and form the sets  $Y_k$  as in 3.13. (Choose  $t_0$  as in Proposition A, but suppose furthermore that  $0 < t_0 \leq 1/(6c)$ , so that  $2\pi c t_0 \leq \pi/3$ .) Then if  $\psi(s) = (\sin(2\pi t_0 s))/(2\pi t_0 s)$ ,  $s \neq 0$  and  $\psi(0) = 1$ , we can apply the calculation in Lemma 3.12 to the probability measures  $\gamma_k$  on the real line and obtain

$$\int_R [1 - \psi(u - \log q_k)] \gamma_k(du) \leq (1/t_0) \int_{-t_0}^{t_0} (1 - |\hat{\gamma}(t)|) dt.$$

Here,  $\gamma_k$  is defined by  $\gamma_k(E) = \mu_k(\{x_k \in X_k : \log f_k(x_k) \in E\})$ . Now, repeating the calculation in Proposition B,

$$\sum_{k=1}^{\infty} \int_{X_k} [1 - \psi(\log(f_k/q_k))] d\mu_k \leq \sum_{k=1}^{\infty} (1/t_0) \int_{-t_0}^{t_0} (1 - |\langle g_k^t, 1 \rangle|) dt \leq 2.$$

Therefore,  $\sum_{k=1}^{\infty} \int_{Y_k} [1 - \psi(\log(f_k/q_k))] d\mu_k \leq 2$ .

But if  $x_k \in Y_k$ ,  $|\log(f_k(x_k)/q_k)| \leq c$ , so by Lemma 3.15(a),

$$[2\pi t_0 \log(f_k(x_k)/q_k)]^2 \leq 12[1 - \psi(\log(f_k(x_k)/q_k))]$$

for all  $k \geq p$ , where  $\mu_k(Y_k) > 0$  if  $k \geq p$ . Hence,  $\sum_{k=1}^{\infty} \int_{Y_k} (\log(f_k/q_k))^2 d\mu_k < \infty$ .

Now applying Lemma 3.15(b) and the fact that  $f_k = d\mu_k/d\nu_k$ ,

$$\sum_{k=1}^{\infty} \int_{Y_k} (\sqrt{q_k} - \sqrt{f_k})^2 dv_k = \sum_{k=1}^{\infty} \int_{Y_k} (\sqrt{q_k}/\sqrt{f_k} - 1)^2 d\mu_k < \infty.$$

But

$$\int_{Y_k} \left[ \sqrt{f_k} - \int_{Y_k} \sqrt{f_k} dv_k / \nu_k(Y_k) \right]^2 dv_k \leq \int_{Y_k} (\sqrt{f_k} - \sqrt{q_k})^2 dv_k,$$

for any index  $k$ . Thus,

$$\sum_{k=1}^{\infty} \int_{Y_k} \left[ \sqrt{f_k} - \int_{Y_k} \sqrt{f_k} dv_k / \nu_k(Y_k) \right]^2 dv_k < \infty,$$

i.e.

$$\sum_{k=1}^{\infty} \left[ \mu_k(Y_k) - \left[ \int_{Y_k} \sqrt{f_k} dv_k / [\nu_k(Y_k)]^{1/2} \right]^2 \right] < \infty.$$

And since from Proposition B,  $\sum_{k=1}^{\infty} [1 - \mu_k(Y_k)] < \infty$ , then

$$\sum_{k=1}^{\infty} \left[ 1 - \left[ \int_{Y_k} \sqrt{f_k} dv_k / [\nu_k(Y_k)]^{1/2} \right]^2 \right] < \infty,$$

i.e.

$$\sum_{k=1}^{\infty} \left[ 1 - \left[ \int_{Y_k} [d\mu_k dv_k]^{1/2} / [\nu_k(Y_k)]^{1/2} \right]^2 \right] < \infty.$$

So,

$$\sum_{k=1}^{\infty} \left[ 1 - \int_{Y_k} [d\mu_k d\nu_k]^{1/2} / [\nu_k(Y_k)]^{1/2} \right] < \infty,$$

as required.

The next proposition shows how we can take the numbers  $\log q_k$  to be the medians of the functions  $\log f_k$  and hence gives a method by which the sets  $Y_k$  could actually be computed, for any  $c > 0$ .

**3.17. PROPOSITION D.** *For each index  $k$ , let  $m_k$  be the median of  $\log f_k$  with respect to  $\mu_k$ . Then Proposition C and hence Theorem 3.9 remain true if we take  $\log q_k = m_k$ , and any  $c > 0$ .*

**Proof.**  $m_k$  is a number such that

$$\mu_k(\{x_k \in X_k : \log(f_k(x_k)) \leq m_k\}) \geq \frac{1}{2}$$

and

$$\mu_k(\{x_k \in X_k : \log(f_k(x_k)) < m_k\}) \leq \frac{1}{2}.$$

Choose the  $q_k$  as in Proposition B. If  $c > 0$ , let

$$Y_k(c) = \{x_k \in X_k : |\log(f_k(x_k)/q_k)| \leq c\}$$

and

$$Z_k(c) = \{x_k \in X_k : |\log(f_k(x_k)) - m_k| \leq c\}.$$

Then for all but a finite number of  $k$ ,  $Y_k(\frac{1}{2}c) \subset Z_k(c) \subset Y_k(2c)$ . For we know that  $\sum_{k=1}^{\infty} [1 - \mu_k(Y_k(\frac{1}{2}c))] < \infty$  and hence that  $\mu_k(Y_k(\frac{1}{2}c)) > \frac{1}{2}$ , for every  $k \geq$  some  $k_0$ . So clearly  $|m_k - \log q_k| \leq \frac{1}{2}c$  if  $k \geq k_0$ . Now if  $x_k \in Y_k(\frac{1}{2}c)$ ,

$$\begin{aligned} |\log(f_k(x_k)) - m_k| &\leq |\log(f_k(x_k)/q_k)| + |\log q_k - m_k| \\ &\leq \frac{1}{2}c + \frac{1}{2}c, \end{aligned}$$

i.e.  $Y_k(\frac{1}{2}c) \subset Z_k(c)$ . Also if  $x_k \in Z_k(c)$ ,

$$\begin{aligned} |\log(f_k(x_k)/q_k)| &\leq |\log f_k(x_k) - m_k| + |\log q_k - m_k| \\ &\leq c + \frac{1}{2}c < 2c. \end{aligned}$$

