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ON SOME STARLIKE AND CONVEX FUNCTIONS

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Abstract. In this paper we study functions of the form $\int_0^z (g(t)/\prod_{k=1}^n (1-tz_k)^{\alpha_k})\,dt$ for $|z|<1$ and show under what conditions such a function is convex, convex in one direction and hence univalent in $|z|<1$. We also study the functions $g(z)$ where $g(0)=1$, $g(z)\neq 0$ and $\operatorname{Re}[zg'(z)/g(z)]\geqq -\alpha$, $0\leqq\alpha<1$, for $|z|<1$.

1. Let S denote the set of all functions of the form

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are holomorphic, univalent and starlike in the disk $D=\{z : |z|<1\}$ in the complex z-plane. It is well known [3, p. 13] that a function f given by equation (1) is in S if and only if $\operatorname{Re}[zf'(z)/f(z)]>0$ for $|z|<1$, or equivalently, by Herglotz's representation [8, p. 570],

$$\frac{zf'(z)}{f(z)} = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\alpha(\theta), \tag{2}$$

where $\alpha(\theta)$ is a nondecreasing function of θ in $[0, 2\pi]$ and $\alpha(2\pi)-\alpha(0)=1$. Equation (2) after simplification may be put in the form

$$f(z) = z\exp\left[-2\int_0^{2\pi}\log(1-e^{-i\theta}z)\,d\alpha(\theta)\right] \tag{3}$$

for $|z|<1$. We choose here that branch of the logarithm which has the value zero at $z=0$.

In particular if we take $\alpha(\theta)$ to be a step function with jumps of $\alpha_k>0$ at $\theta_k\in[0, 2\pi)$ $(k=1, 2, \ldots, n)$ such that $\sum_{k=1}^n \alpha_k=1$ (to meet the requirement that $\alpha(2\pi)-\alpha(0)=1$), in equation (3), it follows that the function

$$w = f(z) = z\Big/\prod_{j=1}^{n}(1-\exp[-i\theta_j]z)^{2\alpha_j} \tag{4}$$

is in S. In fact $f(z)$ maps the disk D onto the w-plane with n radial slits [2, pp. 36–37].

A function h holomorphic in D is said to be convex and univalent in D if and only if there exists a function f in the class S such that $f(z)=zh'(z)$ for $z\in D$. We

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shall denote the class of such functions h by C. It follows from equation (3) now that $h \in C$ if and only if

$$h'(z) = \exp\left[-2\int_0^{2\pi} \log(1-e^{-i\theta}z)\,d\alpha(\theta)\right] \tag{5}$$

for $|z|<1$. Once again that branch of the logarithm is taken in (5) which is zero at $z=0$.

We also have from equation (4) that functions h such that

$$h(z) = \int_0^z \left(1\Big/\prod_{j=1}^n (1-\exp[i\theta_j]t)^{2\alpha_j}\right) dt, \tag{6}$$

with $\sum_{j=1}^n \alpha_j=1$, $\alpha_j>0$, $\theta_j \in [0, 2\pi)$ $(j=1, 2, \ldots, n)$ and $|z|<1$, are members of C. In fact it follows by the Schwarz-Christoffel transformation [4, pp. 192–193] that h maps the disk $D=\{z : |z|<1\}$ onto the interior of an n-sided convex polygon.

We remark here that functions f given by equation (4) are extremal in some sense in S and the functions h given by equation (6) are extremal in the class C [5].

In §2 we intend to study functions which reduce to functions of the type given by equations (4) and (6) as a special case. In §3 we study another class of functions $g(z)$ which are used in §2.

2. It is known that a function $h(z)=z+\sum_{n=2}^\infty a_n z^n$ holomorphic in D is in C if and only if $\text{Re}\,[1+zh''(z)/h'(z)]>0$ for all z in D [3, p. 14]. A function $f(z)$ holomorphic in D is said to be convex in one direction in D if it maps $|z|=r$ for every r near 1 into a contour which is cut by every straight line parallel to this direction in not more than two points. If $h \in C$ then h is convex in every direction in D. We now prove the following result.

