ON BRANCH LOCI IN TEICHMÜLLER SPACE[1]

BY

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Abstract. The branch locus of the ramified covering of the space of moduli of Fuchsian groups with fixed presentation by the corresponding Teichmüller space is decomposed into a union of Teichmüller spaces, each characterised by a description of the action of the conformal self-mappings admitted by the underlying Riemann surfaces. Equivalence classes of subloci under the action of the modular group are studied, and counted in certain simple cases. One may compute as a result the number of conjugacy classes of elements of prime order in the mapping class group of closed surfaces.

1. **Introduction.** The study of Teichmüller spaces may be approached from two different, but essentially equivalent, standpoints. One may consider either classes of marked Riemann surfaces S of fixed genus $g \geq 2$ or classes of Fuchsian groups G of fixed isomorphism type, the connecting link between them being the classical uniformisation theorem for such surfaces and the fact that a conformal mapping f between two surfaces corresponds exactly to an isomorphism between the uniformising Fuchsian groups effected by conjugation with some conformal self-mapping $\tilde{f}$ of the disc obtained by lifting of the original mapping f.

In this paper our interest is primarily directed towards the branch locus Λ_g in the Teichmüller space T_g, which consists of those points which are fixed (points) under the action of the Teichmüller modular group. The survey article of Rauch [12] contains a summary of most of the known facts about this set. It is known that Λ_g is a countable union of complex submanifolds (not disjoint) of T_g, each one being stabilised by a cyclic subgroup of the modular group. Each submanifold is in fact homeomorphic to the Teichmüller space of a Fuchsian group which uniformises quotient surfaces of the classes of surfaces lying in it. We classify all the Fuchsian groups which arise in this way and link them with the geometric nature of the action of the modular group; related results have been obtained by Kuribayashi [5] and Greenberg [2].

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2. **Preliminaries on Fuchsian groups and Teichmüller spaces.** (For details the reader is referred to Macbeath [8] and Bers [1].)

Let U denote the Poincaré upper half plane $\{z \mid \operatorname{Im} z > 0\}$ and let $\mathscr{L} = \mathrm{LF}\,(2, \mathbf{R})$ be the group of linear fractional self-mappings of U. A *Fuchsian group* is a discrete subgroup Γ of $\mathscr{L}$, and if the quotient space U/Γ of Γ-orbits is compact then Γ is isomorphic to an abstract group G with presentation of the form

$$(1) \qquad \begin{aligned} \text{generators:} \quad & x_1, \ldots, x_k;\ a_1, b_1, \ldots, a_\gamma, b_\gamma \\ \text{relations:} \quad & x_1^{m_1} = x_2^{m_2} = \cdots = x_k^{m_k} = 1, \\ & x_1 x_2 \cdots x_k \prod_{j=1}^{\gamma} [a_j, b_j] = 1, \end{aligned}$$

where $[a_j, b_j] = a_j b_j a_j^{-1} b_j^{-1}$. All groups considered here will have compact quotient space. The *signature* of G is the ordered set of integers $(\gamma; m_1, m_2, \ldots, m_k)$. Two such groups are isomorphic if and only if they have the same orbit genus γ and the periods $\{m_i\}$ are the same except for possible reordering [10]. For G as above we define $\mu(G) = 2\gamma - 2 + \sum_{j=1}^{k}(1 - 1/m_j)$. There are Fuchsian groups isomorphic to G if and only if $\mu(G) > 0$, in which case we refer to G as a group of Fuchsian type. We distinguish in particular the group with signature $(g;\ \text{——})$ where $g \geqq 2$, which is denoted by K_g. For each Fuchsian group Γ there is a natural projection map $\pi_\Gamma: U \to U/\Gamma$ defined by $z \mapsto \Gamma z$, the Γ-orbit of the point z. If $\Gamma \cong K_g$ then U/Γ is a compact Riemann surface of genus g.

For a fixed group G of Fuchsian type (1) we consider the set $R(G)$ of all isomorphisms $r: G \to \mathscr{L}$ such that $r(G)$ is Fuchsian (with compact quotient space). $R(G)$ has a natural topology obtained by the correspondence

$$r \mapsto (r(x_1), \ldots, r(x_k), r(a_1), \ldots, r(b_\gamma)) \in \mathscr{L}^{k+2\gamma}.$$

We note that this is the same topology on $R(G)$ as that of pointwise convergence induced on it as a subset of $\prod_{x \in G} \mathscr{L}_x$, $\mathscr{L}_x = \mathscr{L}$ for all $x \in G$. The space $R(G)$ was first introduced by A. Weil [13] in the context of discrete subgroups of any semi-simple Lie group. See also Macbeath [9].

The group $\mathscr{L}$ acts on $R(G)$ by conjugation: for $t \in \mathscr{L}$, $r \in R(G)$, we denote by $t^*(r): G \to \mathscr{L}$ the isomorphism $x \mapsto t^{-1} r(x) t$. The quotient space $R(G)/\mathscr{L}^* = T(G)$ is the *extended Teichmüller space* of the group G. If $G = K_g$ then $T(K_g)$, denoted usually by T_g, consists of two disjoint copies of the usual Teichmüller space $T(S_g)$ of surfaces with genus g, one sheet for each orientation of the surfaces. It is well known that $T(G)$ is homeomorphic to two copies of complex N-space, where $N = 3\gamma - 3 + k$ (Bers [1]). If $\gamma = 0$ and $k = 3$, then $T(G)$ is a pair of points.