Now if  $k \geq k_0$ , let  $Y_k = Y_k(\frac{1}{2}c)$ ,  $Z_k = Z_k(c)$  and  $Y'_k = Y_k(2c)$ . Otherwise, choose  $Y_k \subset Z_k \subset Y'_k$  arbitrarily so that they all have positive finite  $\nu_k$  measure. Put  $Y = \prod_{k \in N} Y_k$ ,  $Z = \prod_{k \in N} Z_k$ ,  $Y' = \prod_{k \in N} Y'_k$ .

By Theorem 3.9, if  $2c \leq 1/(6t_0)$ ,  $\nu^{(Y)} \sim \nu \sim \nu^{(Y')}$ . And  $\nu^{(Y)}(Y') < \infty$ , by Theorem 3.4. But  $Y \subset Z \subset Y'$ , so  $\nu^{(Y)}(Y) \leq \nu^{(Y)}(Z) \leq \nu^{(Y)}(Y')$ . Using 3.4 again, this means  $\nu^{(Y)} \sim \nu^{(Z)}$ . Finally, since the numbers  $m_k$  are independent of  $t_0$ , and since we may choose  $t_0$  arbitrarily small, any  $c > 0$  will do.

**3.18.** This work was initiated by a study of invariant measures on infinite product spaces. We are now in a position to show that certain invariant measures have the product property and to use Theorem 3.9 to obtain a criterion for invariance.

Suppose that for each  $k$ ,  $G_k$  is an ergodic group of automorphisms on  $(X_k, \mathcal{B}_k, \mu_k)$ . Form the direct sum,  $G = \sum_{k=1}^{\infty} \bigoplus G_k$ . Then  $G$  acts on  $(X, \mathcal{B})$  in the obvious way and the groups  $G_k$  may be regarded as subgroups of  $G$ . Furthermore,  $G$  is ergodic with respect to  $\mu$  and acts freely in the sense that if  $g \in G$  and  $g$  is not the identity, then  $\mu(\{x \in X : g \cdot x = x\}) = 0$ . This follows directly from the Zero One Law and the ergodicity of the factor groups.

**3.19. PROPOSITION.** *Let  $\nu$  be a  $\sigma$ -finite measure on  $(X, \mathcal{B}, \mu)$  such that  $\nu \sim \mu$ . Then  $\nu$  is  $G$ -invariant iff  $\nu$  is a product of invariant measures. More precisely,  $\nu$  is  $G$ -invariant iff for each index  $k$  there exists  $\nu_k \sim \mu_k$ ,  $\sigma$ -finite and  $G_k$ -invariant, and for each  $F \in \mathcal{F}$  there exists  $\nu_F^* \sim \mu_F^*$ ,  $\sigma$ -finite and  $G_F^*$ -invariant, such that  $\nu = \nu_F \times \nu_F^*$ . Here  $G_F^* = \sum_{k \notin F} \bigoplus G_k$  and  $\nu_F = \prod_{k \in F} \nu_k$  as before.*

**Proof.** If  $\nu$  can be factorized as above then for any  $k$ ,  $\nu = \nu_k \times \nu_k^*$  which is  $G_k$ -invariant. The converse is more tedious and depends on the ergodicity of the factor groups. Suppose that  $\nu \sim \mu$  and  $\nu$  is  $G$ -invariant. As usual let  $f = d\mu/d\nu$ .

Let  $B_0 = \{x \in X : f(x) = 0 \text{ or } \infty\}$ . Then  $0 = \mu(B_0) = (\mu_1 \times \mu_1^*)(B_0) = \int \mu_1[(B_0)x_1^*] d\mu_1^*$ , where  $(B_0)x_1^* = \{x_1 \in X_1 : (x_1; x_1^*) \in B_0\}$ . Therefore  $\mu_1[(B_0)x_1^*] = 0$  for  $\mu_1^*$ -almost all  $x_1^* \in X_1^*$ , i.e. for  $\mu_1^*$ -almost all  $x_1^* \in X_1^*$   $0 < f(x_1; x_1^*) < \infty$ , for  $\mu_1$ -almost all  $x_1 \in X_1$ .

But this means that we can define a measure on  $(X_1, \mathcal{B}_1, \mu_1)$  which is equivalent to  $\mu_1$  by

$$\nu_1(x_1^*; B_1) = \int_{B_1} f(x_1; x_1^*)^{-1} d\mu_1 \quad \text{where } B_1 \in \mathcal{B}_1.$$

This measure is  $\sigma$ -finite,  $G_1$ -invariant and defined for  $\mu_1^*$ -almost all  $x_1^* \in X_1^*$ . However  $G_1$  is an ergodic group of automorphisms so as in Proposition 2.1 any  $G_1$ -invariant measure is unique to within a positive multiple. Therefore there exists  $\nu_1 \sim \mu_1$ , a  $\sigma$ -finite  $G_1$ -invariant measure and a function  $f_1^*$  defined  $\mu_1^*$ -almost everywhere on  $X_1^*$ , such that  $\nu_1(x_1^*; \cdot) = \nu_1/f_1^*(x_1^*)$ . Of course  $f_1^*$  is positive and finite where defined. Now taking Radon-Nikodym derivatives with respect to  $\mu_1$  we see that  $f(x_1; x_1^*)^{-1} = d\nu_1/d\mu_1 = f_1^*(x_1^*)^{-1} d\nu_1/d\mu_1$  at  $x_1 \in X_1$ .

