TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 154, February 1971

BARYCENTERS OF MEASURES ON CERTAIN NONCOMPACT CONVEX SETS

BY
RICHARD D. BOURGIN

Abstract. Each norm closed and bounded convex subset K of a separable dual Banach space is, according to a theorem of Bessaga and Pełczynski, the norm closed convex hull of its extreme points. It is natural to expect that this theorem may be reformulated as an integral representation theorem, and in this connection we have examined the extent to which the Choquet theory applies to such sets. Two integral representation theorems are proved and an example is included which shows that a sharp result obtains for certain noncompact sets. In addition, the set of extreme points of K is shown to be μ-measurable for each finite regular Borel measure μ, hence eliminating certain possible measure-theoretic difficulties in proving a general integral representation theorem. The last section is devoted to the study of a class of extreme points (called pinnacle points) which share important geometric properties with extreme points of compact convex sets in locally convex spaces. A uniqueness result is included for certain simplexes all of whose extreme points are pinnacle points.

Introduction. In 1966 Bessaga and Pełczynski [1] proved that each norm closed and bounded convex subset of a separable dual Banach space is the norm closed convex hull of its extreme points, thus providing a "Krein-Milman" type theorem for a certain collection of noncompact sets. (A more elementary proof of this result has been given in [11].) For compact convex sets (in locally convex spaces) the Krein-Milman theorem can easily be reformulated as an integral representation theorem, using measures which are supported by the closure of the set of extreme points (see for example [13, p. 6]). This formulation has a large number of applications to analysis, probability, etcetera. A much more precise kind of representation theorem is provided by Choquet theory, where the measures (at least in the metrizable case) are supported by the set of extreme points, not just the closure. (See [5] and [2]. For a comprehensive and readable account see [13].) The Bessaga-Pełczynski theorem would probably find greater applicability if it too could be reformulated as an integral representation theorem. As a step towards solving this problem we investigate to what extent the Choquet theory applies to the sets described in the Bessaga-Pełczynski theorem. We take this opportunity to thank Professor R. R. Phelps for directing the author's thesis from which this paper is

Received by the editors October 17, 1969.

AMS 1970 *subject classifications*. Primary 46B99; Secondary 46G99, 28A40.

Key words and phrases. Barycenters of measures, closed bounded convex sets, integral representation theorem, Choquet theory, Bessaga-Pełczynski theorem, finitely additive measures, extreme points, separable dual Banach spaces.

Copyright © 1971, American Mathematical Society

drawn. In addition we wish to thank Professors R. Blumenthal, H. H. Corson, and V. L. Klee for several fruitful conversations.

In §1 we study analogues in the noncompact case of one of the basic tools of Choquet theory—that of a measure representing a point. §2 contains two integral representation theorems for sets including those described in the Bessaga-Pełczynski theorem, with measures supported by certain boundaries of the sets. Also included in this section is a result which states that the set of extreme points of such sets is nearly a Borel set (in a sense which is made precise). The third section is oriented towards describing certain types of closed and bounded convex sets in certain Banach spaces (including separable dual Banach spaces) which share some important geometric properties with compact convex sets in locally convex spaces. As an application of some of the notions described in §3, a uniqueness result is proved for simplexes having such properties.

NOTATION. Henceforth all topological spaces are assumed to be Hausdorff. For any normal topological space D, denote by $C_b(D)$ the Banach space of continuous real-valued bounded functions on D with the supremum norm. The Banach space $[C_b(D)]^*$ may be identified with the space of regular bounded finitely additive measures on the algebra Σ of sets generated by the closed subsets of D. (See for example [7, Theorem 2, p. 262].) Hence the set $\mathscr{P}(D)$ of all such measures which are nonnegative and integrate the identically 1 function to 1 forms a convex set which is weak* compact. (Note that $\mu \in \mathscr{P}(D)$ implies that $\|\mu\| = 1 = \mu(D)$.) For any convex subset C of a linear space, ex C denotes the set of extreme points of C. If C is a subset of a normed linear space, $n(C)$ denotes the norm closure of C.

1. The Bessaga-Pełczynski theorem may be easily reformulated as an integral representation theorem if finitely additive measures are allowed. One such result is the following. (Similar results occur in [3] and [9].)

PROPOSITION 1.1. *The Bessaga-Pełczynski theorem is equivalent to the following statement*:

Let K be a norm closed and bounded convex set in a separable dual Banach space B^*. Then for each x in K there is a μ in $\mathscr{P}(K)$ such that $\mu(f) = f(x)$ for each f in B^{**} and $\mu(n(\text{ex } K)) = 1$.

Because finitely additive measures have such limited use in analysis, a much more significant theorem would result if finitely additive measures could be replaced by countably additive probability measures (and if $n(\text{ex } K)$ could be replaced by ex K) in Proposition 1.1. As a first step towards improving on Proposition 1.1 the notion of a measure representing a point is discussed in the more general context of noncompact sets.

For a compact convex subset G of a locally convex topological vector space (lcs) V the following result [13, Proposition 1.1, p. 4] is well known: If μ is a (countably additive) probability measure on G there is a (necessarily unique) point x in G for which $\mu(f)=f(x)$ for each f in V^*. (The point x is called the resultant or barycenter of μ.) This theorem provides one of the basic tools of Choquet theory, and in extending Choquet theory to deal with noncompact sets it soon becomes evident that corresponding statements are needed. A closed and bounded convex subset K of a separable dual Banach space B^* has three natural topologies: the weak*, weak, and norm topologies. Each of these topologies on B^* determines a set of continuous linear functionals, and it is natural, in defining what is meant when we say that a (countably additive) probability measure μ on K has barycenter x, to distinguish between these cases. There is yet another collection of functions to consider when defining barycenters: the affine continuous functions on K. (Here again we must distinguish between the affine functions on K which are continuous in the weak*, the weak, and the norm topologies on B^* restricted to K.) Definition 1.3 takes into account these various possibilities.

DEFINITION 1.2. Let C be a convex subset of a lcs E. Denote by $A(C)$ the linear space of affine continuous functions on C.

DEFINITION 1.3. Let C be a closed and bounded convex subset of a lcs E. For μ in $\mathscr{P}(C)$ and x in E say that μ weakly [affinely] represents x and write $\mu\sim x$ [$\mu\approx x$] if $\mu(f)=f(x)$ for each f in E^* [$f\in A(C)$]. If E is a dual Banach space (that is $E=F^*$ for some Banach space F) say μ weak* represents x and write $\mu \sim^* x$ if $\mu(f)=f(x)$ for each f in F. (Note that F has been identified with a subspace of $F^{**}=E^*$. We shall henceforth make this identification whenever it is convenient.)

DEFINITION 1.4. If E is a dual Banach space and $C\subset E$ denote by $w^*(C)$ the weak* closure of C.

The next proposition is used often in what follows. It has been proved in more generality in [3]. A (different) proof is included here for the sake of completeness. (Note that the natural language, when dealing with bounded continuous functions on noncompact sets, is that of finitely additive measures. Because of this we sometimes work with finitely additive measures as well as countably additive ones.)

PROPOSITION 1.5. *Let C be a closed and bounded convex subset of a dual Banach space E^*. For each μ in $\mathscr{P}(C)$ there is a (necessarily unique) point x in $w^*(C)$ for which $\mu \sim^* x$. If $\mu\sim x$, then $x\in C$.*

Proof. The set $J=w^*(C)$ is weak* compact and convex and μ defines a continuous linear functional μ_1 on $C(J)$ by the rule: For $g\in C(J)$ let $\mu_1(g)=\mu(g|_C)$. Evidently μ_1 is a countably additive probability measure on J and hence there is a point x in J such that $\mu_1(f)=f(x)$ for each f in E. Thus $\mu(f)=f(x)$ for f in E and hence $\mu \sim^* x$. If $\mu\sim x$ and $x\notin C$ there is an $f\in E^{**}$ for which $\sup f(C)<f(x)$. For this f we have $f(x)=\mu(f)\leqq\sup f(C)<f(x)$, a contradiction.

If D is a normal topological space and $\mu\in[C_b(D)]^*$ is a countably additive measure on the algebra Σ, the Hahn extension theorem (see, for example [7, p. 136]) guarantees that μ has a unique extension to a countably additive measure on the Borel subsets of D. We shall henceforth identify such a measure with its extension.

DEFINITION 1.6. If D is a normal topological space we denote by $\mathscr{P}_c(D)$ the set of countably additive measures in $\mathscr{P}(D)$.

A stronger version of Proposition 1.5 is available when $\mu \in \mathscr{P}_c(C)$. A similar result has been proved in [9] although the proof differs significantly from the one appearing here.