THEOREM 1. *Let $g(z)$ be holomorphic in D with $g(0)=1$, $g(z)\neq 0$ and* $\text{Re}\,[zg'(z)/g(z)] \geqq -\alpha$ *for $|z|<1$, where $0 \leqq \alpha < 1$. Suppose also that*

$$h(z) = \int_0^z \left(g(t)\Big/\prod_{k=1}^n (1-z_k t)^{\alpha_k}\right) dt = z + \sum_{n=2}^\infty a_n z^n, \tag{7}$$

where $\alpha_k>0$, $|z_k| \leqq 1$ $(1 \leqq k \leqq n)$. Then,

(a) *$h(z)$ is convex in one direction and hence univalent in D if $\sum_{k=1}^n \alpha_k \leqq 3-2\alpha$, and $|a_n| \leqq n$ for all n.*

(b) *$h(z)$ is in C if $\sum_{k=1}^n \alpha_k \leqq 2-2\alpha$ and $|a_n| \leqq 1$ for all n.*

Proof. We have $h'(z)=g(z)/\prod_{k=1}^n (1-zz_k)^{\alpha_k}$ for $|z|<1$, and hence

$$\frac{h''(z)}{h'(z)} = \frac{g'(z)}{g(z)} + \sum_{k=1}^n \frac{\alpha_k z_k}{1-zz_k}.$$

Or,

$$\frac{zh''(z)}{h'(z)} = \frac{zg'(z)}{g(z)} - \frac{1}{2}\sum_{k=1}^{n} \alpha_k \frac{(1-zz_k)-(1+zz_k)}{1-zz_k}$$

$$= \frac{zg'(z)}{g(z)} - \frac{1}{2}\sum_{k=1}^{n} \alpha_k + \frac{1}{2}\sum_{k=1}^{n} \frac{\alpha_k(1+zz_k)}{1-zz_k}$$

$$= \frac{zg'(z)}{g(z)} - \frac{1}{2}\sum_{k=1}^{n} \alpha_k + \frac{1}{2}\sum_{k=1}^{n} \alpha_k \frac{(1-|z|^2|z_k|^2)+2i\,\mathrm{Im}\,(zz_k)}{|1-zz_k|^2}.$$

Therefore,

$$\begin{aligned} \mathrm{Re}\left[1+z\frac{h''(z)}{h'(z)}\right] &\geqq 1-\alpha-\frac{1}{2}\sum_{k=1}^{n}\alpha_k \\ &\quad +\frac{1}{2}\sum_{k=1}^{n}\alpha_k\frac{(1-|z|^2|z_k|^2)}{|1-zz_k|^2} \qquad (8) \\ &> 1-\alpha-\frac{1}{2}\sum_{k=1}^{n}\alpha_k \qquad \text{for } |z|<1. \end{aligned}$$

It follows now by a result of Umezawa [10] that $h(z)$ is convex in one direction in D if $(1-\alpha-\frac{1}{2}\sum_{k=1}^{n}\alpha_k) \geqq -\frac{1}{2}$ or if $\sum_{k=1}^{n}\alpha_k \leqq 3-2\alpha$. That $h(z)$ is univalent in D and $|a_n| \leqq n$ for all n follows from a result of Robertson [6].

Again from inequality (8) we have that $h(z)$ is convex in D if $1-\alpha-\frac{1}{2}\sum_{k=1}^{n}\alpha_k \geqq 0$ or if $\sum_{k=1}^{n}\alpha_k \leqq 2-2\alpha$. Thus $h(z)\in C$ and that $|a_n|\leqq 1$ for all n is well known for functions in C [2, p. 12]. This completes the proof.