Let $M(G)$ denote the group of outer automorphisms of G; $M(G) \cong \mathfrak{A}(G)/\mathfrak{J}(G)$ where $\mathfrak{A}(G)$, $\mathfrak{J}(G)$ are the groups of automorphisms and inner automorphisms of G respectively. Let $\bar{\alpha} \in M(G)$ denote the class of $\alpha \in \mathfrak{A}(G)$. Then $\bar{\alpha}$ acts on $T(G)$

by the rule $[r] \mapsto [r \circ \alpha]$, where $[r]$ signifies $\mathscr{L}^*$-orbit of $r \in R(G)$. It transpires that $M(G)$, which we call the *modular group* of G, is a properly discontinuous group of self homeomorphisms of $T(G)$. We note that an application of the Nielsen theorem that any automorphism of the fundamental group of a surface may be induced by a homeomorphism of the surface (see [11]) implies that the group $M_g = M(K_g)$ is isomorphic to the mapping class group of surfaces of genus g; indeed the action of $\bar{\alpha} \in M_g$ on an element $[r] \in T_g$ is the same as the action of the inducing homotopy class of self-mappings of the base surface S on the element of $T(S)$ corresponding to the surface $U/r(K_g)$ with the marking induced by r. A similar interpretation is possible for any G of Fuchsian type using an extended Nielsen theorem due to Zieschang [14]. The points of $T(G)$ fixed by some nontrivial element of $M(G)$ constitute the *branch locus* $\Lambda(G)$.

Suppose that K', Γ' are Fuchsian groups isomorphic to the abstract groups K, Γ with respective signatures $(g; n_1, \ldots, n_l)$ and $(\gamma; m_1, m_2, \ldots, m_k)$. Thus $K' = r(K)$, $\Gamma' = \rho(r)$ for some $r \in R(K)$, $\rho \in R(\Gamma)$. Let K' be a subgroup of Γ'. A simple argument involving fundamental regions shows that if $[\Gamma'; K'] = n$—it must be finite since the groups have compact quotient spaces—then $\mu(K') = n \cdot \mu(\Gamma')$. If $K = K_g$ we obtain the Riemann-Hurwitz formula

$$(2) \qquad\qquad 2g - 2 = n\left\{2\gamma - 2 + \sum_{i=1}^{k}(1 - 1/m_i)\right\}.$$

Let $K \xrightarrow{i} \Gamma$ be an injection for the above groups K and Γ. We do not assume K normal in Γ, although this will always be so in the sequel. Then there is a natural map from $R(\Gamma)$ into $R(K)$ given by $\rho \mapsto \rho \circ i$, which induces a map $\hat{i}: T(\Gamma) \to T(K)$. Greenberg [2] states the following result.

LEMMA 1. *The map $\hat{i}$ is a real analytic homeomorphism onto the image set I, which is a closed subset of $T(K)$.*

This may be proved by observing that the generators of K can each be expressed as a word in the generators of Γ, thus relating the coordinates of $[\rho \circ i] \in I$ to those of $[\rho] \in T(\Gamma)$ algebraically.

3. **The branch locus.** Let $\bar{\gamma} \in M_g$ be an element of finite order $n > 1$. Then the cyclic group $H = \langle \bar{\gamma} \rangle \cong Z_n$ has nontrivial fixed point set $F(H)$ (see Kravetz [4]). We wish to examine the structure of $F(H)$. We write K for K_g to simplify the notation.

Let $[r] \in F(H)$. Then $[r \circ \gamma] = [r]$, and so there is an element $h_r \in \mathscr{L}$ such that

(i) $h_r^*(r) = r \circ \gamma$, i.e. $r \circ \gamma(k) = h_r^{-1}r(k)h_r$ for each $k \in K$;

(ii) $h_r^\nu \notin r(K)$ for $1 \le \nu \le n-1$, and $h_r^n \in K$.

This is because $h_r^\nu \in r(K) \Leftrightarrow \bar{\gamma}^\nu = 1$.

Now h_r preserves the orbits of $r(K)$ in U and consequently (for details see Macbeath [8]), the Riemann surface represented by the space $U/r(K)$ of $r(K)$-

orbits admits a conformal self map h'_r defined by

$$(3) \qquad\qquad r(K) \cdot z \overset{h'_r}{\longmapsto} h'_r(r(K) \cdot z) = r(K) \cdot (h_r^{-1} z)$$

for $z \in D$.

Condition (ii) implies that h'_r has order n. Let $\Gamma_r = \langle r(K), h_r \rangle$. Then we have an exact sequence

$$(4) \qquad\qquad 1 \longrightarrow r(K) \overset{j}{\longrightarrow} \Gamma_r \overset{\varphi}{\longrightarrow} \langle h'_r \rangle \longrightarrow 1,$$

where j is the inclusion map and φ is the obvious projection map. Notice that the group $\langle h'_r \rangle$ is anti-isomorphic to $H = \langle \bar{\gamma} \rangle$.

Now $\Gamma_r \subset \mathcal{N}(r(K))$, the normaliser of $r(K)$ in $\mathcal{L}$, and as a result Γ_r is Fuchsian with compact orbit space. Let Γ be an abstract group with presentation of type (1) such that $\mathcal{N} \cong \Gamma$, and select an isomorphism $\rho \colon \Gamma \to \Gamma_r$. This determines a *unique* injection $i \colon K \to \Gamma$ such that $\rho \circ i = r$. (It transpires that the particular choice of ρ is unimportant.)