PROPOSITION 1.7. *Let C be a closed and bounded separable convex subset of a Fréchet space E. If $\mu \in \mathscr{P}_c(C)$ there is an x in C for which $\mu \sim x$.*

Proof. A result originally due to Ulam [12] states that a finite Borel measure ν on a complete separable metric space Y is regular with respect to compact subsets, i.e., if P is a Borel subset of Y, then for each $\varepsilon > 0$ there is a compact set $M \subset P$ for which $|\nu|(P \backslash M) < \varepsilon$. (Here $|\nu|$ is the total variation of ν.) Applying this result to the measure μ in $\mathscr{P}_c(C)$, choose a finite or countably infinite sequence $\{V_n\}$ of pairwise disjoint compact subsets of C of positive measure, with $\sum \mu(V_n) = 1$. Let V_n' be the closed convex hull of V_n. Then V_n' is compact and convex. Denote by μ_n the probability measure on V_n' defined by $\mu_n(P) = [\mu(V_n)]^{-1}\mu(V_n \cap P)$ for each Borel set P of V_n'. Then for each n there is a point x_n in V_n' such that $\mu_n \sim x_n$. Moreover, the sequence $\{\sum_{1 \leqq n \leqq k} \mu(V_n)\mu_n\}_{k \geqq 1}$ converges in norm to μ. It is also evident that the sequence $\{\sum_{1 \leqq n \leqq k} \mu(V_n)x_n\}_{k \geqq 1}$ is Cauchy and hence converges to the point $x = \sum \mu(V_n)x_n$. Thus for f in E^* we have $\mu(f) = \sum \mu(V_n)\mu_n(f) = \sum \mu(V_n)f(x_n) = f(x)$ so that $\mu \sim x \in C$.

REMARK 1.8. The only place we used the separability of C in the above proof was in satisfying the hypotheses of Ulam's result. A theorem of Marczewski and Sikorski [10] states that whenever the density character of a metric space Y is a nonmeasurable cardinal, each σ-finite Borel measure on Y is supported by a separable subset. Using this result, the proof of Proposition 1.7 may be easily modified to prove a similar result for closed and bounded convex subsets of a Fréchet space whose density character is a nonmeasurable cardinal.

For G a compact convex subset of a lcs V it is known (see for example [13, Lemma 9.7, p. 68]) that if $\mu \in \mathscr{P}_c(G)$ represents x and if h is an affine upper (lower) semicontinuous function on G then $\mu(h) = h(x)$. We will prove the corresponding result for a collection of noncompact sets in Proposition 1.11. (This proposition will later be applied to prove a uniqueness result about certain noncompact simplexes. See Proposition 3.11.) We need two lemmas.

LEMMA 1.9. *Let h be an affine upper semicontinuous function on a closed convex subset C of a lcs E. Then*

$$h(x) = \inf\{f(x) + a \mid f + a \geqq h, f \in E^*, a \in R\}$$

for each $x \in C$.

Proof. The proof is a standard application of the separation theorem in $E \times R$ and is omitted.

LEMMA 1.10. *Let h be an affine upper semicontinuous function on a closed and bounded separable convex subset C of a Fréchet space E. If $\mu \in \mathscr{P}_c(C)$ then*

$$\mu(h) = \inf\{\mu(g) \mid g \in C_b(C), g \geqq h, \text{ and } g \text{ is concave}\}.$$

Proof. From Lemma 1.9 it follows that

$$h(x) = \inf\{g(x) \mid g \in C_b(C), g \geqq h, \text{ and } g \text{ concave}\}$$

for each x in C. Exactly as in the proof of the first part of Proposition 1.7 observe that μ is regular with respect to compact subsets. The remainder of the proof follows that in Lemma 9.2 of [13, p. 62] using these observations.

PROPOSITION 1.11. *Let C be a closed and bounded separable convex subset of a Fréchet space E. Suppose that h is a bounded upper (lower) semicontinuous affine function on C and that $\mu \in \mathscr{P}_c(C)$ weakly represents x. Then $\mu(h)=h(x)$.*

Proof. If h is lower semicontinuous replace it by $-h$, so assume that h is upper semicontinuous. Exactly as in the proof of Proposition 1.7 there is a finite or countably infinite sequence $\{\mu_n\}$ of measures in $\mathscr{P}_c(C)$ for which $\lim_k \|\mu - \sum_{1 \leqq n \leqq k} a_n \mu_n\| = 0$, each μ_n has compact support, say V_n, and the a_n's are strictly positive real numbers such that $\sum a_n = 1$. The closed convex hull V_n' of V_n is compact and convex and $\mu_n \in \mathscr{P}_c(V_n')$ so that there is a point $x_n \in V_n'$ for which $\mu_n \sim x_n$. Moreover $h|_{V_n'}$ is an upper semicontinuous affine function so that $\mu_n(h) = h(x_n)$. Because h is bounded it follows that $\mu(h) = \sum a_n \mu_n(h) = \sum a_n h(x_n)$. If the sequence $\{\mu_n\}$ is finite we conclude that $\mu(h) = h(\sum a_n x_n)$. It is evident in this case that $\sum a_n x_n = x$ and $\mu(h) = h(x)$ as was to be shown. Thus we shall henceforth deal only with the case of an infinite sequence $\{\mu_n\}$. Since $\{(\sum_{1 \leqq n \leqq k} a_n)^{-1} \sum_{1 \leqq n \leqq k} a_n x_n\}_{k \geqq 1}$ is a Cauchy sequence in C converging to x, it follows that

$$\mu(h) = \sum a_n h(x_n) \leqq h(x).$$

In order to prove the reverse inequality, it suffices to show that for each $\varepsilon > 0$ we have $\mu(h) + \varepsilon > h(x)$. Now given $\varepsilon > 0$ choose g in $C_b(C)$ such that g is concave, $g \geqq h$ and $\mu(g) \leqq \mu(h) + \varepsilon/2$. (This can be done by Lemma 1.10.) Since V_n' is a compact convex metric subset of E, $\mathscr{P}_c(V_n')$ is weak* metrizable and hence μ_n is the weak* limit of a sequence $\{\mu_{n,i}\}_{i \geqq 1}$ of discrete probability measures on V_n' such that $\mu_{n,i} \sim x_n$ for each i [6]. Define, for each positive integer k, the measure $\nu_k \in \mathscr{P}_c(C)$ by $\nu_k = (\sum_{n \geqq k+1} a_n)^{-1} \sum_{n \geqq k+1} a_n \mu_n$. By Proposition 1.6 there is a point $y_k \in C$ such that $\nu_k \sim y$. Define a discrete measure $\lambda_k \in \mathscr{P}_c(C)$ by $\lambda_k = \sum_{1 \leqq n \leqq k} a_n \mu_{n,k} + (\sum_{n \geqq k+1} a_n) \varepsilon_{y_k}$. Evidently $\lambda_k \sim x$ for each k and the weak* limit of $\{\lambda_k\}_{k \geqq 1}$ is μ. It is possible, therefore, to find a discrete probability measure λ in $\mathscr{P}_c(C)$ such that $\lambda \sim x$ and $|\mu(g) - \lambda(g)| < \varepsilon/2$. Writing λ explicitly as

$$\lambda = \sum_{1 \leqq i \leqq l} b_i \varepsilon_{z_i} \quad \left(b_i \geqq 0, \sum_{1 \leqq i \leqq l} b_i = 1, z_i \in C, \sum_{1 \leqq i \leqq l} b_i z_i = x\right)$$

the above inequality becomes $|\mu(g)-\sum_{1\leqq i\leqq l} b_i g(z_i)|<\varepsilon/2$. Since $g(z_i)\geqq h(z_i)$ for $i=1,\ldots,l$ we therefore conclude that

$$\begin{aligned}\mu(h)+\varepsilon/2 &\geqq \mu(g)\\ &> \sum_{1\leqq i\leqq l} b_i g(z_i)-\varepsilon/2\\ &\geqq \sum_{1\leqq i\leqq l} b_i h(z_i)-\varepsilon/2\\ &= h(x)-\varepsilon/2\end{aligned}$$

so that $\mu(h)+\varepsilon>h(x)$ and the proof is complete.

2. In order to obtain a Choquet-type result (with measures supported by ex K) for a set K as in the Bessaga-Pełczynski theorem it would be helpful (and of intrinsic interest) to know whether ex K is a G_δ set or, more generally, a Borel set. While these questions remain open, it is true that ex K is always μ-measurable (in the sense of Carathéodory) for each μ in $\mathscr{P}_c(K)$. Recall that a metrizable subset of a topological space is said to be *analytic* if it is the continuous image of a complete separable metric space (cf. [4, p. 197]).

PROPOSITION 2.1. *Let C be an analytic convex subset of a metric linear space. Then* ex C (*which might be empty*) *is a μ-measurable set* (*in the sense of Carathéodory*) *for each $\mu\in\mathscr{P}_c(C)$.*

Proof. It is clear that the space $F=(0,1)\times C\times C$ is analytic since C is, and that the diagonal $D=\{(t,x,x)\mid t\in(0,1) \text{ and } x\in C\}$ is closed in F. Since C is a separable metric space, it is homeomorphic to a subset (which is necessarily analytic) of a compact metric space M. Using the fact that each analytic subset of M is μ-measurable with respect to each finite Borel measure on M [4, p. 210], it follows that each analytic subset of C is μ-measurable for each μ in $\mathscr{P}_c(C)$. The set $F\backslash D$, being an open subset of an analytic set, is itself analytic, and hence $\psi(F\backslash D)$ is analytic where $\psi\colon F\backslash D\to C$ is the continuous map defined by $\psi(t,x,y)=tx+(1-t)y$. But $\psi(F\backslash D)=C\backslash\text{ex } C$ so that $C\backslash\text{ex } C$, and hence ex C, is μ-measurable for each μ in $\mathscr{P}_c(C)$.