Hence if  $f_1 = d\mu_1/d\nu_1$ ,  $f(x_1; x_1^*) = f_1(x_1)f_1^*(x_1^*)$ . This equation also implies that  $f_1^*$  is measurable. Now we can define  $\nu_1^*$  on  $(X_1^*, \mathcal{B}_1^*)$  by  $d\mu_1^*/d\nu_1^* = f_1^*$ , and  $\nu = \nu_1 \times \nu_1^*$ . Clearly  $\nu_1^*$  is  $\sigma$ -finite and  $G_1^*$ -invariant. Applying the same argument to  $\nu_1^*$  we obtain invariant measures  $\nu_2$  and  $\nu_2^*$  on the appropriate spaces. An obvious induction completes the proof.

The main theorem of this paper is an immediate consequence.

**3.20. THEOREM.** *Suppose that for each positive integer  $k$ , we have a finite measure space  $(X_k, \mathcal{B}_k, \mu_k)$  with  $\mu_k(X_k) = 1$  and an ergodic group of automorphisms  $G_k$  on this space. Let  $G = \sum_{k \in N} \bigoplus G_k$  and  $(X, \mathcal{B}, \mu) = \prod_{k \in N} (X_k, \mathcal{B}_k, \mu_k)$ . Then the following statements are equivalent:*

- (a) *There exists a  $\sigma$ -finite,  $G$ -invariant measure  $\nu$  on  $(X, \mathcal{B}, \mu)$  such that  $\nu \sim \mu$ .*
- (b) *For each index  $k$ , there exists a  $\sigma$ -finite,  $G_k$ -invariant measure  $\nu_k \sim \mu_k$  and there*

exists  $Y_k \in \mathcal{B}_k$ ,  $0 < \nu_k(Y_k) < \infty$ , such that if  $Y = \prod_{k \in N} Y_k$ , then  $\nu^{(Y)}$  is  $G$ -invariant and  $\nu^{(Y)} \sim \mu$ .

(c) For each index  $k$ , there exists a  $\sigma$ -finite,  $G_k$ -invariant measure  $\nu_k \sim \mu_k$  and there exists  $Y_k \in \mathcal{B}_k$ ,  $0 < \nu_k(Y_k) < \infty$ , such that

$$\sum_{k=1}^{\infty} \left[ 1 - \int_{Y_k} [d\mu_k d\nu_k]^{1/2} / [\nu_k(Y_k)]^{1/2} \right] < \infty.$$

There will exist a finite  $G$ -invariant measure  $\nu$  equivalent to  $\mu$  iff for each index  $k$ , there exists a finite  $G_k$ -invariant measure  $\nu_k \sim \mu_k$  such that

$$\sum_{k=1}^{\infty} \left[ 1 - \int_{X_k} [d\mu_k d\nu_k]^{1/2} / [\nu_k(X_k)]^{1/2} \right] < \infty.$$

**Proof.** All the work has been done already. The statement about a finite measure comes from Corollary 3.5 and the rest from Theorem 3.9 and Proposition 3.19.

#### 4. The discrete factor case, Moore's theorem and the examples of Ornstein, Brunel, Arnold and Chacon.

4.1. *The space and the group.* Now each factor space is assumed to be countable. Without loss of generality, for each index  $k$ , either  $X_k = \{1, 2, 3, \dots, n_k\}$  where  $n_k \in N$  or  $X_k = N$  and then we say  $n_k = \infty$ .  $X_k$  is given the usual measure structure with every subset measurable and a unit measure  $\mu_k$  is chosen for  $X_k$  so that each point has positive measure. Thus for each  $k$ , there exists a sequence  $\{p_{k,i} : 1 \leq i \leq n_k\}$  such that  $\mu_k(\{i\}) = p_{k,i} > 0$  and  $\sum_{i=1}^{n_k} p_{k,i} = 1$ .

For the groups  $G_k$  we take  $G_k = Z(n_k)$ , the cyclic group of order  $n_k$ , if  $n_k$  is finite. Otherwise we take  $G_k$  as the integers. Once again  $G = \sum_{k \in N} \bigoplus G_k$ , the direct sum.

To describe the action of  $G_k$  on  $X_k$  let  $g_k$  be the generator. Then, if  $n_k < \infty$ ,  $g_k(i) = i + 1 \pmod{n_k}$ . When  $n_k = \infty$ , the situation is a little complicated since to avoid double summations we have chosen to take  $X_k$  as  $N$  not  $Z$ . The action of  $g_k$  is defined in this instance by  $\dots 7 \rightarrow 5 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \dots$  under  $g_k$ . Now  $G$  acts on  $X$  in the obvious way, i.e. if  $g \in G$ , then  $g = \prod_{k \in F} g_k^{r_k}$  for some  $F \in \mathcal{F}$ , so given  $x = (x_1, x_2, \dots) \in X$ ,

$g \cdot x$  has  $k$ th coordinate  $x_k$  if  $k \notin F$  and

$g \cdot x$  has  $k$ th coordinate  $g_k^{r_k} \cdot x_k$  if  $k \in F$ .

This messy notation is not used in the sequel.

The following theorem was proved by C. C. Moore in [9].

4.2. **THEOREM.** Take  $(X, \mathcal{B}, \mu)$  and  $G$  as above.