THEOREM 2. $F(H) = i(T(\Gamma))$.

Proof. Let $[\sigma] \in T(\Gamma)$, and set $s = \sigma \circ i$. Writing $\theta = \sigma \circ \rho^{-1}$ we see that $\theta \colon \rho(\Gamma) \to \sigma(\Gamma)$ satisfies $\theta | r(K) = s \circ r^{-1}$. Let $h_s = \theta(h_r)$. Then the diagram

$$
\begin{array}{ccccc}
K & \overset{r}{\longrightarrow} & r(K) & \overset{\theta}{\longrightarrow} & s(K) \\
{\scriptstyle \gamma}\downarrow & & {\scriptstyle h_r^*}\downarrow & & {\scriptstyle h_s^*}\downarrow \\
K & \underset{r}{\longrightarrow} & r(K) & \underset{\theta}{\longrightarrow} & s(K)
\end{array}
$$

commutes, and so $[s \circ \gamma] = [s]$. Thus $[s] \in F(H)$.

Suppose conversely that $[s] \in F(H)$; thus there is an element h_s, and a group $\Gamma_s = \langle s(K), h_s \rangle$, such that

$$h_s^*(s \circ r^{-1}) = (s \circ r^{-1})h_r^* \colon r(K) \to s(K).$$

Hence there exists a mapping $\theta \colon \Gamma_r \to \Gamma_s$ extending the isomorphism $s \circ r^{-1}$, defined by $\theta(h_r) = h_s$; one only needs to observe that $W = r(k_1)h_r^\nu = h_r^\nu r(k_2)$ iff $k_1 = \gamma^\nu(k_2)$, which is equivalent to $s(k_1)h_s^\nu = h_s^\nu s(k_2)$, and so θ is a well defined isomorphism. Thus $s = \sigma \circ i$ where $\sigma = \theta \circ \rho$. ∎

It is interesting to note the interpretation of this result with regard to $T(S)$. By a result of Macbeath [10] extending Nielsen's theorem, an isomorphism

$$\theta \colon \rho(\Gamma) \to \sigma(\Gamma)$$

between Fuchsian groups with compact quotient space can be realised geometrically in the sense that there exists a homeomorphism $t \colon U \to U$ such that

$$tut^{-1} = \theta(u) \quad \text{for all } u \in \Gamma.$$

In fact the t constructed in [10] appears as the (essentially unique) quasiconformal self-map of U which covers the Teichmüller map τ between $U/r(K)$ and $U/s(K)$, and one sees that the conformal self maps h'_r, h'_s of the respective surfaces are related by $h'_s = \tau \circ h'_r \circ \tau^{-1}$.

The theorem extends immediately to a description of the fixed point set of any finite subgroup $H \subseteq M_g$ (and indeed to the fixed point set in $T(K)$ of any finite subgroup of $M(K)$ for an arbitrary group K, provided that the set is nonempty).

COROLLARY 3. *Let $H \subseteq M(K)$ be any finite group. Then there exists a group Γ of Fuchsian type and an injection $K \xrightarrow{i} \Gamma$ such that*

(i) $1 \to K \xrightarrow{i} \Gamma \to H' \to 1$ *is exact with H' anti-isomorphic to H;*

(ii) $F(H) = \hat{i}(T(\Gamma))$.

Proof. The anti-isomorphism comes from the correspondence

$$\gamma \mapsto h'_r(\gamma)$$

where $[r]$ is a typical element of $F(H)$. One obtains

$$\gamma_1 \gamma_2 \mapsto h'_r(\gamma_1 \gamma_2) = h'_r(\gamma_2) \circ h'_r(\gamma_1). \qquad \blacksquare$$

Next we consider the reverse situation. Let K, Γ be groups with Fuchsian presentation such that

$$(5) \qquad\qquad 1 \longrightarrow K \xrightarrow{\ i\ } \Gamma \longrightarrow H' \longrightarrow 1$$

is exact for some finite group H'.

PROPOSITION 4. *The set $\hat{i}(T(\Gamma)) = \mathrm{Im}\,(\hat{i})$ is the fixed point set of some subgroup $H \subset M(K)$ with H anti-isomorphic to H'.*

Proof. Let $r \in R(K)$, $\rho \in R(\Gamma)$, with $r = \rho \circ i$. Then for each $h' \in H'$ there is an $h_r \in \rho(\Gamma)$ such that

(a) $h_r^n \in r(K)$, $h_r^k \notin r(K)$ for $1 \le k \le n-1$;

(b) $h_r^{-1} r(K) \cdot h_r = r(K)$.