A function $f(z)$ holomorphic for $|z|<1$ is said to be starlike in one direction if f maps $|z|=r$ for every r near 1 onto a contour C which is cut by a straight line passing through the origin in two, and not more than two points. From this definition, Theorem 1 and the relation between the members of S and C we have the following corollary.

COROLLARY 2. *If $g(z)$ and z_k $(1\leqq k\leqq n)$ are subject to the same conditions as in Theorem* 1, $\alpha_k>0$ $(1\leqq k\leqq n)$ *and*

$$f(z) = zg(z)\Big/\prod_{k=1}^{n}(1-zz_k)^{\alpha_k} = z+\sum_{n=2}^{\infty} b_n z^n,$$

then,

(a) *$f(z)$ is starlike in one direction if $\sum_{k=1}^{n}\alpha_k\leqq 3-2\alpha$ and $|b_n|\leqq n^2$ for all n;*
(b) *$f(z)\in S$ if $\sum_{k=1}^{n}\alpha_k\leqq 2-2\alpha$ and $|b_n|\leqq n$ for all n.*

Proof. We have $f(z)=zh'(z)$, where h is given by equation (7). That $f(z)$ is starlike in one direction if $\sum_{k=1}^{n}\alpha_k\leqq 3-2\alpha$ follows from part (a) of Theorem 1 and a result of Robertson [6]. Again from equation (7) we have $b_n=na_n$ for all n. Hence from (a) of Theorem 1 we have $|b_n|\leqq n^2$ for all n. (b) follows from (b) of Theorem 1.

If we take, in particular, $g(z)=1$ for $|z|<1$, $\alpha_k=m$ $(1\leqq k\leqq n)$ in Corollary 2 we have the following result of Rubinstein [7].

COROLLARY 3. *The function* $f(z)=z/\prod_{k=1}^{n}(1-zz_k)^m$ *is univalent* (*and starlike*) *in* D *if* $m\leqq 2/n$ *and* $|z_k|\leqq 1$ $(1\leqq k\leqq n)$.

In the following result no restriction is put on the α_k except that these are positive. This in turn, however, shrinks the domain on which $f(z)$ is starlike or convex.

THEOREM 4. *Let* $g(z)$ *be holomorphic in* D *with* $g(0)=1$ *and* $g(z)\neq 0$ *and*

$$\operatorname{Re}\,[zg'(z)/g(z)] \geqq -\alpha, \qquad 0\leqq \alpha < 1.$$

Suppose that

$$h(z) = \int_0^z \left(g(t)\Big/\prod_{k=1}^{n}(1-tz_k)^{\alpha_k}\right)dt, \tag{9}$$

with $\alpha_k>0$, $|z_k|\leqq 1$ $(1\leqq k\leqq n)$, *and* $R=\max_{1\leqq k\leqq n}|z_k|$. *Then*

(a) $h(z)$ *is convex in one direction and hence univalent for*

$$|z| \leqq (3-2\alpha)\Big/R\left(2\sum_{k=1}^{n}\alpha_k+2\alpha-3\right);$$

(b) $h(z)$ *is convex* (*and univalent*) *for*

$$|z| \leqq (1-\alpha)\Big/R\left(\sum_{k=1}^{n}\alpha_k+\alpha-1\right).$$

Proof. As in the proof of Theorem 1, we have

$$\operatorname{Re}\left[1+\frac{zh''(z)}{h'(z)}\right] \geqq 1-\alpha-\frac{1}{2}\sum_{k=1}^{n}\alpha_k+\frac{1}{2}\sum_{k=1}^{n}\alpha_k\frac{1-|z|^2|z_k|^2}{|1-zz_k|^2}. \tag{10}$$

It is clear from the proof of Theorem 1 that $f(z)$ is convex in one direction and univalent for $|z|\leqq R_{1/2}$ if the right side of the inequality (10) is $\geqq -\frac{1}{2}$ for $|z|\leqq R_{1/2}$ and that $f(z)$ is convex for $|z|<R_0$ if the right side of the inequality (10) $\geqq 0$ for $|z|<R_0$.