(a) There exists a finite  $G$ -invariant measure equivalent to  $\mu$  iff for each index  $k$ ,  $n_k < \infty$ , and

$$\sum_{k=1}^{\infty} \sum_{i=1}^{n_k} [\sqrt{p_{k,i}} - 1/\sqrt{n_k}]^2 < \infty.$$



(b) Assume that for each  $k$ ,  $p_{k,i}$  is decreasing with respect to  $i$  and there exists  $a > 0$  such that all  $p_{k,1} \geq a$ . Then there is a  $\sigma$ -finite,  $G$ -invariant measure equivalent to  $\mu$  iff for some  $c > 0$ ,

$$\sum_{k=1}^{\infty} \sum_{i=1}^{n_k} p_{k,i} |p_{k,1}/p_{k,i} - 1|_c^2 < \infty,$$

where  $|t|_c$  means  $\min \{|t|, c\}$  for any  $t \in \mathbb{R}$ .

REMARK. Notice that if we define  $s_k \leq n_k$  by, for each index  $k$ ,  $1 \leq i \leq s_k$  iff  $p_{k,1}/p_{k,i} - 1 \leq c$ , then the convergence of the series in (b) is equivalent to the convergence of the two series,

$$\sum_{k=1}^{\infty} \sum_{i=1}^{s_k} p_{k,i} (p_{k,1}/p_{k,i} - 1)^2$$

and

$$\sum_{k=1}^{\infty} \sum_{i=s_k+1}^{n_k} (p_{k,i}).$$

Also, in this theorem that the condition  $\{p_{k,1} : k \in \mathbb{N}\}$  has a uniform, positive lower bound is immediately suspect since it does not apply in the finite case. In fact, if for each  $k$ ,  $n_k = k$ , and  $p_{k,i} = 1/k$ ,  $1 \leq i \leq k$ , then we have an example in which a finite invariant measure exists by 4.2(a) and yet  $p_{k,1} \rightarrow 0$ .

The original purpose of this paper was to free Moore's theorem from this condition. However, once this had been achieved it became clear that the improved result also applied to the more general situation described in Theorem 3.20. So as a corollary to 3.20 we have the following:

4.3. THEOREM. If  $(X, \mathcal{B}, \mu)$  and  $G$  are described as in 4.1 then the following conditions are equivalent:

(a) There exists a  $\sigma$ -finite,  $G$ -invariant measure  $\nu$  equivalent to  $\mu$ .

(b) For each index  $k$ , there exists  $Y_k \subset X_k$  such that  $|Y_k| = r_k < \infty$  and

$$\sum_{k=1}^{\infty} \left[ 1 - \left( \sum_{i \in Y_k} \sqrt{p_{k,i}} \right) / \sqrt{r_k} \right] < \infty.$$

(c) For each index  $k$ , there exists  $Y_k \subset X_k$  such that  $|Y_k| = r_k < \infty$  and both

$$\sum_{k=1}^{\infty} \sum_{i \notin Y_k} (p_{k,i}) < \infty$$

and

$$\sum_{k=1}^{\infty} \sum_{i \in Y_k} [\sqrt{p_{k,i}} - 1/\sqrt{r_k}]^2 < \infty.$$

There will exist a finite invariant measure equivalent to  $\mu$  iff for each  $k$ ,  $n_k < \infty$  and

$$\sum_{k=1}^{\infty} \left[ 1 - \left( \sum_{i=1}^{n_k} \sqrt{p_{k,i}} \right) / \sqrt{n_k} \right] < \infty.$$

**Proof.** This is just Theorem 3.20 restated in the discrete factor case. To note the equivalence of the conditions in (b) and (c) look at the equality

$$2\left[1 - \left(\sum_{i \in Y_k} \sqrt{p_{k,i}}\right) / \sqrt{r_k}\right] = \left[1 - \sum_{i \in Y_k} p_{k,i}\right] + \sum_{i \in Y_k} [\sqrt{p_{k,i}} - 1/\sqrt{r_k}]^2.$$

**4.4. REMARK.** A long and tedious calculation shows that Moore's theorem can be derived directly from Theorem 4.3. Furthermore, the single condition "there exists  $c > 0$  such that  $\sum_{k=1}^{\infty} \sum_{i=1}^{n_k} p_{k,i} |p_{k,1}/p_{k,i} - 1|_c^2 < \infty$ ," while a sufficient condition for the existence of a  $\sigma$ -finite,  $G$ -invariant measure,  $\nu$  equivalent to  $\mu$ , is not a necessary condition except under the additional hypothesis "there exists  $a > 0$  such that  $p_{k,1} \geq a$  for all  $k \in N$ ." These results are not used in what follows and for further details the reader is referred to the author's thesis. We turn now to an interesting application of Theorem 4.3.

**4.5. A class of transformations which lack an invariant measure.** Theorem 4.3 can be used in a rather nice way to construct a class of ergodic automorphisms on the unit interval which have no invariant  $\sigma$ -finite measure equivalent to Lebesgue measure. As special cases, this class includes the transformations constructed by D. S. Ornstein [10], A. Brunel [2], L. K. Arnold [1] and R. V. Chacon [3]. Moreover, Theorem 4.3 shows directly why these transformations have the required property and reveals their common features.

We commence by setting up a correspondence between the unit interval and a product space of finite factors, which identifies the product structure with an easily visualized family of subintervals. Let  $I$  be the unit interval  $[0, 1)$ ,  $\mathcal{B}_0$  the Borel sets, and  $m$  Lebesgue measure. Let  $\{n_k : k \in N\} \subset N$ . Successively partition the unit interval as follows:

*Stage 1.* We divide  $I$  into  $n_1$  equal subintervals.