Hence there exists $\gamma = \gamma(r, h_r) \in \mathfrak{A}(K)$ defined by $\gamma = r^{-1} \circ h_r^* \circ r$. Evidently

$$H = \langle \overline{\gamma(r, h_r)} \mid h' \in H' \rangle$$

is the required group by Theorem 2. $\quad\blacksquare$

We are led to consider the family of all sequences (5) for fixed groups K, Γ and H'. Having fixed an injection i we shall examine the effect of automorphisms $\alpha \in \mathfrak{A}(K)$, $\beta \in \mathfrak{A}(\Gamma)$ on the induced mapping $\hat{i}$. The essential action is rather that of $M(K)$ and $M(\Gamma)$:

LEMMA 5. *Let i, j be two injections of K into Γ with $j = \beta \circ i \circ \alpha$, where $\alpha \in \mathfrak{A}(K)$, $\beta \in \mathfrak{A}(\Gamma)$. Then $\mathrm{Im}\,\hat{j} = \bar{\alpha}(\mathrm{Im}\,\hat{i})$.*

Proof. For each $[r] \in \operatorname{Im} \hat{\imath}$, $r = \rho \circ i$ for some $\rho \in R(\Gamma)$ and so $\bar{\alpha}[r] = [\rho \circ i \circ \alpha]$ $= [\rho \circ \beta^{-1} \circ j] \in \operatorname{Im} \hat{\jmath}$. ∎

We note that in the above situation $\operatorname{Im} \hat{\imath} = F(H)$ for some $H \subset M(K)$ implies that $\operatorname{Im} \hat{\jmath} = F(\bar{\alpha} H \bar{\alpha}^{-1})$.

In order to give a systematic description of the branch locus we introduce the following notations. Let $I(K, \Gamma, H')$ denote the set of all injections $i \colon K \to \Gamma$ with $\Gamma / i(K) \cong H'$. Every such i is characterised up to automorphisms of K and Γ by fixing a surjection of Γ onto H' (a fixed abstract group). Let $\Phi(K, \Gamma, H')$ be the family of all equivalence classes of surjections $\varphi \colon \Gamma \to H'$ with $\ker \varphi \cong K$, modulis the actions of $\mathfrak{A}(\Gamma)$ and $\mathfrak{A}(H')$. Note that Φ is finite since Γ is finitely generated. Lemma 5 shows that the image sets of injections characterised by a fixed class of surjections in Φ coincide. Let i_φ be any injection corresponding to a class $\{\varphi\} \in \Phi(K, \Gamma, H')$; thus $i_\varphi \colon K \to \ker \varphi$. We denote by $\Lambda(K, \Gamma, H')$ the set $\{[r] \in T(K) : [r] \in \operatorname{Im}(\hat{\imath})$, $i \in I(K, \Gamma, H')\}$. It follows that

$$(6) \qquad \Lambda(K, \Gamma, H') = \bigcup_{\bar{\alpha} \in M(K)} \alpha \left(\bigcup_{\{\varphi\} \in \Phi(K, \Gamma, H')} \operatorname{Im} \hat{\imath}_\varphi \right),$$

the choice of representatives φ and i_φ being unimportant.

As an illustration, let Γ be the group with signature $(0; 2^{(2g+2)})$ with $g \geq 2$, and $H' \cong Z_2$. Then $\Phi(K_g, \Gamma, Z_2)$ contains one equivalence class of surjections, and $\Lambda(K_g, \Gamma, Z_2)$ represents the hyperelliptic branch locus in T_g. Kravetz [4] showed that if $g > 2$ this set is a countable union of component submanifolds. In general however the components are much more complicated, as distinct image sets $\bar{\alpha}_1(\operatorname{Im} i_\varphi)$, $\bar{\alpha}_2(\operatorname{Im} i_\varphi)$ may intersect without coinciding.

We observe that the sets $\Lambda(K_g, \Gamma, Z_n)$ afford us a simple description of Λ_g. Let n range over all possible orders for an element of M_g—in fact $2 \leq n \leq 4g + 2$ (see [3])—and for each n let $\mathfrak{F}_{n,g}$ denote the set of isomorphism classes of group Γ of Fuchsian type with $\Phi(K_g, \Gamma, Z_n)$ nonempty. Then

$$(7) \qquad \Lambda_g = \bigcup_n \bigcup_{\Gamma \in \mathfrak{F}_{n,g}} \Lambda(K_g, \Gamma, Z_n),$$

and we have decomposed Λ_g into a countable union of submanifolds, each consisting of a family of surface classes admitting a conformal self-map of order n such that the quotient surface has a fixed uniformising group Γ. We shall find that the nature of the conformal map is fixed by the pair $(\Gamma, \{\varphi\})$.

4. Conformal self-mappings of Riemann surfaces. Let K be a Fuchsian group uniformising a compact surface S of genus $g \geq 2$, and let $\tau \colon S \to S$ be a conformal mapping. Then, as seen in §2, there corresponds an element $t \in \mathscr{L}$ such that $t^{-1}Kt = K$ and t induces the self-mapping τ by $\tau = t'$, the action on the K-orbits. Thus, writing $\Gamma = \langle K, t \rangle$, we have $U/\Gamma = S_1 \cong S/\langle \tau \rangle$ and an exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma \overset{\varphi}{\longrightarrow} \langle t' \rangle \longrightarrow 1,$$

with $\langle t' \rangle \cong Z/nZ$, where n is the order of $\langle \tau \rangle$. There is an obvious extension of this to the case of an arbitrary group H' of self-maps.

The presentation of the group Γ indicates the geometric nature of the action of τ, in that if Γ has signature $(\gamma; m_1, \ldots, m_k)$ then γ is the genus of S_1 and the $\{m_i\}$ are the branching orders of the covering $S \to S_1$. In particular, the number of m_i equal to n is the number of fixed points of τ. We show now the interrelation between the surjection φ and the behavior of τ at the branch points.