For any point x of K (K as in the Bessaga-Pełczynski theorem) it is obvious that $\varepsilon_x\sim x$ and hence there is a measure in $\mathscr{P}_c(K)$ supported by the trivial boundary—K itself—which weakly represents x. More generally we shall call a subset D of K a *boundary* if, for each x in K, there is a measure $\mu\in\mathscr{P}_c(K)$ such that $\mu\sim x$ and $\mu(D)=1$. The major question of whether or not ex K is a boundary of K remains unanswered except in certain very special cases. (See Example 2.3. Note also that when K is weak* compact, ex K is a boundary by the Choquet theorem, using Proposition 1.7 and the elementary fact—see the proof of Proposition 2.2—that the weak* and norm Borel subsets of K coincide.) However, as is shown in Proposition 2.2, the following subset $b(K)$ of K is a boundary. Let $T_1=w^*(K)\backslash K$ and

$T_2 = w^*(T_1 \cup \text{ex } w^*(K))$. Then $b(K) = T_2 \cap K$. Observe that $b(K) = w^*(\text{ex } K)$ when K is weak* compact and, for certain noncompact sets K, we have $b(K) = \text{ex } K$ (Example 2.3). Consequently Proposition 2.2 does provide a Choquet theorem in some nontrivial cases. (On the other hand, for $K = \{x = (x_i) \in l_1 \mid x_j \geqq 0 \text{ for each } j, \|x\| = 1\}$ it is easy to see that $b(K) = K$ and hence Proposition 2.2 provides no information in this case.)

PROPOSITION 2.2. *Let K be a closed bounded convex subset of a separable dual Banach space B^*. Then $b(K)$ (as defined above) is a boundary for K. That is, for each x in K there is a measure $\mu \in \mathscr{P}_c(K)$ for which $\mu \sim x$ and $\mu(b(K)) = 1$.*

Proof. Since $J = w^*(K)$ (in the weak* topology) is compact convex and metrizable, there is a probability measure $\mu \in [C(J)]^*$ (defined, therefore, on the weak* Borel subsets of J) for which $\mu \sim^* x$ and $\mu(\text{ex } w^*(K)) = 1$. Since $\text{ex } w^*(K) \subset T_2$ the set $H = \{\mu \in [C(J)]^* \mid \mu \text{ is a probability measure, } \mu \sim^* x \text{ and } \mu(T_2) = 1\}$ is nonempty. If μ_1 and μ_2 are measures in H write $\mu_1 \succ \mu_2$ provided $\mu_1(g) \geqq \mu_2(g)$ for each convex function $g \in C(J)$. Because T_2 is weak* compact, a standard Zorn's lemma argument shows that there is a *minimal* $\mu_0 \in H$ with respect to $\succ$. We will show that μ_0 is a probability measure on the Borel subsets of J induced by the norm topology, and that $\mu_0(b(K)) = 1$.

Each norm closed ball of B^* is weak* compact (and is hence a weak* Borel set) and each norm open subset of B^* is a countable union of closed balls. It follows that each norm open set (and hence each norm Borel set) is a weak* Borel set. (The reverse inclusion is immediate.) Thus, considering J in the norm topology we have $\mu_0 \in \mathscr{P}_c(J)$. In particular, the norm closed set K is μ_0-measurable.

Suppose now that $\mu_0(T_2 \backslash K) > 0$. For each point y in $T_2 \backslash K$ let U_y be a closed ball centered at y such that $U_y \cap K = \varnothing$. Because $T_2 \backslash K$ (in the norm topology) is a separable metric space there is a countable subcollection of the U_y's covering $T_2 \backslash K$. Since μ_0 is countably additive there is a point $y_0 \in T_2 \backslash K$ for which (denoting U_{y_0} by U_0) $\mu_0(U_0 \cap J) = \alpha > 0$. Let $\nu_1 \in \mathscr{P}_c(J)$ be defined by

$$\nu_1(P) = \alpha^{-1} \mu_0(P \cap U_0 \cap J)$$

for each Borel subset P of J. Since $U_0 \cap J$ is a norm closed and bounded convex set, Proposition 1.7 guarantees that there is a point $y_1 \in U_0 \cap J \subset J \backslash K \subset T_2$ such that $\nu_1 \sim y_1$. Now define $\nu_2 \in \mathscr{P}_c(J)$ by $\nu_2(P) = \mu_0(P \backslash U_0) + \alpha \varepsilon_{y_1}(P)$. Evidently $\nu_2 \sim x$ and $\nu_2(T_2) = 1$ and each weak* continuous convex function g on $U_0 \cap J$ satisfies the inequality $g(y_1) = \varepsilon_{y_1}(g) \leqq \nu_1(g)$ [13, p. 25]. It follows from this inequality that $\mu_0 \succ \nu_2$. Hence $\mu_0 = \nu_2$.

To complete the proof it suffices to produce a measure $\nu \in H$ such that $\nu \neq \mu_0$ and $\mu_0 \succ \nu$. Evidently $\alpha < 1$ since otherwise $\mu_0 \sim y_1 \neq x$, a contradiction. Defining $\nu_3 \in \mathscr{P}_c(J)$ by $\nu_3(P) = (1-\alpha)^{-1} \mu_0(P \backslash \{y_1\})$, Proposition 1.9 shows that there is a point $x_1 \in J$ such that $\nu_3 \sim x_1$ and hence $x = \alpha y_1 + (1-\alpha) x_1$. Since $y_1 \notin K$ there is a point $z \neq y_1$ on the segment between x and y_1 which is not in K (hence is in $T_2 \backslash K$)

and can be written as $z=\alpha_0 y_1+(1-\alpha_0)x_1$ with $\alpha<\alpha_0<1$. It is readily checked that $x=\alpha\alpha_0^{-1}z+(1-\alpha\alpha_0^{-1})x_1$ and that the measure $\nu\in\mathscr{P}_c(J)$ defined by $\nu(P)=\alpha\alpha_0^{-1}\varepsilon_z(P)+(1-\alpha\alpha_0^{-1})\nu_3(P)$ satisfies the conditions $\nu_3(T_2)=1$ and $\nu_3\sim x$. From $\nu(\{y_1\})=0$ it follows that $\nu\neq\mu_0$, and the proof will be completed by showing that $\mu_0\succ\nu$; that is, if g is a convex function in $C(J)$ then we must show that $\nu(g)\leqq\mu_0(g)$. Recall the earlier expressions for μ_0 and ν; then this inequality becomes

$$\alpha\alpha_0^{-1}g(z)+(1-\alpha\alpha_0^{-1})\nu_3(g) \leqq \alpha g(y_1)+(1-\alpha)\nu_3(g)$$

and a further rewriting reduces the problem to showing that

$$(1-\alpha_0)^{-1}g(z)-\alpha_0(1-\alpha_0)^{-1}g(y_1) \leqq \nu_3(g).$$

But we have $g(z)\leqq\alpha_0 g(y_1)+(1-\alpha_0)g(x_1)$ since g is convex and hence

$$(1-\alpha_0)^{-1}g(z)-\alpha_0(1-\alpha_0)^{-1}g(y_1) \leqq g(x_1) \leqq \nu_3(g)$$

as was to be shown.

EXAMPLE 2.3. As mentioned earlier, examples may be constructed of subsets K of l_1 which are norm closed bounded and convex but not weak* closed for which $b(K)=\text{ex }K$. These examples are all subsets of Q, the positive cone of l_1, and are of the form $\{x\in Q \mid f(x)\geqq 1\}\cap K_1$ for certain continuous real-valued concave functions f on Q and norm closed halfspaces K_1. (Of course by l_1 we mean the Banach space of absolutely summable sequences of real numbers with $\|(x_i)\|=\sum_{i\geqq 1}|x_i|$. Then $l_1=c_0^*$ where c_0 is the Banach space of sequences of real numbers converging to zero, with the supremum norm.) We will need the following lemma.

LEMMA 2.4. *Let $\{f_i\}_{i\geqq 1}$ be a sequence of continuous real-valued nonnegative functions on* $[0,\infty)$ *with the property that $(f_i(x_i))\in Q$ whenever $(x_i)\in Q$. (Examples of such sequences are given in Lemma* 2.5.) *For $x=(x_i)\in Q$ define $f(x)=\sum_{i\geqq 1}f_i(x_i)$. Then f is continuous on Q.*

Proof. Since f is the supremum of the sequence $\{\sum_{1\leqq i\leqq n}f_i\}_{n\geqq 1}$ the function f is lower semicontinuous. Suppose now that f is not upper semicontinuous. Then there is a sequence $\{x^n\}_{n\geqq 1}=\{(x_i^n)\}_{n\geqq 1}$ in Q and $x=(x_i)\in Q$ for which $\|x^n-x\|\to 0$ and for some $r\in R$ and $\varepsilon>0$ we have $f(x^n)\geqq r$ for each n, and $f(x)<r-\varepsilon$. First observe that this supposition implies that for each positive integer M there is a positive integer N such that

$$\text{(I)}\qquad \sum_{i\geqq M}|f_i(x_i^n)-f_i(x_i)| > \varepsilon/2 \quad \text{provided } n \geqq N.$$

A contradiction is now obtained by using (I) to construct a point $y=(y_i)\in Q$ such that $(f_i(y_i))\notin l_1$. Choose N_1 so that (denoting N_1+1 by a)

(1) $\sum_{i\geqq a}f_i(x_i)<\varepsilon/8$, and $\sum_{i\geqq a}x_i<\frac{1}{4}$ and $\|x^n-x\|<\frac{1}{4}$ for $n\geqq N_1$, hence

(2) $\sum_{i\geqq a}x_i^n<2^{-1}$ for each $n\geqq N_1$.