Put  $P_{1,j} = [(j-1)/n_1, j/n_1)$ ,  $1 \leq j \leq n_1$ , and put  $\mathcal{P}_1 = \{P_{1,j} : 1 \leq j \leq n_1\}$ .

*Stage 2.* We divide each  $P_{1,j}$  into  $n_2$  equal subintervals.

Put  $P_{2,j} = [(j-1)/m_2, j/m_2)$ ,  $1 \leq j \leq m_2 = n_1 n_2$ , and put  $\mathcal{P}_2 = \{P_{2,j} : 1 \leq j \leq m_2\}$ .

By an induction argument we have an increasing (with respect to refinement) sequence of partitions  $\{\mathcal{P}_k : k \in N\}$  such that  $P_{k,j} = [(j-1)/m_k, j/m_k)$ ,  $1 \leq j \leq m_k$ , and  $\mathcal{P}_k = \{P_{k,j} : 1 \leq j \leq m_k\}$ , where  $m_0 = 1$  and  $m_k = n_1 n_2 \cdots n_k$ .  $\mathcal{P}_k$  is obtained by dividing each of the subintervals of  $\mathcal{P}_{k-1}$  into  $n_k$  equal parts.

Clearly, for each  $t \in I$ , we have a unique sequence  $\{P_{k,j}(t) : k \in N\}$  such that  $\{t\} = \bigcap_{k \in N} P_{k,j}(t)$  and the smallest  $\sigma$ -algebra containing all the  $P_{k,j}$  is just  $\mathcal{B}_0$ .

*Action of  $G$  on  $(I, \mathcal{B}_0)$ .* As in 4.1,  $G = \sum_{k=1}^{\infty} \bigoplus G_k$ , where  $G_k$  is the cyclic group of order  $n_k$  and  $g_k$  is the generator.

Now, for all  $t \in I$ ,  $g_1(t) = t + 1/n_1 \pmod{1}$ , and  $g_k(t) = t + 1/m_k \pmod{1/m_{k-1}}$ , for  $k > 1$ , i.e.  $g_k$  translates  $\pmod{1/m_{k-1}}$  the elements of each  $\mathcal{P}_k \cap P_{k-1,j}$ .

*Definition of the map  $T$ .* A map  $T: I \rightarrow I$  is defined in terms of the generators of  $G$ .  $T$  will have the property that elements of  $G$  can be expressed in terms of

powers of  $T$ . Firstly, we construct the countable partition  $\mathcal{R} = \{R_k : k \in \mathbb{N}\}$ , of  $I$  where

$$R_k = [(m_{k-1} - 1)/m_{k-1}, (m_k - 1)/m_k] \quad \text{for all } k \in \mathbb{N}.$$

Then  $T(t) = g_1(t) = t + 1/n_1$  if  $t \in R_1$ . And  $T(t) = g_{k-1}g_k(t) = t + 1/m_{k-1} + 1/m_k \pmod{1}$  if  $t \in R_k$  and  $k > 1$ . What  $T$  does is to reverse the order of the subintervals  $R_k$ , while leaving their length and direction unchanged. A map  $T$  defined as above is said to be of type  $\{n_k : k \in \mathbb{N}\}$ .

*Compatible measures for  $T$ .* Take a set of positive numbers

$$\{p_{k,i} : 1 \leq i \leq n_k, k \in \mathbb{N}\},$$

with the property that for all  $k$ ,  $\sum_{i=1}^{n_k} p_{k,i} = 1$ . Using these we define a measure  $\mu'$  on  $(I, \mathcal{B}_0)$  by giving the values of  $\mu'$  on the sets  $P_{k,j}$ .

*Stage 1.*  $\mu'(P_{1,j}) = p_{1,j}$ ,  $1 \leq j \leq n_1$ , i.e. we divide  $I$  into  $n_1$  equal subintervals and give them measures in the ratio  $p_{1,1} : p_{1,2} : \cdots : p_{1,n_1}$ .

*Stage  $k$ .* If  $k > 1$ , divide each of the  $P_{k,j}$  into  $n_k$  equal subintervals and assign them measures in the ratio  $p_{k,1} : p_{k,2} : \cdots : p_{k,n_k}$ . A measure  $\mu'$  defined as above is said to be compatible with respect to the map  $T$ .

*The correspondence between  $(X, \mathcal{B}, \mu)$  and  $(I, \mathcal{B}_0, \mu')$ .* Let  $T$  and  $\mu'$  be defined as above. Form the product space  $(X, \mathcal{B}, \mu)$  in the usual way and let  $U: X \rightarrow I$  be defined by  $U(x) = \sum_{k=1}^{\infty} (x_k - 1)/m_k$  for all  $x \in X$ .

**4.6. PROPOSITION.** (a) *The map  $U$  is measurable, invertible and preserves the action of  $G$ . Furthermore,  $\mu = \mu'U$  and  $G$  acts on  $(I, \mathcal{B}_0)$  as an ergodic group of automorphisms.*

(b)  *$T$  is an ergodic automorphism of  $(I, \mathcal{B}_0, \mu')$ .*

(c) *If  $\nu$  is a  $\sigma$ -finite measure on  $(I, \mathcal{B}_0)$ , then  $\nu$  is  $G$ -invariant iff  $\nu$  is  $T$ -invariant.*

**Proof.** (a) This is an easy generalization of the case when all the  $n_k$  are equal which is well known (e.g. decimals). The statement about  $G$  is an immediate consequence of the properties of  $U$ .

