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GROUPS OF EMBEDDED MANIFOLDS

BY
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Abstract. This paper defines a group $\theta(M^n, \nu_\varphi)$ which generalizes the group of framed homotopy n-spheres in S^{n+k}. Let M^n be an arbitrary 1-connected manifold satisfying a weak condition on its homology in the middle dimension and let ν_φ be the normal bundle of some imbedding $\varphi: M^n \to S^{n+k}$, where $2k \geqq n+3$. Then $\theta(M^n, \nu_\varphi)$ is the set of h-cobordism classes of triples (F, V^n, f), where $F: S^{n+k} \to T(\nu_\varphi)$ is a map which is transverse regular on M, $V^n = F^{-1}(M^n)$, and $f = F|V^n$ is a homotopy equivalence. ($T(\nu_\varphi)$ is the Thom complex of ν_φ.) There is a natural group structure on $\theta(M^n, \nu_\varphi)$, and $\theta(M^n, \nu_\varphi)$ fits into an exact sequence similar to that for the framed homotopy n-spheres.

This paper attempts to generalize in a natural way a well-known exact sequence concerning framed homotopy spheres which is contained in the work of Novikov [11], Kervaire-Milnor [7], and Levine [10]. The author stumbled onto these results partly because of his efforts to prove imbedding theorems for manifolds in the metastable range, and partly because of his recent work on Browder-Novikov theory for maps of degree d, $|d| \neq 0$ (see [2]).

§2 describes the basic constructions used in this paper. The "group of embedded manifolds", $\theta(M, \nu_\varphi)$, is defined in §3. A fairly simple description of that group is given towards the end of that section. §3 also contains the main results about $\theta(M, \nu_\varphi)$. In §4 we discuss a few interesting open problems. The author would like to thank the referee for some helpful suggestions.

1. **Notation.** All manifolds will be C^∞, compact, and oriented. Maps will be transverse to boundaries.

If M^n is a connected closed manifold, let $[M] \in H_nM$ denote the orientation class. Recall that $f: V^n \to M^n$ is said to have degree d, i.e., $\deg f = d$, if $f_*([V]) = d[M]$, where $f_*: H_nV \to H_nM$ is the map induced by f on the integral homology groups.

As usual, D^k denotes the closed unit ball in Euclidean k-space $\boldsymbol{R}^k$, i.e., $D^k = \{(y_1, \ldots, y_k) \in \boldsymbol{R}^k \mid y_1^2 + \cdots + y_k^2 \leqq 1\}$. $S^k = \partial D^{k+1} = D^k_+ \cup D^k_-$, where $D^k_+ = \{(y_1, \ldots, y_{k+1}) \in \boldsymbol{R}^{k+1} \mid y_1^2 + \cdots + y_{k+1}^2 = 1, y_1 \geqq 0\}$ and $D^k_- = \{(y_1, \ldots, y_{k+1}) \in \boldsymbol{R}^{k+1} \mid y_1^2 + \cdots + y_{k+1}^2 = 1, y_1 \leqq 0\}$. We have natural inclusions $S^k \subseteq S^{k+1}$ and $D^k \subseteq D^{k+1}$. Let $e = (1, 0) \in S^0 \subseteq S^k$.

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If $f\colon V^n \to W^{n+q}$ is an imbedding, we shall consider f as an inclusion map and identify the total space $E=E(\nu_f)$ of the normal disk bundle ν_f with a tubular neighborhood of V in W. $T(\nu_f)=E/\partial E$ is the Thom complex of ν_f, and $T_f\colon W \to T(\nu_f)$ is the natural collapsing map. Given $g\colon U^m \to T(\nu_f)$ which is transverse regular on V so that $N=g^{-1}(V)$ is an $(m-q)$-submanifold of U, we shall always assume that a tubular neighborhood T of N has been given a fixed bundle structure which is the pullback of ν_f under g. We also assume that N is given an orientation which is induced from the orientation of V.

If $V^n \subseteq W^{n+q}$, then by a framing of V in W, or by a framing of a tubular neighborhood T of V, we shall mean a diffeomorphism $\mathscr{F}\colon V \times D^q \to T$ such that $\mathscr{F}(x, 0)=x$. Two framed submanifolds $(V_1^n, \mathscr{F}_1)$ and $(V_2^n, \mathscr{F}_2)$ in W^{n+q} are framed cobordant if there is a framed submanifold $(N^{n+1}, \mathscr{G})$ in $W \times [1, 2]$ such that $(N, \mathscr{G}) \cap (W \times i)=(V_i, \mathscr{F}_i) \times i$, $i=1, 2$. They are framed h-cobordant if N is an h-cobordism.

2. **Preliminaries.** Throughout this paper we shall make the following assumptions: M^n is a 1-connected oriented manifold with $n \geqq 5$. Let $t=[n/2]$. Then either $n \equiv 0 \pmod 4$, or $H_t M=0$ and $H_{t-1}M$ is torsion-free. φ is an imbedding of M in S^{n+k}, where $2k \geqq n+3$.

DEFINITION.

$$\mathscr{T}_0^+(M, \nu_\varphi) = \{(F, V^n, f) \mid F\colon S^{n+k} \to T(\nu_\varphi) \text{ is a map which is transverse regular on } M \text{ with } V^n = F^{-1}(M) \text{ 1-connected and } f = F \mid V\colon V \to M\}.$$

$$\mathscr{S}_0^+(M, \nu_\varphi) = \{(F, V, f) \in \mathscr{T}_0^+(M, \nu_\varphi) \mid \deg f > 0 \text{ and } f_*\colon H_i V \to H_i M \text{ is an isomorphism for } 0 \leqq i \leqq [n/2]\}.$$

NOTE. $\mathscr{S}_0^+(M, \nu_\varphi) \neq \varnothing$ since it contains $(T_\varphi, M, \text{identity})$.