Using (I) choose $n_1\geqq N_1$ such that $\sum_{i\geqq a}|f_i(x_i^{n_1})-f_i(x_i)|>\varepsilon/2$. It follows that for some k_1 we have (denoting N_1+k_1 by b) the inequality $\sum_{a\leqq i\leqq b}|f_i(x_i^{n_1})-f_i(x_i)|>\varepsilon/2$. Let $y_1=0,\ldots,y_{N_1}=0, y_a=x_a^{n_1},\ldots,y_b=x_b^{n_1}$. Then $y_i\geqq 0$, $\sum_{1\leqq i\leqq b}y_i<2^{-1}$ by (2) and $\sum_{1\leqq i\leqq b}f_i(y_i)\geqq\sum_{a\leqq i\leqq b}[|f_i(x_i^{n_1})-f_i(x_i)|-f_i(x_i)]>\varepsilon/2-\varepsilon/8>\varepsilon/4$. Suppose now that y_i has been chosen for $i=1,\ldots,N_1+k_1,\ldots,N_2+k_2,\ldots,N_j+k_j$ so that $y_i\geqq 0$ for each i and (letting $b'=N_j+k_j$) $\sum_{1\leqq i\leqq b'}y_i<2^{-1}+2^{-2}+\cdots+2^{-j}=1-2^{-j}$ and $\sum_{1\leqq i\leqq b'}f_i(y_i)>j\varepsilon/4$. Choose $N_{j+1}>N_j+k_j$ so that (for $a''=N_{j+1}+1$) we have

(1′) $\sum_{i\geqq a''}f_i(x_i)<\varepsilon/8$ and

(2′) $\sum_{i\geqq a''}x_i^n<2^{-(j+1)}$ for $n\geqq N_{j+1}$.

Then choose k_{j+1} as was k_1 and (letting $b''=N_{j+1}+k_{j+1}$) choose $y_{b'+1},\ldots,y_{N_{j+1}}$, $y_{a''},\ldots,y_{b''}$ as were $y_1,\ldots,y_{N_1},y_a,\ldots,y_b$. It is immediate that $(y_i)\in Q$ but that $\sum_{i\geqq 1}f_i(y_i)=\infty$. Thus f is upper semicontinuous and the lemma follows.

Suppose that the functions f_i satisfy the further conditions that for each i, $f_i(0)=0$, that each f_i is strictly concave and that $\{f_i^{-1}(1)\}_{i\geqq 1}$ is a bounded subset of R. It follows immediately that f is strictly concave and $f(0)=0$ so that $f^{-1}[1,\infty)$ is a closed convex subset of Q not containing 0. Observe next that $\{f_i^{-1}(1)\}_{i\geqq 1}$ is bounded away from 0. (Indeed, if this were not the case a subsequence $\{f_{n_i}^{-1}(1)\}_{i\geqq 1}$ could be constructed for which $\sum_{i\geqq 1}f_{n_i}^{-1}(1)<\infty$. We are led to the contradiction $f(x)=\infty$ where $x=(x_i)\in Q$ is the point with $x_{n_i}=f_{n_i}^{-1}(1)$ and $x_j=0$ whenever $j\neq n_i$ for each i.) Let $\alpha_i=f_i^{-1}(1)$, and from above it follows that $(\alpha_i^{-1})\in l_\infty$. The set K_1 mentioned at the beginning of this example is defined to be $K_1=\{(x_i)\in l_1 \mid \sum\alpha_i^{-1}x_i\leqq 1\}$. Finally, let $K=f^{-1}[1,\infty)\cap K_1$. Because $\{f_i^{-1}(1)\}_{i\geqq 1}$ is bounded, the points $f_i^{-1}(1)\delta_i\in K$ converge in the weak* topology to 0 so that K is not weak* closed. (Here δ_i is the sequence with 1 in the ith position and 0 elsewhere.) From the strict concavity of the f_i's and the definition of K it follows that $\operatorname{ex}K=f^{-1}(1)$. If $x=(x_i)\in K$ and $f(x)>1$, there is an integer N such that $\sum_{1\leqq i\leqq N}f_i(x_i)>1$. It is an elementary matter to check that $w^*(K)=K_1\cap Q$ and hence $w^*(K)\backslash K=f^{-1}[0,1)$. Thus there is no point $(y_i)\in(w^*(K)\backslash K)\cup\operatorname{ex}w^*(K)$ for which $\sum_{1\leqq i\leqq N}f_i(y_i)>1$, so that $x\notin b(K)$. Hence $b(K)=\operatorname{ex}K$. (There are, of course, many possible modifications in the above construction which yield sets K with similar properties.)

This example is concluded with the construction of one possible collection of f_i's. We wish to thank Mr. J. Deeter for pointing out the elementary proof of this next lemma.

LEMMA 2.5. *Let $\{\theta_i\}_{i\geqq 1}$ be a summable sequence of real numbers such that $0<\theta_i<1$ for each i, and let $\lambda_i=1-\theta_i$. Then the functions $f_i(x)=x^{\lambda_i}$ satisfy the conditions of the construction.*

Proof. The only part requiring checking is that $\sum_{i\geqq 1}x_i^{\lambda_i}<\infty$ whenever $(x_i)\in Q$. Let $A=\{i\mid x_i\geqq\theta_i\}$ and $B=\{i\mid x_i<\theta_i\}$. Now

$$\begin{aligned}\sum_{i\in A}x_i^{\lambda_i}&=\sum_{i\in A}x_ix_i^{-\theta_i}\\&\leqq\sum_{i\in A}x_i\theta_i^{-\theta_i}\leqq 3\sum_{i\in A}x_i<\infty.\end{aligned}$$

(The number 3 arises from the inequality $x^{-x} \leqq e^{1/e} < 3$ for $x > 0$.) Also

$$\sum_{i \in B} x_i^{\lambda_i} \leqq \sum_{i \in B} \theta_i^{\lambda_i} < 3 \sum_{i \in B} \theta_i < \infty$$

and putting these inequalities together we obtain the desired result.

In the proof of Proposition 2.2 we showed that a minimal measure on a certain weak* compact set T_2 was in fact supported by $K \cap T_2$. It is clear that whenever D is a weak* compact subset of $w^*(K)$ such that $D \supset \text{ex } w^*(K)$, there are always minimal measures on D representing a given point of K. The difficulty arises in showing that such a measure is supported by $K \cap D$. This difficulty was overcome in Proposition 2.2 by using the fact that for each point y of $T_2 \backslash K$ there is a convex (norm) neighborhood of y entirely contained in $T_2 \backslash K$, thus utilizing a type of "local convexity" for $T_2 \backslash K$. Moreover, examples may be given to show that without some sort of "local convexity" for $D \backslash K$ (D a weak* compact set containing ex $w^*(K)$) it need not be true that minimal measures on D are supported by $D \cap K$. (This eliminates the possibility, then, of adapting the proof of Proposition 2.2 to prove an integral representation theorem for the sets K as in the Bessaga-Pełczynski theorem, with measures supported by $w^*(\text{ex } K) \cap K$.)

Now relax the condition that a boundary of K be a subset of K, and require only that it be a subset of some compactification of K. Under this more general notion of boundary we are able to prove the following representation theorem.

Proposition 2.6. *Let C be a bounded convex subset of a normed linear space N and assume that C is the norm closed convex hull of* ex C. *Let βC denote the Stone-Čech compactification of C considered as a subset of N in the weak topology (induced by N^*). Then for each point x of C there is a probability measure μ in* $[C(\beta C)]^*$ *such that $\mu(f') = f(x)$ for each f in N^* (where f' is the extension of $f|_C$ to an element of $C(\beta C)$) and $\mu(P) = 0$ whenever P is a Borel subset of βC disjoint from $(\beta C \backslash C) \cup$ (weak closure* ex C).

Proof. Let χ denote the canonical embedding of N in N^{**}. The set $w^*(\chi C)$ is a weak* compact convex subset of N^{**} and hence there is a maximal probability measure μ_1 on $w^*(\chi C)$ such that $\mu_1(f) = f(x)$ for each f in N^*. Denote by X the linear subspace of $C(\beta C)$ of functions f such that $f|_C$ is uniformly continuous (in the relative uniformity on C induced by the weak uniformity on N). For f in X the function $f|_C \circ \chi^{-1}$ defined on χC is then uniformly continuous (in the relative uniformity on χC induced by the weak* uniformity on N^{**}). Since $f|_C \circ \chi^{-1}$ is bounded on χC it has a unique extension to a continuous function on $w^*(\chi C)$, henceforth denoted by $^{\#}f$. Define the linear functional L_1 on X by $L_1(f) = \mu_1(^{\#}f)$. Then $L_1(1) = 1$ and $L_1 \geqq 0$ so that $\|L_1\| = 1$. Extend L_1 to a linear functional L on $C(\beta C)$ with $\|L\| = \|L_1\| = 1 = L(1)$. Then L corresponds to a probability measure $\mu \in [C(\beta C)]^*$. For f in N^* it follows that $\mu(f') = L(f') = L_1(f')$ since $f|_C$ is weakly uniformly continuous. But $L_1(f') = \mu_1(^{\#}f') = \mu_1(f) = f(x)$. Hence $\mu(f') = f(x)$ for each f in N^*.