(b) From the way  $T$  was defined it is clearly both measurable and invertible. To show that  $T$  is nonsingular let  $B \in \mathcal{B}_0$  and suppose that  $\mu'(B) = 0$ . Then  $B = \bigcup_{k \in \mathbb{N}} B \cap R_k$  and these sets are disjoint. Let  $g_0$  be the identity on  $I$ . Then

$$\begin{aligned} \mu'(TB) &= \mu'\left(T \bigcup_{k \in \mathbb{N}} B \cap R_k\right) = \mu'\left(\bigcup_{k \in \mathbb{N}} g_{k-1}g_k(B \cap R_k)\right) \\ &= \sum_{k \in \mathbb{N}} \mu'(g_{k-1}g_k(B \cap R_k)) = 0, \end{aligned}$$

since each  $g_k$  is an automorphism.

To show that  $T$  is ergodic, suppose that  $B \in \mathcal{B}_0$  is a  $T$ -invariant set. Then,  $U^{-1}B$  is  $U^{-1}TU$ -invariant in  $(X, \mathcal{B}, \mu)$ . But invariant sets of the map  $U^{-1}TU$  are of the type to which we can apply the Zero One Law, hence have measure zero or one.

(c) This is the most interesting part of the previous proposition. Suppose that  $\nu$  is a  $G$ -invariant measure on  $(I, \mathcal{B}_0, \mu')$ . Let  $B \in \mathcal{B}_0$ . Then,  $B = \bigcup_{k \in N} B \cap R_k$  and these sets are disjoint. So,

$$\nu(TB) = \sum_{k \in N} \nu(g_{k-1}g_k(B \cap R_k)) = \sum_{k \in N} \nu(B \cap R_k) = \nu(B).$$

To get the converse we must express the generators of  $G$  in terms of powers of  $T$ . Look at  $g_1$ .

$$\begin{aligned} g_1(t) &= T(t) & \text{if } t \in R_1, \\ &= T^{-n_1}(t) & \text{if } t \notin R_1. \end{aligned}$$

In the general case if we let, for each index  $k$ ,

$$Q_k = \bigcup \{P_{k,jm_k} : 1 \leq j \leq m_k - 1\},$$

then

$$\begin{aligned} g_k(t) &= T^{m_k-1}(t) & \text{if } t \notin Q_k, \\ &= T^{-m_k}(t) & \text{if } t \in Q_k. \end{aligned}$$

Now suppose that  $\nu$  is  $T$ -invariant. If  $B \in \mathcal{B}_0$  and  $k \in N$ ,

$$\begin{aligned} \nu(g_k B) &= \nu(T^{m_k-1}(B \cap Q_k)) + \nu(T^{-m_k}(B \setminus Q_k)) \\ &= \nu(B \cap Q_k) + \nu(B \setminus Q_k) = \nu(B). \end{aligned}$$

It follows that  $\nu$  is  $G$ -invariant also.

The last step in this construction is to map  $I$  into itself in such a way that the measure  $\mu$  is exchanged with Lebesgue measure. Suppose that  $T$  is a transformation of type  $\{n_k : k \in N\}$  and  $\mu$  is a compatible measure. Corresponding to  $\{\mathcal{P}_k : k \in N\}$ , we define another sequence of increasing partitions  $\{\mathcal{P}_k(\mu) : k \in N\}$  by the following process.

*Stage 1.* Divide  $I$  into  $n_1$  subintervals  $\mathcal{P}_1(\mu) = \{P_{1,j}(\mu) : 1 \leq j \leq n_1\}$ . These subintervals are half open on the right and have lengths in the ratio  $p_{1,1} : p_{1,2} : \cdots : p_{1,n_1}$ , i.e.  $P_{1,1}(\mu) = [0, p_{1,1})$ ,  $P_{1,j}(\mu) = [\sum_{i=1}^{j-1} p_{1,i}, \sum_{i=1}^j p_{1,i})$  if  $1 < j \leq n_1$ .

*Stage  $k$ .* We divide each of the  $P_{k-1,j}(\mu)$  into  $n_k$  subintervals half open on the right and with lengths in the ratio  $p_{k,1} : p_{k,2} : \cdots : p_{k,n_k}$ . This partition is called  $\mathcal{P}_k(\mu) = \{P_{k,j}(\mu) : 1 \leq j \leq m_k\}$ .

*The map  $V_\mu$ .* We define a map  $V_\mu : \mathcal{B}_0 \rightarrow \mathcal{B}_0$  by  $V_\mu(P_{k,j}) = P_{k,j}(\mu)$ ,  $1 \leq j \leq m_k$  and  $k \in N$ . This clearly extends to a unique map on  $\mathcal{B}_0$  and a point to point transformation on  $I$  if  $\mu$  is nonatomic.

**4.7. PROPOSITION.** Suppose that  $\mu$  is nonatomic.

(a)  $V_\mu$  is a measurable, invertible transformation on  $(I, \mathcal{B}_0, m)$  and the measures are related by  $mV_\mu = \mu$ .

(b) If  $T_\mu = V_\mu T V_\mu^{-1}$ , then  $T_\mu$  is an ergodic automorphism of  $(I, \mathcal{B}_0, m)$ .

**Proof.** (a) Let  $t \in I$ . Then for each index  $k$ , there exists a unique  $P_{k,j(t)}(\mu)$  containing  $t$ . If  $\mu$  is nonatomic, then

$$\{t\} = \bigcap \{P_{k,j(t)}(\mu) : k \in N\}.$$

That  $V_\mu$  is a measurable, invertible, point to point transformation follows immediately. Also, for any  $P_{k,j}$ ,  $mV_\mu(P_{k,j}) = m(P_{k,j}(\mu)) = \mu(P_{k,j})$ , so  $mV_\mu = \mu$ .

(b) is obvious.

4.8. **REMARK.** It is the transformations  $T_\mu$  in which we are mainly interested.  $T_\mu$  acts linearly with respect to the  $P_{k,j}(\mu)$  in the same way that  $T$  translated the  $P_{k,j}$ .