We intend to find R_β such that

$$1-\alpha-\frac{1}{2}\sum_{k=1}^{n}\alpha_k+\frac{1}{2}\sum_{k=1}^{n}\alpha_k\frac{1-|z|^2|z_k|^2}{|1-zz_k|^2} \geqq -\beta \quad \text{for } |z|\leqq R_\beta, \tag{11}$$

where $\beta=\frac{1}{2}$ or 0. Let $z=re^{i\theta}$, $z_k=r_k\exp[i\theta_k]$, $1\leqq k\leqq n$. The inequality (11) is true if

$$1-\alpha-\frac{1}{2}\sum_{k=1}^{n}\alpha_k+\frac{1}{2}\sum_{k=1}^{n}\alpha_k\frac{1-r^2r_k^2}{1-2rr_k\cos(\theta+\theta_k)+r^2r_k^2} \geqq -\beta$$

for $0\leqq r\leqq R_\beta$, $0\leqq\theta\leqq 2\pi$. The last inequality holds if

$$1-\alpha-\frac{1}{2}\sum_{k=1}^{n}\alpha_k+\frac{1}{2}\sum_{k=1}^{n}\frac{\alpha_k(1-r^2r_k^2)}{1+r^2r_k^2+2rr_k} \geqq -\beta \quad \text{for } 0\leqq r\leqq R_\beta,$$

or, if

$$1-\alpha-\frac{1}{2}\sum_{k=1}^{n}\alpha_k+\frac{1}{2}\sum_{k=1}^{n}\frac{\alpha_k(1-rr_k)}{1+rr_k} \geqq -\beta \quad \text{for } 0 \leqq r \leqq R_\beta.$$

If we let $F_k(r)=(1-rr_k)/(1+rr_k)$ $(1\leqq k\leqq n)$ then the last inequality becomes

$$2(1-\alpha+\beta)-\sum_{k=1}^{n}\alpha_k+\sum_{k=1}^{n}\alpha_kF_k(r) \geqq 0 \quad \text{for } 0 \leqq r \leqq R_\beta. \tag{12}$$

For a fixed r_k, $0\leqq r_k\leqq R$, $F_k(0)=1$ and $F_k(r)$ is a decreasing function of r, and hence $\min_{0\leqq r\leqq R_\beta} F_k(r)=(1-R_\beta r_k)/(1+R_\beta r_k)=F_k(R_\beta)$. Again $F_k(R_\beta)$, when treated as a function of r_k, is a decreasing function of r_k and hence

$$\min_{0\leqq r_k\leqq R} F_k(R_\beta) = (1-R_\beta R)/(1+R_\beta R).$$

Thus the inequality (12) holds if

$$2(1-\alpha+\beta)-\sum_{k=1}^{n}\alpha_k+\left(\frac{1-R_\beta R}{1+R_\beta R}\right)\left(\sum_{k=1}^{n}\alpha_k\right) \geqq 0,$$

or, if

$$2(1+RR_\beta)(1-\alpha+\beta)-2\sum_{k=1}^{n}\alpha_k(RR_\beta) \geqq 0,$$

or if

$$R_\beta \leqq (1-\alpha+\beta)\Big/R\left(\sum_{k=1}^{n}\alpha_k-1+\alpha-\beta\right).$$

Hence by taking $\beta=\frac{1}{2}$ and 0 we have that $h(z)$ is convex in one direction and univalent for

$$|z| \leqq (3-2\alpha)\Big/R\left(2\sum_{k=1}^{n}\alpha_k+2\alpha-3\right)$$

and $h(z)$ is convex for

$$|z| \leqq (1-\alpha)\Big/R\left(\sum_{k=1}^{n}\alpha_k+\alpha-1\right).$$

We deduce immediately from Theorem 4 the following.