It remains to be shown that each Borel subset P of βC disjoint from $(\beta C\backslash C) \cup$ (weak closure ex C) has μ-measure 0. If $\mu(P)>0$ for some such P, the regularity of μ guarantees the existence of a compact subset M of P such that $\mu(M)>0$. It is evident that M is a weakly compact subset of C and hence there is a $y_0 \in M$ such that the intersection with M of each weak neighborhood of y_0 has positive measure. It is possible therefore [8, p. 7] to find a weakly uniformly continuous bounded nonnegative function f_0 on C such that $f_0(y_0)=1$ and $f_0(z)=0$ whenever $z \in$ (weak closure ex C). Hence $f_0' \in C(\beta C)$ satisfies the conditions $f_0' \geqq 0$ and

$$\mu(\{y \in \beta C \mid f_0'(y) > \tfrac{1}{2}\}) \geqq \mu(\{y \in M \mid f_0(y) > \tfrac{1}{2}\}) > 0.$$

It follows that $\mu(f_0')>0$.

On the other hand $w^*(\chi(\text{ex } C))=w^*(\text{ex } \chi C)\supset \text{ex } w^*(\chi C)$ by the Milman converse of the Krein-Milman theorem. The function f_0 is weakly uniformly continuous on C so that $f_0' \in X$. Since $f_0 \circ \chi^{-1}(\chi(\text{ex } C))=\{0\}$ we conclude that ${}^{\#}f_0'$ is identically 0 on ex $w^*(\chi C)$ and hence on $w^*(\text{ex } w^*(\chi C))$. But μ_1 is supported by $w^*(\text{ex } w^*(\chi C))$ and hence $\mu_1({}^{\#}f_0')=0$. This yields the contradiction $0=\mu_1({}^{\#}f_0')=L_1(f_0')=\mu(f_0')>0$. Thus $\mu(M)=0$ and hence $\mu(P)=0$ as was to be shown.

3. For a compact convex subset G of a lcs V it is well known [6] that each μ in $\mathscr{P}_c(G)$ is the weak* limit of a net of discrete probability measures on G, each of whose barycenters coincides with that of μ. (In the proof of Proposition 1.11 it was shown that, in fact, a similar result holds for a measure $\mu \in \mathscr{P}_c(C)$ where C is a closed and bounded separable convex subset of a Fréchet space.) We shall be concerned here with the more general problem of determining geometric conditions on C under which a measure μ in $\mathscr{P}(C)$ which weakly represents a point x of C is the weak* limit of a net of discrete measures $(\nu_\gamma)_{\gamma\in\Gamma}$ in $\mathscr{P}_c(C)$ such that $\nu_\gamma \sim x$ for each γ in Γ. (Note that for any normal topological space D, it is easy to see that $\mathscr{P}(D)=w^*(\text{co }(\{\varepsilon_y \mid y \in D\}))$ and hence each $\mu \in \mathscr{P}(D)$ is the weak* limit of *some* net of discrete probability measures on D.)

Our first result says, in effect, that we can under quite general circumstances do the next best thing to requiring that the resultants of the ν_γ's coincide with that of μ.

PROPOSITION 3.1. *Let C be a normal subset of a lcs E (with topology $\mathfrak{J}$) and suppose that C is bounded and convex. Let $\mu \in \mathscr{P}(C)$ and suppose that $\mu \sim x$ for some x in C. Then there is a net $(\nu_\gamma)_{\gamma\in\Gamma}$ of discrete measures in $\mathscr{P}_c(C)$ such that if x_γ is the resultant of ν_γ it follows that*

(1) *$(\nu_\gamma)_{\gamma\in\Gamma}$ converges weak* to μ; and*

(2) *$(x_\gamma)_{\gamma\in\Gamma}$ converges (in the $\mathfrak{J}$ topology) to x.*

Proof. Let βC denote the Stone-Čech compactification of C (C in the $\mathfrak{J}$ topology). Then μ corresponds to a (countably additive) probability measure μ' in $[C(\beta C)]^*$ by the rule $\mu'(f)=\mu(f|_C)$ for each f in $C(\beta C)$, and hence μ' lies in the weak* closure of co $(\{\varepsilon_y \mid y \in \beta C\})$. Let $(U_\lambda)_{\lambda\in\Lambda}$ be a downward directed convex neighborhood base of $0 \in [C(\beta C)]^*$ in the weak* topology. For each λ in Λ choose

$\mu_\lambda \in \text{co}(\{\varepsilon_y \mid y \in \beta C\})$ such that $u_\lambda \in \mu' + U_\lambda$. Now each μ_λ is of the form $\mu_\lambda = \sum_{1 \leq i \leq n} a_i \varepsilon_{y_i}$ where $a_i > 0$, $\sum_{1 \leq i \leq n} a_i = 1$, and $\{y_i\}_{1 \leq i \leq n} \subset \beta C$. Since $\{\varepsilon_y \mid y \in C\}$ is weak* dense in $\{\varepsilon_y \mid y \in \beta C\}$ for each i there is a point $x_i \in C$ such that $\varepsilon_{x_i} \in \varepsilon_{y_i} + U_\lambda$. Since U_λ is convex if we let $\nu_\lambda = \sum_{1 \leq i \leq n} a_i \varepsilon_{x_i}$, then $\nu_\lambda \in \mu_\lambda + U_\lambda \subset \mu' + U_\lambda + U_\lambda$. But for each α in Λ there is a λ in Λ such that $U_\lambda + U_\lambda \subset U_\alpha$, and hence the net $(\nu_\alpha)_{\alpha \in \Lambda}$ converges weak* to μ'.

Each f in E^* is bounded on C, hence let f' denote the extension of $f|_C$ to βC. It is clear that $\lim_\lambda \nu_\lambda(f) = \mu'(f') = \mu(f) = f(x)$. Letting $z_\lambda \in C$ denote the barycenter of ν_λ this shows that $(z_\lambda)_{\lambda \in \Lambda}$ converges in the weak topology on E (induced by E^*) to x. Hence each subnet of $(z_\lambda)_{\lambda \in \Lambda}$ converges weakly to x. Let $(V_\delta)_{\delta \in \Delta}$ be a downward directed $\mathfrak{J}$-neighborhood base of x in E. For each $\lambda \in \Lambda$, the net $(z_\alpha)_{\alpha \succ \lambda}$ converges weakly to x and by the Hahn-Banach theorem there is, therefore, a net $(\nu_{\lambda,\delta})_{\delta \in \Delta}$ in co $\{\nu_\alpha \mid \alpha \in \Lambda, \alpha \succ \lambda\}$ such that the resultant of $\nu_{\lambda,\delta}$ is in V_δ. Let Γ denote the cartesian product $\Lambda \times \Delta$, and observe that Γ is a directed set under the ordering $(\lambda_1, \delta_1) \gg (\lambda, \delta)$ if $\lambda_1 \succ \lambda$ and $\delta_1 \succ \delta$. It is easy to check that the net $(\nu_\gamma)_{\gamma \in \Gamma}$ satisfies (1) and (2) of the proposition.

COROLLARY 3.2. *Let C be a normal bounded convex subset of a lcs E and suppose that $\mu \in \mathscr{P}(C)$ weakly represents some point x in C. Then $\mu(h) = h(x)$ for each bounded function h in $A(C)$. In particular $\mu \approx x$ whenever $A(C) \subset C_b(C)$.*

Note that if C is a closed and bounded convex subset of a Fréchet space, then $A(C) \subset C_b(C)$. Indeed, if $h \in A(C)$ is unbounded (say, from above) then for each n choose a point $x_n \in C$ for which $h(x_n) \geqq n$. Let $\{\alpha_n\}_{n \geqq 1}$ be a sequence of nonnegative real numbers such that $\sum_{n \geqq 1} \alpha_n = 1$ and $\sum_{n \geqq 1} n\alpha_n = \infty$. Because C is complete the point $\sum_{n \geqq 1} \alpha_n x_n = x$ belongs to C, and $h(x) = \sum_{n \geqq 1} \alpha_n h(x_n) \geqq \sum_{n \geqq 1} n\alpha_n = \infty$, a contradiction.