Suppose that Γ has presentation (1). We recall that for $u \in \Gamma$, $\varphi(u)$ is defined to be the map u' of orbits

$$Kz \overset{u'}{\longmapsto} Ku^{-1}(z), \qquad z \in U.$$

The group $\langle \tau \rangle$ will be written as the integers modulo n with $\tau \equiv 1 \bmod n$. Thus $\varphi(u) = 1$ means that $u' = \tau$. Let φ be specified by setting $\varphi(x_i) \equiv \xi_i \bmod n$, $i = 1, \ldots, k$ and $\varphi(a_j) \equiv A_j$, $\varphi(b_j) \equiv B_j$, $j = 1, \ldots, \gamma$. Thus $x_i' = \tau^{\xi_i}$.

LEMMA 6. *The mapping φ is a surjection with kernel $K \cong K_g$ if and only if*
(a) $\sum_{i=1}^{k} \xi_i \equiv 0 \bmod n$,
(b) *for each $i = 1, \ldots, k$, g.c.d. $(\xi_i, n) = n/m_i$.*

Proof. (a) is a trivial consequence of the homomorphism condition and (b) corresponds to the requirement that $\ker \varphi$ have no torsion. ∎

Each elliptic generator x_i has a unique fixed point z_i in U, and the orbit Γz_i is precisely the set of points fixed by the conjugates of x_i in Γ. We write $\text{stab}_\Gamma(z) = \{\gamma \in \Gamma \mid \gamma z = z\}$; it follows that if $z \in \Gamma z_i$, $\text{stab}_\Gamma(z) \cong Z_{m_i}$.

Consider the image set of Γz_i under the projection $\pi_K : U \to S$. It is the inverse image under the covering $S \to S_1$ of the point represented by Γz_i. Since $\Gamma = \bigcup_{j=0}^{n-1} Kt^j$, we have $\Gamma z_i = \bigcup_j Kt^j(z_i)$ and so $\Gamma z_i = \bigcup_j \tau^{-j}(Kz_i)$. Now $x_i' = \tau^{\xi_i}$ and so the sets $\tau^{-j}(Kz_i)$ in U fall into a family of n/m_i K-orbits (points of S)

$$\{\tau^{-j}(Kz_i), j = 0, 1, \ldots, n/m_i - 1\},$$

since $(\xi_i, n) = n/m_i$. These are the fixed points of $\langle x_i' \rangle = \langle \tau^{n/m_i} \rangle$. If $m_i = n$ there is of course only one point of S corresponding to Γz_i, fixed by the whole group $\langle \tau \rangle$.

If $T : S \to S$ is a conformal self-mapping of order n which fixes a point P, then in local coordinates z near P we must have $T(z) = \varepsilon z$ where ε is a primitive nth root of unity. Thus T is locally a rotation at P through angle $2\pi\nu/n$ where $(\nu, n) = 1$.

THEOREM 7. *The rotation angles of the automorphism group $\langle \tau \rangle$ are determined by the homomorphism φ.*

Proof. Let P be a point of $\pi_K(\Gamma z_i)$. We shall determine the rotation angle of τ^{-n/m_i} at P. We assume that the element $x_i \in \Gamma$ is a counterclockwise rotation about z_i through angle $2\pi/m_i$. There is no loss of generality here since for any Fuchsian group Γ such generators may be chosen by taking a suitable canonical fundamental

polygon for Γ (see Macbeath [8]). Now $x_i' = \tau^{\xi_i}$ and so, near $P = Kz_i$, τ^{ξ_i} represents a mapping

$$Kz \mapsto Kx_i^{-1}(z),$$

(i.e. $z \mapsto x_i^{-1}(z)$ for z near to z_i). If $P = Kt^j(z_i)$ then writing

$$\tau^{\xi_i} = (t^j x_i t^{-j}),$$

we see that τ^{ξ_i} represents a rotation about P through angle $-(2\pi/m_i)$. Let η_i be the integer satisfying $\xi_i \cdot \eta_i = (n/m_i) \bmod n$, $0 < \eta_i < m_i$, $(\eta_i, m_i) = 1$. Then τ^{-n/m_i} is locally a rotation at each $P \in \pi_K(z_i)$ through an angle $2\pi\eta_i/m_i$. ∎

Thus the surjection φ characterises completely the action of τ on S. One may deduce, for example, the representations of $\langle \tau \rangle$ as linear transformations on the spaces $A_q(S)$ of holomorphic q-differentials on S. If $\omega \in A_1(S)$, the Abelian differentials on S, and $\omega = f(z)\, dz$ near $P \in S$, then $f(z \circ \tau^{-1})\, d(z \circ \tau^{-1})$ represents an Abelian differential near $Q = \tau(P)$ which we denote by ω_τ. A choice of basis for $A_1(S)$ results in a complex $g \times g$ matrix representation $\rho(\tau)$ of the map $\omega \mapsto \omega_\tau$. The eigenvalues of the unitary matrix $\rho(\tau)$ are complex nth roots of unity since $(\rho(\tau))^n = \mathrm{Id}$, and the multiplicity N_j of the value ε^j ($\varepsilon = \exp 2\pi i/n$) may be calculated using the Riemann-Roch theorem [6] giving

$$N_0 = \gamma,$$

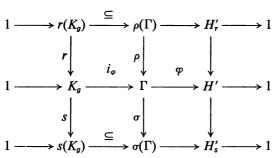$$N_j = \gamma - 1 + \sum_{l=1}^{k} c_{lj}\left(1 - \left\langle \frac{j \cdot \xi_l}{n} \right\rangle\right)$$

where $c_{lj} = 0$ if $j \cdot \xi_l \equiv 0 \bmod n$, and 1 otherwise, and $\langle y \rangle = y - [y]$ denotes the fractional part of y. Similar formulae hold for the action induced on $A_q(S)$.