Corollary 5. *If $g(z)$ and z_k $(1\leqq k\leqq n)$ and R are the same as in Theorem* 4 *and $f(z)=zg(z)/\prod_{k=1}^{n}(1-zz_k)^{\alpha_k}$ with $\alpha_k>0$ $(1\leqq k\leqq n)$. Then*

(a) *$f(z)$ is starlike in one direction for*

$$|z| \leqq (3-2\alpha)\Big/R\left(2\sum_{k=1}^{n}\alpha_k+2\alpha-3\right),$$

(b) *$f(z)$ is starlike (and univalent) for*

$$|z| \leqq (1-\alpha)\Big/R\left(\sum_{k=1}^{n}\alpha_k+\alpha-1\right).$$

By taking $g(z)=1$ for $|z|<1$, $\alpha_k=1$ $(1\leqq k\leqq n)$ in Corollary 5 we obtain the following result of Rubinstein [7].

COROLLARY 6. *The function* $z/\prod_{k=1}^{n}(1-zz_k)$ *is univalent* (*and starlike*) *for* $|z|<1/R(n-1)$ *where* $R=\max_{1\leqq k\leqq n}|z_k|\leqq 1$.

We use Theorem 1 to prove the next result.

THEOREM 7. *If* $g(z)$ *is holomorphic in* D, $g(0)=1$ *and* $g(z)\neq 0$, $\operatorname{Re}[zg'(z)/g(z)]\geqq-\alpha$ *where* $0\leqq\alpha\leqq 1$, *for* $|z|<1$, *then the function*

$$F(z) = zg(z)\Big/\left(\prod_{k=1}^{n}(1-zz_k)^{\alpha}\int_0^z\left(g(t)\Big/\prod_{k=1}^{n}(1-z_kt)^{\alpha_k}\right)dt\right),$$

where $\alpha_k>0$, $|z_k|\leqq 1$, $1\leqq k\leqq n$, *and* $\sum_{k=1}^{n}\alpha_k\leqq 2-2\alpha$, *is holomorphic in* $|z|<1$ *and if* $F(z)=1+\sum_{n=1}^{\infty}b_nz^n$ *then* $|b_n|\leqq 1$ *for all* n.

Proof. By Theorem 1, $h(z)=\int_0^z(g(t)/\prod_{k=1}^{n}(1-z_kt)^{\alpha_k})\,dt$ is holomorphic, univalent and convex for $|z|<1$. It follows by a result of Strohäcker [9] that

$$\operatorname{Re}[zh'(z)/h(z)] \geqq \tfrac{1}{2},$$

or equivalently, $\operatorname{Re}F(z)\geqq\frac{1}{2}$ for $|z|<1$. Since $h(0)=0$, $h(z)$ is univalent for $|z|<1$, it follows that $h(z)\neq 0$ for $0<|z|<1$ and hence $F(z)$ is holomorphic for $|z|<1$. Let $G(z)=F(z)-1$. Then $G(0)=0$, $G(z)=\sum_{n=1}^{\infty}b_nz^n$ and $\operatorname{Re}(G(z))\geqq-\frac{1}{2}$ for $|z|<1$. The function $w=z/(1-z)=\sum_{n=1}^{\infty}z^n$ is holomorphic, univalent and convex for $|z|<1$ and maps $|z|<1$ onto the half-plane $\operatorname{Re}(w)>-\frac{1}{2}$. It follows now by a known result [2, p. 12] that $|b_n|\leqq 1$ for all n.