Proof of Corollary 3.2. Let $(\nu_\gamma)_{\gamma \in \Gamma}$ be a net of discrete measures in $\mathscr{P}_c(C)$ converging in the weak* sense to μ, such that their respective resultants x_γ converge to x. If $h \in A(C)$ is bounded (and hence $h \in C_b(C)$) then

$$\mu(h) = \lim_\gamma \nu_\gamma(h) = \lim_\gamma h(x_\gamma) = h(x).$$

COROLLARY 3.3. *Let C be a bounded convex subset of a normed linear space E. If x is an interior point of C and if $\mu \in \mathscr{P}(C)$ weakly represents x, then there is a net $(\nu_\gamma)_{\gamma \in \Gamma}$ of discrete measures in $\mathscr{P}_c(C)$ such that*

(1) *$(\nu_\gamma)_{\gamma \in \Gamma}$ converges in the weak* sense to μ; and*

(2) *$\nu_\gamma \sim x$ for each γ in Γ.*

Proof. Choose a net $(\mu_\delta)_{\delta \in \Delta}$ of discrete measures in $\mathscr{P}_c(C)$ converging weak* to μ, whose resultants form a net $(x_\delta)_{\delta \in \Delta}$ converging in norm to x. Since x is an interior point it may be assumed that the points x_δ are all inside some fixed ball centered at x, and hence $2x - x_\delta \in C$ for each $\delta \in \Delta$. For each positive integer k, let $(k+1)x - kx_\delta$ be denoted by $y_{\delta,k}$, and let $\varepsilon_{\delta,k}$ denote the point mass at $y_{\delta,k}$. For each

δ in Δ define, inductively, a sequence $\{\mu_{\delta,n}\}_{n\geqq 1}$ of measures as follows: Let $\mu_{\delta,1}=\frac{1}{2}(\mu_\delta+\varepsilon_{\delta,1})$ and let $\mu_{\delta,k}=(k+1)^{-1}(k\mu_\delta+\varepsilon_{\delta,k})$ if $y_{\delta,k}\in C$ and $\mu_{\delta,k}=\mu_{\delta,k-1}$ otherwise. Using the fact that for each k the net $((k+1)x-kx_\delta)_{\delta\in\Delta}$ converges to x, it follows that the net $(\mu_{\delta,k})_{(\delta,k)\in\Delta\times N}$ (N the integers) converges in the weak* sense to μ. Furthermore, for any $h\in A(C)$ we have $\mu_{\delta,k}(h)=k(k+1)^{-1}h(x_\delta)+(k+1)^{-1}h(y_{\delta,k})=h(x)$ if $y_{\delta,k}\in C$; otherwise $\mu_{\delta,k}(h)=\mu_{\delta,k-1}(h)$. Since $\mu_{\delta,1}(h)=h(x)$, clearly $\mu_{\delta,k}\sim x$ for each $(\delta,k)\in\Delta\times N$ so that for $\Gamma=\Delta\times N$ conditions (1) and (2) are satisfied.

The next definition and proposition have exact analogies in the compact case.

DEFINITION 3.4. For C a normal convex subset of a lcs E and $f\in C_b(C)$ we define the upper envelope $\bar f$ of f on C by $\bar f(x)=\inf\{h(x)\mid h\geqq f \text{ on } C,\ h\in A(C)\}$ for each x in C.

PROPOSITION 3.5. *For C a normal convex subset of a lcs and for f in $C_b(C)$, the function $\bar f$ is concave, bounded, and upper semicontinuous; moreover $f\leqq\bar f$ and $f=\bar f$ iff f is concave. Finally, $\bar f(x)=\sup\{\mu(f)\mid \mu\in\mathscr{P}(C),\ \mu\sim x\}$ for each x in C.*

Proof. The proof follows that in [13, p. 19, and Proposition 3.1, p. 21] using finitely additive instead of countably additive measures.

Suppose that for some point x in C (C a normal convex subset of a lcs E) each $\mu\in\mathscr{P}(C)$ for which $\mu\sim x$ is the weak* limit of a net $(\nu_\gamma)_{\gamma\in\Gamma}$ of discrete measures in $\mathscr{P}_c(C)$ such that $\nu_\gamma\sim x$ for each γ in Γ. Then for each function f in $C_b(C)$ it follows that

$$(*)\qquad \bar f(x)=\sup\{\mu(f)\mid \mu \text{ discrete in } \mathscr{P}_c(C),\ \mu\sim x\}.$$

(Note that Corollary 3.3 shows that each interior point of a bounded convex subset of a normed linear space is such a point.) Such a description of $\bar f$ has several applications (see for example, Corollary 3.8; see also Proposition 3.11). A second class of points for which (*) holds for a large collection of functions f in $C_b(C)$ is described by the next proposition. A definition is needed first and in order to motivate it observe that, for G a compact convex subset of a lcs, a point x of G is extreme if and only if x belongs to the closure of $\{x_\gamma\}_{\gamma\in\Gamma}\cup\{y_\gamma\}_{\gamma\in\Gamma}$ whenever $(x_\gamma)_{\gamma\in\Gamma}$ and $(y_\gamma)_{\gamma\in\Gamma}$ are nets in G (with the same directed set Γ) such that the net $(\frac{1}{2}x_\gamma+\frac{1}{2}y_\gamma)_{\gamma\in\Gamma}$ converges to x. This characterization is no longer valid in noncompact sets (see Example 3.9 below) and it turns out to be helpful to identify those extreme points for which it is satisfied. This will be done with the aid of the next definition.

DEFINITION 3.6. Let C be a closed and bounded convex subset of a dual Banach space E^*. A point $x\in\operatorname{ex} C$ is a pinnacle point if whenever $(x_\gamma)_{\gamma\in\Gamma}$ and $(y_\gamma)_{\gamma\in\Gamma}$ are nets in C (with the same directed set Γ) for which the net $(\frac{1}{2}x_\gamma+\frac{1}{2}y_\gamma)_{\gamma\in\Gamma}$ converges in the weak topology on E^* (induced by E^{**}) to x, it follows that

$$x\in w^*(\{x_\gamma\}_{\gamma\in\Gamma}\cup\{y_\gamma\}_{\gamma\in\Gamma}).$$

Recall that for a normal topological space D, the set Σ is the algebra of sets generated by the closed subsets of D.

PROPOSITION 3.7. *Let E^* be the dual of a separable Banach space and suppose that $C \subset E^*$ is a closed and bounded convex set. Let $x_0 \in \operatorname{ex} C$. Then the following are equivalent*:

(1) *x_0 is a pinnacle point*;

(2) *if $\mu \in \mathscr{P}(C)$ satisfies $\mu \sim x_0$ then for each set $S \in \Sigma$ for which $\mu(S) > 0$ it follows that $x_0 \in w^*(S)$*;

(3) *$\bar{f}(x_0) = f(x_0)$ for each function f defined and weak* continuous on $w^*(C)$.*

Proof. We will prove (2) $\Rightarrow$ (3) and not (2) $\Rightarrow$ not (1) $\Rightarrow$ not (2) $\Rightarrow$ not (3).

(2) $\Rightarrow$ (3). Suppose that (2) holds and let $f \in C(w^*(C))$. Given $\varepsilon > 0$ choose $S \in \Sigma$ to be a relative weak* neighborhood of x_0 in C for which $y \in S$ implies $|f(y) - f(x_0)| < \varepsilon$. Then $x_0 \notin w^*(C \backslash S)$ so that $\mu(C \backslash S) = 0$. It follows that $\int_C f\, d\mu = \int_S f\, d\mu$ and hence $|f(x_0) - \int_C f\, d\mu| \leqq \int_C |f(x_0) - f(y)|\, d\mu \leqq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, clearly $f(x_0) = \int_C f\, d\mu$. Thus $f(x_0) = \bar{f}(x_0)$ follows now using Proposition 3.5.

not (2) $\Rightarrow$ *not* (1). First of all a norm closed convex set A in Σ is constructed such that $C \backslash A$ is convex, $\mu(A) > 0$, and $x_0 \notin w^*(A)$. Let D' denote the image in E^{***} of a subset D of E^*. There is a measure μ_1' on $w^*(C')$ corresponding to μ, and because $w^*(C')$ is compact, μ_1' is countably additive. It is shown that $0 < \mu_1'(w^*(A')) < 1$ and the nets contradicting (1) are obtained by working with the resultants of the normalized restrictions of the measure μ_1' to $w^*(A')$ and to $w^*(C') \backslash w^*(A')$. The detailed argument follows.

Choose S in Σ for which $\mu(S) > 0$ and $x_0 \notin w^*(S)$. For each point y of $w^*(S)$ let U_y be a weak* closed convex neighborhood of y in E^* such that $x_0 \notin U_y$. Since $S \subset C$ is a bounded set, $w^*(S)$ is weak* compact and hence admits a finite subcover $U_{y_1}, \ldots, U_{y_n}$. Since $U_{y_i} \cap C \in \Sigma$ and since $\mu(S) > 0$, one of these sets has positive measure, say $\mu(U_{y_1} \cap C) > 0$. Choose h in E and l and l' in R for which $h(x_0) > l > l' > \sup h(U_{y_1})$. Let

$$A_1 = h^{-1}(-\infty, l'] \cap C$$

and

$$A = h^{-1}(-\infty, l] \cap C.$$

Then $x_0 \notin w^*(A)$, A is convex, norm closed and bounded, $C \backslash A$ is convex, and $\mu(A) \geqq \mu(A_1) > 0$.

Denote by χ the natural embedding of E^* in E^{***}, and let $D' = \chi D$ for each subset D of E^*. We now construct the measure μ_1' on $w^*(C')$. Let $\mu' = \mu \circ \chi^{-1}$. Then μ' is a finitely additive measure on C' and since χ^{-1} is an affine (uniform) homeomorphism between C' in the weak* topology and C in the weak topology, each weak* continuous bounded function on C' is μ'-integrable. Define μ_1' on $C(w^*(C'))$ by the rule $\mu_1'(g) = \mu'(g|_{C'})$ for each g in $C(w^*(C'))$. Then $\mu_1' \geqq 0$ and $\mu_1'(1) = 1$; hence μ_1' is a (countably additive) probability measure on $w^*(C')$.