The results in this section extend to the situation where any group of automorphisms acts on a surface.

5. Submanifolds of Λ_g.

If we view a Teichmüller class $[r] \in T_g$ as a class of marked surfaces $[S, \tilde{r}]$, where $\tilde{r}$ is a homeomorphism from a fixed surface X to $S = U/r(K_g)$ and equivalence between two marked surfaces $(S_1, \tilde{r}_1)$ and $(S_2, \tilde{r}_2)$ is a conformal mapping $c: S_1 \to S_2$ such that $\tilde{r}_2 = c \circ \tilde{r}_1$, then in the light of §4 we may characterise the fixed submanifolds of Λ_g as follows.

PROPOSITION 8. *Let $\{\varphi\} \in \Phi(K_g, \Gamma, H')$, and let $i_\varphi : K \to \ker \varphi$. Let $[r] \in \mathrm{Im}\, \hat{\imath}_\varphi$. Then the surface class $[S, \tilde{r}]$ admits an automorphism group $H_r' \cong H$ with the geometric character implied by φ, and $\mathrm{Im}\, \hat{\imath}_\varphi$ is the set of all classes $[s] = [S', \tilde{s}]$ admitting an automorphism group $H_s' \cong H'$ such that there is a homeomorphism $\theta: S \to S'$ with $H_r' = \theta^{-1} H_s' \theta$.*

Proof. It only remains to observe that the geometric information supplied by φ is characteristic of the class $\{\varphi\}$, and that if $r = \rho \circ i_\varphi$, $s = \sigma \circ i_\varphi$, then we have a commutative diagram with exact rows

$$
\begin{array}{ccccccc}
1 & \longrightarrow & r(K_g) & \overset{\subseteq}{\longrightarrow} & \rho(\Gamma) & \longrightarrow & H'_r & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \rho} & & \downarrow \\
& & & i_\varphi & & \varphi \\
1 & \longrightarrow & K_g & \longrightarrow & \Gamma & \longrightarrow & H' & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \sigma} & & \downarrow \\
1 & \longrightarrow & s(K_g) & \overset{\subseteq}{\longrightarrow} & \sigma(\Gamma) & \longrightarrow & H'_s & \longrightarrow & 1
\end{array}
$$

where $\subseteq$ denotes inclusion map. The homeomorphism θ is obtained by realising geometrically the isomorphism $s \circ r^{-1}$ between the fundamental groups $r(K_g)$ and $s(K_g)$.

We single out the subloci with $H' \simeq Z_n$ for closer attention. One finds [3] that in order for a group Γ with signature $(\gamma; m_1, \ldots, m_k)$ to possess Z_n as factor group with torsion-free factor, it is necessary and sufficient that

(i) l.c.m. $(m_1, \ldots, m_k) = m = $ l.c.m. $(m_1, \ldots, \hat{m}_i, \ldots, m_k)$ for all $i = 1, \ldots, k$, where $\hat{m}_i$ denotes omission of m_i;

(ii) m divides n, and $m = n$ if $\gamma = 0$;

(iii) $k \neq 1$, and if $\gamma = 0$ then $k \geq 3$;

(iv) if m is even then the number of periods m_i divisible by the maximal power of 2 dividing m is even.

To this we add the condition implied by the Riemann-Hurwitz formula:

(v) $(2g - 2)/n = 2\gamma - 2 + \sum_{i=1}^{k} (1 - 1/m_i)$.

PROPOSITION 9. *The family $\mathfrak{F}_{n,g}$ consists of isomorphism classes of groups whose signatures satisfy* (i)–(v).

Note. (1) Two signatures correspond to isomorphic groups if and only if the genera are equal and the periods are reorderings of the same set of integers.

(2) $\mathfrak{F}_{n,g}$ is nonempty if and only if M_g contains elements of order n (in other words some surface of genus g has a conformal self-map of order n).

THEOREM 10. *Let p be an odd prime. Then $\mathfrak{F}_{p,g}$ $(g \geq 2)$ is nonempty if and only if at least one of the following statements holds*:

(a) $g \equiv 1 \bmod p$;

(b) *there is an integer l such that $l(p-1)/2 \leq g \leq lp/2$.*

Proof. The Riemann-Hurwitz formula reads $2g - 2 = p(2\gamma - 2) + k(p - 1)$, and we seek suitable solutions for γ and k to the equation $g = \gamma p + (k - 2)((p - 1)/2)$. If $k = 0$ there is a solution for those g satisfying $g = \gamma p - p + 1$, where $\gamma \geq 2$, and condition (a) follows. Otherwise, since $k \geq 2$, $\gamma \geq 0$ we seek nonnegative integer solutions for $X = \gamma$, $Y = k - 2$, to $g = pX + ((p - 1)/2)Y$. The general solution is

$$X = g - ((p-1)/2) \cdot l, \qquad Y = pl - 2g,$$

for l an integer. A nonnegative solution exists if and only if (b) occurs.

We remark that analogous sufficient conditions are immediate for composite n, but they are then by no means necessary.