It would of course have been possible to construct  $T$  directly on the product space (i.e.  $U^{-1}TU$ ) and then proceeded straight to  $(I, \mathcal{B}_0, m)$ . However, the map is more easily pictured on the interval and the notation less messy.

The best that can be done in the atomic case is to take an atomic subalgebra  $\mathcal{B}_\mu$  of  $\mathcal{B}_0$  such that  $\mathcal{B}_\mu(m) \stackrel{V_\mu}{\approx} \mathcal{B}_0(\mu)$ . Then  $V_\mu$  and  $T_\mu$  will not be point to point transformations, but they will still be invertible transformations of the algebras. However, we might just as well use  $(I, \mathcal{B}_0, \mu, T)$  as the model for the system.

If  $\mu$  is atomic there will always be a  $\sigma$ -finite,  $T$ -invariant measure equivalent to  $\mu$ . This measure is finite only when all but a finite number of the  $n_k = 1$  and the infinite product collapses down to a finite product. This situation is still covered by the criterion in the next theorem.

4.9. **THEOREM.** Let  $T$  be a transformation of type  $\{n_k : k \in N\}$ , and let  $\mu$  be the compatible measure for  $T$  determined by  $\{p_{k,i} : 1 \leq i \leq n_k, k \in N\}$ . Form  $T_\mu$  as above.

(a) There exists  $\nu \sim m$ ,  $\sigma$ -finite and  $T_\mu$ -invariant iff for each  $k \in N$ , there exists  $Y_k \subset \{1, 2, \dots, n_k\}$ ,  $|Y_k| = r_k$ , such that

$$\sum_{k=1}^{\infty} \left[ 1 - \left( \sum_{i \in Y_k} \sqrt{p_{k,i}} \right) / \sqrt{r_k} \right] < \infty.$$

(b) Assume that for each index  $k$ ,  $p_{k,i}$  is decreasing in  $i$  and  $p_{k,1} \geq a > 0$ , for some  $a > 0$ . Then there exists  $\nu \sim m$ ,  $\sigma$ -finite and  $T_\mu$ -invariant iff

$$\sum_{k=1}^{\infty} \sum_{i=1}^{n_k} p_{k,i} |p_{k,1}/p_{k,i} - 1|_c^2 < \infty \quad \text{for some } c > 0.$$

(c) There exists  $\nu \sim m$ , finite and  $T_\mu$ -invariant, iff

$$\sum_{k=1}^{\infty} \left[ 1 - \left( \sum_{i=1}^{n_k} \sqrt{p_{k,i}} \right) / \sqrt{n_k} \right] < \infty.$$

**Proof.** If  $\nu$  is a  $\sigma$ -finite measure on  $(I, \mathcal{B}_0)$ , so too is  $\nu V_\mu$  and  $\nu \sim m$  iff  $\nu V_\mu \sim m V_\mu$ . Now we apply 4.2 and 4.3.

4.10. *The transformations of Ornstein, Brunel, Arnold and Chacon.* The first example of an automorphism with no  $\sigma$ -finite invariant measure was constructed

by D. S. Ornstein in 1960 [10]. Later examples have been found by A. Brunel [2], R. V. Chacon [3] and L. K. Arnold [1]. In addition, Arnold demonstrated that both Ornstein's and Brunel's transformations could be approached more simply using his method. However, we do not want to go into the details of the original constructions of these transformations here. We wish only to indicate how they can be regarded as special cases of the transformations  $T_\mu$ .

*D. Ornstein's Transformation.* For each index  $k$ , let  $n_k \geq 3$ ,  $p_{k,1} = \frac{1}{2}$ , and  $p_{k,i} = 1/(2n_k - 2)$ ,  $1 < i \leq n_k$ . We apply the criterion in Theorem 4.9(a) using the fact that convergence of

$$\sum_{k=1}^{\infty} \left[ 1 - \left( \sum_{i \in Y_k} \sqrt{p_{k,i}} \right) / \sqrt{r_k} \right]$$

is equivalent to the joint convergence of the series

$$\sum_{k=1}^{\infty} \sum_{i \in Y_k} p_{k,i} \quad \text{and} \quad \sum_{k=1}^{\infty} \sum_{i \in Y_k} (\sqrt{p_{k,i}} - 1/\sqrt{r_k})^2.$$

Suppose that we can find sets  $Y_k \subset \{1, 2, \dots, n_k\}$  such that  $\sum_{k=1}^{\infty} \sum_{i \in Y_k} p_{k,i} < \infty$ . Then clearly, there is a  $k_0 \in \mathbb{N}$  such that if  $k \geq k_0$ ,  $1 \in Y_k$  and

$$\sum_{k=k_0}^{\infty} (n_k - r_k)/(2n_k - 2) = \sum_{k=k_0}^{\infty} \sum_{i \in Y_k} p_{k,i} < \infty.$$

In particular  $r_k/n_k \rightarrow 1$  and  $r_k \geq 3$  for  $k \geq$  some  $k_1 \in \mathbb{N}$ . Hence

$$\sum_{k=k_1}^{\infty} \sum_{i \in Y_k} (\sqrt{p_{k,i}} - 1/\sqrt{r_k})^2 \geq \sum_{k=k_1}^{\infty} (1/\sqrt{2} - 1/\sqrt{3})^2 = \infty.$$

So there does not exist any  $\sigma$ -finite measure on  $I$  which is  $T_\mu$ -invariant and equivalent to  $m$ .

*A. Brunel's Transformation.* For each index  $k$ , let  $n_k = 3$ , and  $p_{k,1} = \frac{1}{4} = p_{k,3}$ , while  $p_{k,2} = \frac{1}{2}$ .