The next step is to show that $0 < \mu_1'(w^*(A')) < 1$. Since $w^*(C)$ is a weak* compact set and h (used in the definition of A) is weak* continuous, Urysohn's lemma guarantees that there is a weak* continuous function g on w^*C $(0 \leqq g \leqq 1)$ such that $g=1$ on $h^{-1}(-\infty, l'] \cap w^*(C)$ and $g=0$ on $w^*(C) \cap h^{-1}[l, \infty)$. The function g is weak* uniformly continuous on $w^*(C)$ and hence on C and consequently $g|_C$ is a weakly uniformly continuous function on C. Thus $g'=g|_C \circ \chi^{-1}$ is uniformly continuous on C' in the weak* topology and there is a (unique) function g_1' in $C(w^*(C'))$ which extends g'. Since $\{y \in C' \mid g'(y) \neq 0\} \subset A'$ it follows that

$$\{y \in w^*(C') \mid g_1'(y) \neq 0\} \subset w^*(A')$$

and hence $\int_{w^*(C')} g_1'\, d\mu_1' = \int_{w^*(A')} g_1'\, d\mu_1'$. Referring to the definition of μ_1' we have

$$\int_{w^*(C')} g_1'\, d\mu_1' = \int_{C'} g_1'|_{C'}\, d\mu' = \int_{C'} g'\, d\mu' = \int_C g|_C\, d\mu \geqq \int_{A_1} g\, d\mu = \mu(A_1) > 0.$$

It follows that $\int_{w^*(A')} g_1'\, d\mu_1' > 0$ and thus $\mu_1'(w^*(A')) > 0$.

We show next that $\mu_1'(w^*(A')) < 1$. Indeed, if $\mu_1'(w^*(A'))=1$ then since $w^*(A')$ is weak* compact and convex there would be a point of $w^*(A')$ which μ_1' weak* represents. Now $\int_{w^*(C')} h^*\, d\mu_1' = \int_C h^*\, d\mu = h^*(x_0)$ for each h^* in E^{**} and hence μ_1' would represent $\chi(x_0)=x_0'$. It follows that $x_0' \in w^*(A')$. Hence there would be a net in A' converging in the weak* topology to x_0'. This leads to a contradiction since the corresponding net in A would converge to x_0 in the weak topology (which is impossible since A is weakly closed in C and $x_0 \notin A$). Thus letting $a=\mu_1'(w^*(A'))$ it follows that $0<a<1$.

Define measures μ_2' and μ_3' on $w^*(C')$ by $\mu_2'(P)=a^{-1}\mu_1'(P \cap w^*(A'))$ and $\mu_3'(P)=(1-a)^{-1}\mu_1'(P \cap (w^*(C')\backslash w^*(A')))$ for each weak* Borel set P of E^{***}. Since μ_2' and μ_3' are probability measures on $w^*(C')$ there are points F_2^* and F_3^* in $w^*(C')$ such that $\mu_i' \sim^* F_i^*$ for $i=2, 3$. Furthermore, $F_2^* \in w^*(A')$ since $\mu_2'(w^*(A'))=1$, and it is evident that $x_0'=aF_2^*+(1-a)F_3^*$. Define points F_2 and F_3 in $w^*(C')$ as follows: If $a \geqq \frac{1}{2}$ let $F_2=F_2^*$ and $F_3=2x_0'-F_2^*$. If $a<\frac{1}{2}$ let $F_3=F_3^*$ and $F_2=2x_0'-F_3^*$. Evidently $[F_2, F_3] \subset [F_2^*, F_3^*] \subset w^*(C')$ and $x_0'=\frac{1}{2}(F_2+F_3)$. Let $(U_\lambda)_{\lambda \in \Lambda}$ be a downward directed convex neighborhood base at $0 \in E^{***}$ in the weak* topology. If $a \geqq \frac{1}{2}$ choose $x_\lambda \in A$ such that $x_\lambda' \in F_2+U_\lambda$. Similarly, choose $y_\lambda \in C$ such that $y_\lambda' \in F_3+U_\lambda$. (If $a<\frac{1}{2}$ choose $w_\lambda \in A$ such that $w_\lambda' \in F_2^*+U_\lambda$. Then let $x_\lambda=a(1-a)^{-1}(w_\lambda-x_0)+x_0$. Evidently $x_\lambda \in C$ and $x_\lambda' \in F_2+U_\lambda$. Choose $y_\lambda \in C$ such that $y_\lambda' \in F_3+U_\lambda$.) Thus the net $(\frac{1}{2}x_\lambda+\frac{1}{2}y_\lambda)_{\lambda \in \Lambda}$ converges weakly to x_0. Since $x_0 \notin w^*(A)$ clearly $x_0 \notin w^*(\{x_\lambda\}_{\lambda \in \Lambda})$ even if $a<\frac{1}{2}$. It is easy to see that some subnet $(y_\gamma)_{\gamma \in \Gamma}$ of $(y_\lambda)_{\lambda \in \Lambda}$ satisfies the condition that its point set does not contain x_0 in its weak* closure. Then the nets $(x_\gamma)_{\gamma \in \Gamma}$ and $(y_\gamma)_{\gamma \in \Gamma}$ satisfy all but the last part of Definition 3.6, thus completing this portion of the proof.

not (1) $\Rightarrow$ *not* (2). Choose nets $(x_\gamma)_{\gamma \in \Gamma}$ and $(y_\gamma)_{\gamma \in \Gamma}$ such that $(\frac{1}{2}x_\gamma+\frac{1}{2}y_\gamma)_{\gamma \in \Gamma}$ converges weakly to x_0 and yet $x_0 \notin w^*(\{x_\gamma\}_{\gamma \in \Gamma} \cup \{y_\gamma\}_{\gamma \in \Gamma})$. By going to subnets if necessary assume that there is a point F in $w^*(C')$ such that $(x_\gamma')_{\gamma \in \Gamma}$ converges weak* to F (and hence $(y_\gamma')_{\gamma \in \Gamma}$ converges weak* to $2x_0'-F$). Let βC denote the Stone-Čech compactification of C (C considered in the norm topology). Again by going to subnets if necessary, assume that $(x_\gamma)_{\gamma\in\Gamma}$ converges in βC to v and that $(y_\gamma)_{\gamma\in\Gamma}$ converges in βC to w.

For each g in $C_b(C)$ denote by $g^\#$ its extension to a function in $C(\beta C)$. Define linear functionals ν_1 and ν_2 on $C_b(C)$ by the rules $\nu_1(g)=g^\#(v)$ and $\nu_2(g)=g^\#(w)$ for each g in $C_b(C)$; evidently $\nu_i\in\mathscr{P}(C)$. Let $\nu=\frac{1}{2}(\nu_1+\nu_2)$. For each h^* in E^{**} it follows that

$$\begin{aligned}\nu(h^*) &= \tfrac{1}{2}[\nu_1(h^*)+\nu_2(h^*)] = \tfrac{1}{2}[h^{*\#}(v)+h^{*\#}(w)]\\ &= \tfrac{1}{2}\lim_\gamma\,[h^*(x_\gamma)+h^*(y_\gamma)] = \tfrac{1}{2}\lim_\gamma\,[x'_\gamma(h^*)+y'_\gamma(h^*)]\\ &= \tfrac{1}{2}[F(h^*)+(2x'_0-F)(h^*)] = x'_0(h^*) = h^*(x_0).\end{aligned}$$

That is, $\nu\sim x_0$. Let $M=w^*(\{x_\gamma\}_{\gamma\in\Gamma}\cup\{y_\gamma\}_{\gamma\in\Gamma})$ and let U be a relative weak* open subset of $w^*(C)$ for which $x_0\notin w^*(U)$ and $U\supset M$. Choose a weak* continuous function g^* on $w^*(C)$ such that $g^*=1$ on M and $g^*=0$ on $w^*(C)\backslash U$ $(0\leqq g^*\leqq 1)$. Then we have

$$\begin{aligned}\int_{U\cap C} g^*\,d\nu &= \int_C g^*\,d\nu = \tfrac{1}{2}\int_C g^*\,d\nu_1+\tfrac{1}{2}\int_C g^*\,d\nu_2\\ &= \tfrac{1}{2}\lim_\gamma\,[g^*(x_\gamma)+g^*(y_\gamma)] = 1.\end{aligned}$$

It follows that $\nu(U\cap C)=1$ and since $x_0\notin w^*(U)$ (and thus $x_0\notin w^*(U\cap C)$) the measure ν and the set $U\cap C$ contradict (2).