3. In this section we study the functions $g(z)$ which we have used in the last section.

THEOREM 8. *If* $g(z)$ *is holomorphic in* D, $g(0)=1$, *and* $g(z)\neq 0$, $\operatorname{Re}[zg'(z)/g(z)]\geqq-\alpha$, *with* $0\leqq\alpha<1$, *for* $|z|<1$, *then*

$$g(z) = \exp\left[2(\alpha+\rho)\int_0^{2\pi}\log\left(\frac{e^{i\theta}}{e^{i\theta}-z}\right)d\beta(\theta)\right],$$

for some $\rho>0$, *where* $\beta(\theta)$ *is a nondecreasing function of* θ *in* $[0,2\pi]$ *with* $\beta(2\pi)-\beta(0)=1$. (*Here that branch of the logarithm is chosen which vanishes at* $z=0$.) *Also*

$$\frac{1}{(1+r)^{2\alpha}} \leqq |g(re^{i\theta})| \leqq \frac{1}{(1-r)^{2\alpha}} \quad \textit{for } 0\leqq r<1,\ 0\leqq\theta\leqq 2\pi.$$

Proof. For $0\leqq\alpha<1$, we have that $\operatorname{Re}[zg'(z)/g(z)]\geqq-\alpha>-(\alpha+\rho)$ for any $\rho>0$ and hence $\operatorname{Re}[zg'(z)/(\alpha+\rho)g(z)+1]>0$. Let

$$\frac{zg'(z)}{(\alpha+\rho)g(z)}+1 = F(z) = \sum_{n=0}^{\infty}b_nz^n \quad \text{for } |z|<1. \tag{13}$$

It is easy to see that $F(0)=1$ and by Herglotz's representation [8, p. 570] for such functions F we have

$$\frac{zg'(z)}{(\alpha+\rho)g(z)}+1 = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\beta(\theta) \quad \text{for } |z| < 1,$$

where $\beta(\theta)$ is nondecreasing in $[0, 2\pi]$ and $\beta(2\pi)-\beta(0)=1$. Thus

$$\frac{zg'(z)}{g(z)} = (\alpha+\rho)\int_0^{2\pi} \left(\frac{e^{i\theta}+z}{e^{i\theta}-z}-1\right) d\beta(\theta)$$

or,

$$\frac{g'(z)}{g(z)} = 2(\alpha+\rho)\int_0^{2\pi} \frac{1}{e^{i\theta}-z}\,d\beta(\theta).$$

Integrating the last equation from 0 to z, $|z|<1$, with respect to z, we obtain

$$\begin{aligned}\log g(z)-\log g(0) &= 2(\alpha+\rho)\int_0^{z}\int_0^{2\pi} \frac{1}{e^{i\theta}-z}\,d\beta(\theta)\,dz\\ &= 2(\alpha+\rho)\int_0^{2\pi}\int_0^{z} \frac{1}{e^{i\theta}-z}\,dz\,d\beta(\theta)\\ &= 2(\alpha+\rho)\int_0^{2\pi} \log\left(\frac{e^{i\theta}}{e^{i\theta}-z}\right) d\beta(\theta).\end{aligned}$$

Hence,

$$g(z) = \exp\left[2(\alpha+\rho)\int_0^{2\pi} \log\left(\frac{e^{i\theta}}{e^{i\theta}-z}\right) d\beta(\theta)\right], \tag{14}$$

for $|z|<1$. In equation (14) we take that branch of the logarithm in the integral which vanishes at $z=0$.

To prove the other assertion we have from equation (13) that $F(z)=1+\sum_{n=1}^{\infty} b_n z^n$ and $\operatorname{Re}(F(z))>0$ for $|z|<1$ and hence by a known result [1, p. 44]

$$\frac{1-r}{1+r} \leqq \operatorname{Re}(F(re^{i\theta})) \leqq \frac{1+r}{1-r} \quad \text{for } 0 \leqq r < 1,\ 0 \leqq \theta \leqq 2\pi.$$

In terms of g these inequalities become

$$\frac{-2(\alpha+\rho)r}{1+r} \leqq \operatorname{Re}\left(\frac{re^{i\theta}g'(re^{i\theta})}{g(re^{i\theta})}\right) \leqq \frac{2(\alpha+\rho)r}{1-r}, \qquad 0 \leqq r < 1,\ 0 \leqq \theta \leqq 2\pi.$$