COROLLARY 11. *If $\mathfrak{F}_{p,g}$ is nonempty, then p divides $(2g+1)\{(g+1)!\}$.*[2]

Proof. Certainly $p \leq 2g+1$. If $g+2 \leq p \leq 2g-1$, then (a) cannot hold. Moreover $2g/p > 1$ and $2g/(p-1) < 2$, and so there is no integer in the interval $[2g/p, 2g/(p-1)]$, which contradicts (b).

COROLLARY 12. *$\mathfrak{F}_{n,g}$ is nonempty for every $n \leq (2g+1)^{1/2}$.*

Proof. If $n \leq (2g+1)^{1/2}$ then $g \geq n(n-1)/2$, and so the interval $[2g/n, 2g/(n-1)]$ always contains an integer since it has length at least 1.

COROLLARY 13. *The largest value of g for which $\mathfrak{F}_{p,g}$ is empty is $(p^2-4p+1)/2$.*

One can compute the cardinality of $\mathfrak{F}_{p,g}$ directly from the Riemann-Hurwitz equation $g = \gamma p + (k-2)((p-1)/2)$.

Let ν_g be the number of classes in $\mathfrak{F}_{p,g}$, and let $F_p(z) = \sum_{g \geq 2} \nu_g z^g$. If $p \neq 2, 3$ we find that

$$
\begin{aligned}
F_p(z) &= \sum_{\gamma,k} z^{\gamma p + (k-2)(p-1)/2} \\
&= z^{(p-1)/2}(1-z^p)^{-1}(1-z^{(p-1)/2})^{-1} + z^p(1+z)(1-z^p)^{-1},
\end{aligned}
$$

with similar expressions for $F_2(z)$ and $F_3(z)$. Lloyd [7] has found generating functions $F_n(z)$ for all values of n using more advanced combinatorial techniques.

We next study the set $\Phi(K_g, \Gamma, Z_n)$.

THEOREM 14. *Let Γ be a group of Fuchsian type with signature $(\gamma; m_1, \ldots, m_k)$ and l.c.m. $(m_i) = m$. Each equivalence class of surjections in $\Phi(K_g, \Gamma, Z_n)$ contains a surjection φ which satisfies $\varphi(a_j) = \mathrm{id}$ for $j = 1, \ldots, \gamma$, $\varphi(b_1) = B_1$, $\varphi(b_j) = \mathrm{id}$ for $j = 2, \ldots, \gamma$, where B_1 generates the subgroup $Z_{n/m}$.*

If $k = 0$, all surjections of $\Gamma = K_\gamma$ onto Z_n are equivalent.

Proof. We list the automorphisms of Γ which will be used, giving their actions on the generators of Γ:

	$x_1 \cdots$	$x_{k-1} x_k$	a_1	a_2	$a_3 \cdots a_\gamma$	b_1	b_2	$b_3 \cdots b_\gamma$
$\mathfrak{A}_1$	$x_1, \ldots$	x_k	$a_1 b_1,$	$a_2, \ldots$	a_γ	$b_1,$	$b_2 \cdots$	b_γ
$\mathfrak{A}_2$	$x_1, \ldots$	x_k	$a_1 b_1,$	$a_2, \ldots$	a_γ	$a_1^{-1},$	$b_2, \ldots$	b_γ
$\mathfrak{A}_3$	$a_2 x_1 a_2^{-1}, \ldots,$	$a_2 x_k a_2^{-1}$	$a_2 a_1,$	$b_1 a_2 b_1^{-1},$	$a_3, \ldots a_\gamma$	$b_1,$	$a_2 b_2 a_2^{-1} b_1^{-1},$	$b_3, \ldots b_\gamma$
$\mathfrak{A}_4$	$x_1, \ldots, x_{k-1},$	$a_1^{-1} x_k a_1$	$[a_1^{-1}, x_k^{-1}] a_1,$	$a_2, \ldots, a_\gamma$		$b_1 a_1^{-1} x_k a_1, b_2, \ldots$		b_γ

---

(2) In fact, if $p < g-1$, $\mathfrak{F}_{p,g}$ is nonempty iff $p|g-1$ or $p = 2l+1$ with $1|g$.

We also need the automorphisms $\mathfrak{B}_j, j=1, \ldots, \gamma-1$, defined by

$$x_i \to x_i, \qquad i = 1, \ldots, k;$$
$$a_i \to a_i, \qquad b_i \to b_i, \qquad i \neq j, j+1;$$
$$a_j \to a_{j+1}, \qquad b_j \to b_{j+1};$$
$$a_{j+1} \to c_{j+1}^{-1} a_j c_{j+1}, \qquad b_{j+1} \to c_{j+1}^{-1} b_j c_{j+1},$$

where $c_{j+1} = [a_{j+1}, b_{j+1}]$.