not (2) $\Rightarrow$ *not* (3). Let μ be a measure for which (2) fails. Using Proposition 3.5 it suffices to construct a function f which is weak* continuous on $w^*(C)$ such that $\mu(f)>f(x_0)$. For the function f choose any weak* continuous strictly convex function on $w^*(C)$. (Such a function exists since $w^*(C)$ is a metrizable compact convex set in the weak* topology.) If $S\in\Sigma$ is a set for which $\mu(S)>0$ and $x_0\notin w^*(S)$, construct, exactly as in the first part of the proof of not (2) $\Rightarrow$ not (1), a set A of the form $h^{-1}(-\infty,l]\cap C$ (for some h in E) such that $x_0\notin w^*(A)$ and $\mu(A)>0$. Since $\mu\sim x_0$ it is evident (using Proposition 1.5) that $b=\mu(A)<1$. Define measures λ_1 and λ_2 on C by the conditions $\lambda_1(P)=b^{-1}\mu(A\cap P)$ and $\lambda_2(P)=(1-b)^{-1}\mu(P\cap(C\backslash A))$ for each P in Σ. It is easy to see that λ_1 and λ_2 belong to $\mathscr{P}(C)$. Then there exist x_i in $w^*(C)$ (by Proposition 1.5) such that $\lambda_i\sim^* x_i$ and since $x_1\in w^*(A)$ we have $x_1\neq x_0$. It follows that $x_0=bx_1+(1-b)x_2$. The measure λ_i corresponds to a measure $\lambda_i^*\in[C(w^*(C))]^*$ by the rule $\lambda_i^*(g)=\lambda_i(g|_C)$ for g in $C(w^*(C))$. Since λ_i^* is a (countably additive) probability measure on $w^*(C)$ it is the weak* limit of a net of discrete measures $(\nu_{\gamma,i})_{\gamma\in\Gamma_i}$ in $\mathscr{P}_c(w^*(C))$ such that $\nu_{\gamma,i}\sim^* x_i$ for each γ in Γ_i and $i=1,2$. Hence $\lambda_i(f)=\lambda_i^*(f)=\lim_{\gamma\in\Gamma_i}\nu_{\gamma,i}(f)\geqq f(x_i)$ from the convexity and weak* continuity of f. Finally, from the strict convexity of f we conclude that $\mu(f)=b\lambda_1(f)+(1-b)\lambda_2(f)\geqq bf(x_1)+(1-b)f(x_2)>f(x_0)$.

COROLLARY 3.8. *Let C be a closed and bounded convex subset of the dual E^* of a separable Banach space. Then $\{x\in\text{ex }C\mid x \text{ is a pinnacle point}\}$ is a G_δ set. In particular if each point x in* ex C *is a pinnacle point then* ex C *is a G_δ set.*

Proof. Let f be a bounded weak* continuous strictly convex function on $w^*(C)$. Using Proposition 3.7 it is easy to check that $\{x \in \text{ex}\, C \mid x \text{ is a pinnacle point}\} = \{x \in C \mid \bar{f}(x) = f(x)\}$ and since $\bar{f}$ is upper semicontinuous the conclusion follows.

It is not difficult to construct a closed and bounded convex set K in l_1 having an extreme point which is not a pinnacle point, as the following shows.

EXAMPLE 3.9. Let

$$A = \{\delta_1 + \delta_i \mid i = 2, 3, \ldots\} \quad \text{and} \quad B = \{-\delta_1 + i^{-1}\delta_{i-1} - \delta_i \mid i = 3, 4, \ldots\}$$

and let $k = n(\text{co}\,(A \cup B))$. The point 0 is in K since

$$\|\tfrac{1}{2}(\delta_1 + \delta_i) + \tfrac{1}{2}(-\delta_1 + i^{-1}\delta_{i-1} - \delta_i)\| = \tfrac{1}{2}i^{-1}$$

for $i = 3, 4, \ldots$. In fact, some calculation shows that $0 \in \text{ex}\, K$. Since the sequences $(\delta_1 + \delta_i)_{i \geqq 3}$ and $(-\delta_1 + i^{-1}\delta_{i-1} - \delta_i)_{i \geqq 3}$ satisfy all but the last part of Definition 3.6, evidently 0 is not a pinnacle point.

With the benefit of Example 3.9 the following elementary argument shows that Proposition 3.1 cannot in general be improved. If C is a closed and bounded separable convex subset of a Fréchet space, then each measure $\mu \in \mathscr{P}_c(C)$ is regular with respect to compact subsets (see the beginning of the proof of Proposition 1.7). Using this fact it is possible to prove, exactly as in the compact case (see, for example, [13, Proposition 1.4, p. 8]) that if $x \in \text{ex}\, C$ and $\mu \in \mathscr{P}_c(C)$ weakly represents x then $\mu = \varepsilon_x$. In particular, if K is the set described in Example 3.9, the only measure in $\mathscr{P}_c(K)$ weakly representing 0 is ε_0. Since 0 is not a pinnacle point, there is a μ in $\mathscr{P}(K)$ such that $\mu \sim 0$ and μ does not satisfy (2) of Proposition 3.7. Evidently $\mu \neq \varepsilon_0$ and hence μ is not the weak* limit of a net of discrete measures in $\mathscr{P}_c(K)$ each of which represents 0.

Finally, it should be mentioned in connection with Proposition 3.7 that there is a closed and bounded convex subset K of l_1 which has a pinnacle (extreme) point x such that $\chi(x) \notin \text{ex}\, w^*(\chi(K))$. (Here χ denotes the natural embedding of l_1 in l_1^{**}.) Thus, although in a convex set C (satisfying the conditions of Proposition 3.7) an extreme point of C whose image under χ is an extreme point of $w^*(\chi C)$ must necessarily be a pinnacle point, these two notions are distinct.

As an application of some of the foregoing ideas we prove a uniqueness result for a certain collection of simplexes.

DEFINITION 3.10. Let C be a convex subset of a lcs E. Then C is a simplex if the cone with vertex 0 over the set $C \times \{1\} \subset E \times R$ generates a lattice ordering on the linear span of $C \times \{1\}$.

PROPOSITION 3.11. *Let K be a closed and bounded simplex in a separable dual Banach space B^* such that*

$$(*) \qquad \bar{f}(x) = \sup \{\mu(f)) \mid \mu \text{ a discrete measure in } \mathscr{P}_c(K),\ \mu \sim x\}$$

whenever $x \in K$ and f is a convex function in $C(w^*(K))$ ($w^*(K)$ in the weak topology). Then given x in C there is at most one measure μ in $\mathscr{P}_c(K)$ such that $\mu \sim x$ and $\mu(\text{ex}\, K) = 1$.*

Proof. Observe first that ex K is a G_δ set (Corollary 3.8) so that $\mu(\text{ex } K)=1$ makes sense. It follows from (*) and the fact that the only measure $\nu \in \mathscr{P}_c(K)$ such that $\mu \sim z \in \text{ex } K$ is $\nu=\varepsilon_z$ that $f(y)=\bar{f}(y)$ whenever $f \in C(w^*(K))$ and $y \in \text{ex } K$. Suppose that $\mu \in \mathscr{P}_c(K)$ is a measure such that $\mu \sim x$ and $\mu(\text{ex } K)=1$. Let $J=w^*(K)$. For each weak* continuous convex function f in $C(J)$ we have $\mu(f)=\mu(\bar{f})$, and exactly as in the compact case [13, p. 67] the fact that K is a simplex implies that $\bar{f}$ is affine on K. Hence since $\bar{f}$ is upper semicontinuous (Proposition 3.5) we conclude that $\mu(f)=\mu(\bar{f})=\bar{f}(x)$ (Proposition 1.11). If $\nu \in \mathscr{P}_c(K)$ is also a measure for which $\nu(\text{ex } K)=1$ and $\nu \sim x$ then $\mu(f)=\bar{f}(x)=\nu(f)$ for each convex function f in $C(J)$. But as was observed in the proof of Proposition 2.2, the weak* and norm Borel subsets of J coincide, and hence μ and ν may be considered as elements of $[C(J)]^*$. Since the linear span of the cone of convex functions in $C(J)$ is a dense subspace of $C(J)$, and since μ and ν agree on this subspace, $\mu=\nu$ and this completes the proof.

Bibliography

1. C. Bessaga and A. Pełczynski, *On extreme points in separable conjugate spaces*, Israel J. Math. **4** (1966), 262–264. MR **35** #2126.

2. E. Bishop and K. de Leeuw, *The representations of linear functionals by measures on sets of extreme points*, Ann. Inst. Fourier (Grenoble) **9** (1959), 305–331. MR **22** #4945.

3. W. Bonnice and V. L. Klee, *The generation of convex hulls*, Math. Ann. **152** (1963), 1–29. MR **27** #6108.

4. N. Bourbaki, *General topology*. Part 2, Addison-Wesley, Reading, Mass., 1966. MR **34** #5044b.

5. G. Choquet, *Existence et unicité des représentations intégrales au moyen des points extrémaux dans les cônes convexes*, Séminaire Bourbaki 1956/57, Exposé 139, Secrétariat mathématique, Paris, 1959. MR **28** #1090.

6. G. Choquet and P.-A. Meyer, *Existence et unicité des représentations intégrales dans les convexes compacts quelconques*, Ann. Inst. Fourier (Grenoble) **13** (1963), 139–154. MR **26** #6748; errata, MR **30** p. 1203.

7. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

8. J. R. Isbell, *Uniform spaces*, Math. Surveys, no. 12, Amer. Math. Soc., Providence, R. I., 1964. MR **30** #561.

9. S. S. Khurana, *Measures and barycenters of measures on convex sets in locally convex spaces*. I, J. Math. Anal. Appl. **27** (1969), 103–115. MR **39** #4631.

10. E. Marczewski and R. Sikorski, *Measures in non-separable metric spaces*, Colloq. Math. **1** (1948), 133–139. MR **10**, 23.

11. I. Namioka, *Neighborhoods of extreme points*, Israel J. Math. **5** (1967), 145–152. MR **36** #4323.

12. J. C. Oxtoby and S. Ulam, *On the existence of a measure invariant under a transformation*, Ann. of Math. (2) **40** (1939), 560–566.

13. R. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N. J., 1966. MR **33** #1690.

State University of New York at Buffalo,
Amherst, New York 14226