Since

$$\operatorname{Re}\left(\frac{re^{i\theta}g'(re^{i\theta})}{g(re^{i\theta})}\right) = r\frac{\partial}{\partial r}\operatorname{Re}(\log g(re^{i\theta})),$$

we have

$$\frac{-2(\alpha+\rho)}{1+r} \leqq \frac{\partial}{\partial r}\log|g(re^{i\theta})| \leqq \frac{2(\alpha+\rho)}{1-r}, \qquad 0 < r < 1,\ 0 \leqq \theta \leqq 2\pi.$$

Integrating with respect to r from 0 to r, we get

$$-2(\alpha+\rho)\log(1+r) \leqq \log|g(re^{i\theta})| \leqq -2(\alpha+\rho)\log(1-r),$$
$$0 < r < 1,\ 0 \leqq \theta \leqq 2\pi.$$

Or,

$$\frac{1}{(1+r)^{2(\alpha+\rho)}} \leqq |g(re^{i\theta})| \leqq \frac{1}{(1-r)^{2(\alpha+\rho)}}, \qquad 0 \leqq r < 1,\ 0 < \theta < 2\pi, \tag{15}$$

since the relations hold trivially for $r=0$. Since inequalities (15) hold for any $\rho>0$, we let $\rho \to 0$ and we finally get

$$\frac{1}{(1+r)^{2\alpha}} \leqq |g(re^{i\theta})| \leqq \frac{1}{(1-r)^{2\alpha}}, \qquad 0 \leqq r < 1,\ 0 \leqq \theta \leqq 2\pi.$$

THEOREM 9. *If* $g(z)$ *is subject to the same conditions as in Theorem* 8 *and* $g(z) = 1+\sum_{n=1}^{\infty} a_n z^n$, *then* $|a_n| \leqq n+1$ *for all positive integral values of* n.

Proof. Choose a positive number ρ such that $\alpha+\rho \leqq 1$, and let

$$\frac{zg'(z)}{(\alpha+\rho)g(z)}+1 = F(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{16}$$

As in the proof of Theorem 8, we have $\mathrm{Re}\,(F(z))>0$ for $|z|<1$, $F(0)=b_0=1$ and hence by a known result of Carathéodory $|b_n| \leqq 2$ for $n=1, 2, 3, \dots$. Substituting in equation (16) the power series expansion for $g(z)$ we obtain

$$\sum_{n=1}^{\infty} na_n z^n = (\alpha+\rho)\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} b_k a_{n-k}\right)z^n \quad \text{with } a_0 = 1.$$

Equating the coefficients of z^n we have

$$na_n = (\alpha+\rho)\sum_{k=1}^{n} b_k a_{n-k} \qquad (n = 1, 2, 3, \dots). \tag{17}$$

For $n=1$, equation (17) gives, $|a_1| = |\alpha+\rho|\ |b_1 a_0| \leqq |b_1 a_0| \leqq 2$. For $n=2$, we have $2|a_2| = |\alpha+\rho|\ |b_1 a_1 + b_2 a_0| \leqq 2|a_1| + 2|a_0| \leqq 4+2=6$ and hence $|a_2| \leqq 3$.

Suppose now that $|a_n| \leqq n+1$ for $1 \leqq n \leqq j$, then again from equation (17) we have

$$\begin{aligned}(j+1)|a_{j+1}| &\leqq |\alpha+\rho| \sum_{k=1}^{j+1} |b_k|\ |a_{j+1-k}| \\ &\leqq 2\sum_{k=1}^{j+1} |a_{j+1-k}| \leqq 2\sum_{k=1}^{j+1} (j+2-k) \\ &= 2[(j+1)+j+\cdots+1] = (j+1)(j+2).\end{aligned}$$

Thus $|a_{j+1}| \leqq j+2$ and the result follows by induction.

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