Let φ be defined by $x_i \mapsto \xi_i$, $a_j \mapsto A_j$, $b_j \mapsto B_j$. Then the effect of the automorphisms $\mathfrak{A}_1, \ldots, \mathfrak{A}_4$ is as follows:

	$\xi_1 \cdots \xi_k$	A_1	A_2	A_γ	B_1	B_2	B_γ
$\mathfrak{A}_1$	$\xi_1, \ldots \xi_k,$	$A_1+B_1,$	$A_2, \ldots$	$A_\gamma,$	$B_1, \ldots$		B_γ
$\mathfrak{A}_2$	$\xi_1, \ldots \xi_k,$	$B_1,$	$A_2, \ldots$	$A_\gamma,$	$-A_1,$	$B_2, \ldots$	B_γ
$\mathfrak{A}_3$	$\xi_1, \ldots \xi_k,$	$A_1+B_1,$	$A_2, \ldots$	$A_\gamma,$	$B_1,$	$B_2-B_1, B_3 \cdots$	B_γ
$\mathfrak{A}_4$	$\xi_1, \ldots \xi_k,$	$A_1, \ldots$		$A_\gamma,$	$B_1+\xi_k,$	$B_2, \ldots$	B_γ

Also $\mathfrak{B}_j$ induces the mapping

$$\xi_i \to \xi_i, \qquad\qquad i = 1 \cdots k;$$
$$(A_i, B_i) \to (A_i, B_i), \qquad i \neq j, j+1$$
$$(A_j, B_j) \to (A_{j+1}, B_{j+1}), \qquad (A_{j+1}, B_{j+1}) \to (A_j, B_j).$$

We reduce φ systematically. By application of $\mathfrak{A}_1$, and, if needed, $\mathfrak{A}_2$ we reduce A_1 to 0. Using $\{\mathfrak{B}_j\}$ we bring each pair (A_i, B_i) to the position of (A_1, B_1) and reduce A_i to 0. We may now assume that the pair $(0, B_1')$ in position 1 is such that B_1' generates $\langle A_i, B_i, i=1, \ldots, \gamma \rangle$. Using $\mathfrak{A}_3$ we may reduce the second-position B-element to 0, and employing $\{\mathfrak{B}_j\}$ we kill all other elements B_i', $i \geq 2$, in the same way. Finally we reduce B_1' to a generator of Z_n/Z_m by applying $\mathfrak{A}_4$, together with automorphisms which reorder the periodic generators $x_1 \cdots x_k$. ∎

We are able now to count the number of classes in $\Phi(K_g, \Gamma, Z_p)$. It suffices to consider those Γ with orbit genus $\gamma=0$ unless $k=0$. Let $N(k, p)$ be the cardinality of $\Phi(K_g, \Gamma, Z_p)$ when Γ has signature $(\gamma; p, \ldots, p)$ with k periods, and let $G_p(z) = \sum_k N(k, p) z^k$. It follows from a result of Lloyd [7] that

$$G_p(z) = \frac{1}{p-1} \left\{ \frac{1}{p} \left[(1-z)^{-p+1} + (p-1)(1-z)(1-z^p)^{-1} \right] + \sum_l \phi(l)(1-z^l)^{-(p-1)/l} \right\},$$

the summation being over l such that $1 \neq l | (p-1)$, and $\phi(l)$ denoting the Euler totient function. Note that if $p=2$ we get $G_2(z) = 1/(1-z^2)$; there is one class of mapping onto Z_2 when k is even and none when k is odd.

There is an interesting link here with the structure of the modular group M_g.

PROPOSITION 15. *Let* $\lambda(p, g)$ *denote the number of conjugacy classes of elements of* M_g *which have prime order* p.

Then $L_p(z) = \sum_{g \geq 2} \lambda(p, g) z^g$ *is given by*

$$L_p(z) = z^{1-p} \cdot (1 - z^p)^{-1} G_p(z^{(p-1)/2}) - z^{1-p} - z - 2, \quad \text{if } p > 3;$$

$$L_3(z) = z^{-2}(1 - z^3)^{-1} G_3(z) - z^{-2} - 2z - 1;$$

$$L_2(z) = z^{-1}(1 - z^2)^{-1}(1 - z)^{-1} - z^{-1} - 2z - 1.$$

Proof. It follows from §3 that $\lambda(p, g) = \sum |\Phi(K_g, \Gamma, Z_p)|$, where $|\ |$ denotes "number of elements in", the summation running over classes of group Γ in $\mathfrak{F}_{p,g}$. We must evaluate $\sum_{\gamma, k} N(k, p) z^g$, where $g = \gamma p + (k-2)((p-1)/2)$. One finds that, for terms in z^2 and higher powers of z, $L_p(z)$ coincides with the function

$$z^{1-p} \sum_{\gamma \geq 0} \left\{ \sum_{k \geq 0} N(k, p) z^{k(p-1)/2} \right\} z^{\gamma p} = z^{1-p} \cdot G_p(z^{(p-1)/2})(1 - z^p)^{-1}.$$

It would seem to be of interest to know $\lambda(n, g)$ for arbitrary n, but this may be a difficult question.

REMARK. The study of Λ_g sheds light on the structure of the quotient $\mathfrak{K}_g = T_g/M_g$, which is the *space of moduli of surfaces of genus g*. This is a complex space with singular set Λ_g/M_g, and using our dissection of Λ_g one can visualize the ramification a little more clearly. Each set $\Lambda(K_g, \Gamma, Z_n)$ is mapped into itself by M_g, and the quotient of this set is made up of subsets in 1-1 correspondence with classes $\{\varphi\} \in \Phi(K_g, \Gamma, Z_n)$. We note that when Γ has signature $(0; m_1, \ldots, m_k)$ and φ is specified by the integers $\xi_1, \ldots, \xi_k$ less than n, the surfaces underlying the points of the set corresponding to $\{\varphi\}$ are characterised by the algebraic curve

$$Y^n = \prod_{i=1}^{k} (X - y_i)^{\xi_i},$$

where $y_1, \ldots, y_k$ are distinct complex numbers.

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