Interchanging  $p_{k,1}$  and  $p_{k,2}$ , and using exactly the same argument as above, we arrive at the same conclusion.

*L. K. Arnold's Transformation.* For each index  $k$ , let  $n_k = 2$ ,  $p_{k,1} = 1/(\alpha + 1) > \frac{1}{2}$ , and  $p_{k,2} = \alpha/(\alpha + 1)$ , where  $\alpha \in (0, 1)$ . Clearly,  $\sum_{k=1}^{\infty} \sum_{i \in Y_k} p_{k,i} < \infty$  iff  $Y_k = \{1, 2\}$ , for all  $k \geq$  some  $k_0$ . Then

$$\sum_{k=k_0}^{\infty} \sum_{i \in Y_k} (\sqrt{p_{k,i}} - 1/\sqrt{r_k})^2 \geq \sum_{k=k_0}^{\infty} (1/\sqrt{(\alpha + 1)} - 1/\sqrt{2})^2 = \infty.$$

Once again, there is no  $\sigma$ -finite  $T_\mu$ -invariant measure on  $I$  equivalent to  $m$ .

It is obvious from the form of the criterion in Theorem 4.9 that if the factor spaces are taken to be identical as in the last two transformations above, there can not exist any  $\sigma$ -finite invariant measure unless  $p_{k,i} = 1/n_k$ ,  $1 \leq i \leq n_k$ , and then there is a finite invariant measure.

*R. V. Chacon's Transformation.* Chacon proves a result (Lemma 2 in his paper) repeated applications of which can be used to construct an automorphism with the required properties. He does not go into details of the construction and there would appear to be more than one way to achieve this. However, by what seems to be the most simple interpretation we get the following transformation of the sort  $T_\mu$ . Details of his construction not essential to our purpose have been omitted.

Let  $\varepsilon \in (0, 1)$  and let  $\{b_k : k \in N\}$  be a positive sequence such that  $b_k \rightarrow \infty$ . Take  $n_1 > 2$  and an arbitrary set of positive numbers  $\{p_{1,j} : 1 \leq j \leq n_1\}$  summing to 1. For  $k > 1$ , take  $p_{k,1} = \varepsilon$  and  $0 < p_{k,j} \leq 1/b_k$  if  $1 < j \leq n_k$ . Here all the  $n_k > 2$ .

Now suppose that we can find sets  $Y_k \subset \{1, 2, \dots, n_k\}$  with  $|Y_k| = r_k$ , such that  $\sum_{k=1}^\infty \sum_{i \in Y_k} p_{k,i} < \infty$ . Then, except for a finite number of values of  $k$ ,  $1 \in Y_k$ . Also, if

$$\sum_{k=1}^\infty \sum_{i \in Y_k} (\sqrt{p_{k,i}} - 1/\sqrt{r_k})^2 < \infty,$$

then  $p_{k,i} \rightarrow \varepsilon$  uniformly ( $i \in Y_k$ ). But from our hypotheses,  $p_{k,i} \rightarrow 0$  uniformly ( $i > 1$ ).

It follows that  $Y_k = \{1\}$  for all but a finite number of values of  $k$ , and

$$\sum_{k=1}^\infty \sum_{i \in Y_k} p_{k,i} = \infty.$$

From Chacon's basic lemma, we can also construct a more complicated automorphism which, though not itself of the  $T_\mu$  type, induces a  $T_\mu$  type transformation on a subinterval of  $I$ . This induced transformation is somewhat similar to the above and can be reached by our criterion.

In general, this criterion can always be applied to a system  $(X, \mathcal{B}, \mu, T)$  with the following properties.

- (a)  $T$  is an ergodic automorphism of the space.
- (b) There exists a countable family of independent, finite, measurable partitions of  $(X, \mathcal{B}, \mu)$ ,  $\{\mathcal{J}_k : k \in N\}$  which generate  $\mathcal{B}$ .
- (c) For each  $n \in N$ ,  $\bigvee_{k=1}^n \mathcal{J}_k$  is a  $T$ -invariant partition.

Under these conditions,  $T$  will be isomorphic to some  $T_\mu$  on  $(I, \mathcal{B}_0, m)$ .

#### BIBLIOGRAPHY

1. L. K. Arnold, *On  $\sigma$ -finite invariant measures*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **9** (1968), 85–97.
2. A. Brunel, *Sur les mesures invariantes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **6** (1966), 300–303.
3. R. V. Chacon, *A class of linear transformations*, Proc. Amer. Math. Soc. **15** (1964), 560–564.
4. N. Dunford and J. Schwartz, *Linear operators. I: General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.
5. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1950. MR **11**, 504.

6. P. R. Halmos, *Lectures on ergodic theory*, Publ. Math. Soc., Japan, no. 3, Math. Soc. Japan, 1956; reprint, Chelsea, New York, 1960. MR 20 #3958; MR 22 #2677.
7. A. Ionescu Tulcea, *On the category of certain classes of transformations in ergodic theory*, Trans. Amer. Math. Soc. **114** (1965), 261–279. MR 31 #3575.
8. S. Kakutani, *On equivalence of infinite product measures*, Ann. of Math. (2) **49** (1948), 214–224. MR 9, 340.
9. C. C. Moore, *Invariant measures on product spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965), vol. II, part 2, Univ. of California Press, Berkeley, Calif., 1967, pp. 447–459. MR 37 #2951.
10. D. S. Ornstein, *On invariant measures*, Bull. Amer. Math. Soc. **66** (1960), 297–300. MR 26 #3872.

YALE UNIVERSITY,  
NEW HAVEN, CONNECTICUT 06520  
UNIVERSITY OF OTAGO,  
DUNEDIN, NEW ZEALAND