If $\alpha_i=(F_i, V_i, f_i) \in \mathscr{T}_0^+(M, \nu_\varphi)$, define $\alpha_1 \sim \alpha_2$ if there is a map $H\colon S^{n+k} \times [1, 2] \to T(\nu_\varphi)$ which is transverse regular on M such that $H|S^{n+k} \times i=F_i \times i$ and $H^{-1}(M)$ is an h-cobordism between $V_1 \times 1$ and $V_2 \times 2$. Clearly, $\sim$ is an equivalence relation. We set

$$\mathscr{T}^+(M, \nu_\varphi) = \mathscr{T}_0^+(M, \nu_\varphi)/\sim \quad \text{and} \quad \mathscr{S}^+(M, \nu_\varphi) = \mathscr{S}_0^+(M, \nu_\varphi)/\sim \; \subseteq \mathscr{T}^+(M, \nu_\varphi).$$

If $\alpha \in \mathscr{T}_0^+(M, \nu_\varphi)$, we shall also write α for the equivalence class that α determines in $\mathscr{T}^+(M, \nu_\varphi)$.

Suppose that $\alpha_i=(F_i, V_i, f_i) \in \mathscr{T}^+(M, \nu_\varphi)$. Let ν_i be the normal disk bundle of V_i in S^{n+k} and let $D_+^n \times D^k$ and $D_-^n \times D^k$ be canonical tubular neighborhoods of D_+^n and D_-^n in D_+^{n+k} and D_-^{n+k}, respectively. Without loss of generality we may assume that $D_+^n \subseteq V_1$, $D_-^n \subseteq V_2$, $E(\nu_1|S^{n-1})=E(\nu_2|S^{n-1}) \subseteq S^{n+k-1}$, $V_1 - D_+^n \subseteq D_-^n \times D^k$, $V_2 - D_-^n \subseteq D_+^n \times D^k$, $f_1(D_+^n)=f_2(D_-^n)=x_0 \in M$, and $f_1|S^{n+k-1}=f_2|S^{n+k-1}$. Now define $F_3\colon S^{n+k} \to T(\nu_\varphi)$ by $F_3|D_+^{n+k}=f_2|D_+^{n+k}$ and $F_3|D_-^{n+k}=f_1|D_-^{n+k}$. Let $V_3 = F_3^{-1}(M)$, $f_3=F_3|V_3$, and $\alpha_1 \# \alpha_2=(F_3, V_3, f_3)$. Then $\deg f_3=\deg f_1+\deg f_2$, $V_3 = V_1 \# V_2$, and $\alpha_1 \# \alpha_2$ is a well defined element of $\mathscr{T}^+(M, \nu_\varphi)$.

Next, define $\theta_f^{n+k,n}$ to be the group of h-cobordism classes of framed homotopy n-spheres in S^{n+k}. If Σ^n is a homotopy n-sphere in S^{n+k} with a framing $\mathscr{F}$ of its normal disk bundle ν_Σ, we let $[\Sigma, \mathscr{F}]$ denote the element it determines in $\theta_f^{n+k,n}$. (Note that in analogy with $\theta_f^{n+k,n}$ we can think of $\mathscr{T}^+(M, \nu_\varphi)$ and $\mathscr{S}^+(M, \nu_\varphi)$ as h-cobordism classes of certain submanifolds $V^n \subseteq S^{n+k}$ with a given bundle map from the normal disk bundle of V in S^{n+k} to ν_φ.)

Keeping the notation of the two previous paragraphs, let $\sigma = [\Sigma, \mathscr{F}] \in \theta_f^{n+k,n}$. Assume that $D_-^n \subseteq \Sigma$, $E(\nu_\Sigma | S^{n-1}) = E(\nu_1 | S^{n-1})$, and $X = \Sigma - D_-^n \subseteq D_+^n \times D^k$. Define $F' : S^{n+k} \to T(\nu_\varphi)$ by $F' | D_-^{n+k} = f_1 | D_-^{n+k}$, $F'(\mathscr{F}(y, u)) = f_1(\mathscr{F}(e, u))$, for $(y, u) \in X \times D^k$, and $F'(D_-^{n+k} - \mathscr{F}(X \times D^k)) =$ canonical base point of $T(\nu_\varphi)$. Set $V' = (F')^{-1}(M)$, $f' = F' | V'$, and $\alpha_1 \# \sigma = (F', V', f')$. Then $\alpha_1 \# \sigma$ is a well defined element of $\mathscr{T}^+(M, \nu_\varphi)$, $\deg f' = \deg f$, and $V' = V_1 \# \Sigma$.

One can easily check that if $\alpha_i \in \mathscr{T}^+(M, \nu_\varphi)$ and $\sigma_i \in \theta_f^{n+k,n}$, then the two connected sum operations defined above have the following properties:

(1) $(\alpha_1 \# \alpha_2) \# \alpha_3 = \alpha_1 \# (\alpha_2 \# \alpha_3)$,

(2) $\alpha_1 \# \alpha_2 = \alpha_2 \# \alpha_1$,

(3) $(\alpha_1 \# \alpha_2) \# \sigma_1 = \alpha_1 \# (\alpha_2 \# \sigma_1)$,

(4) $(\alpha_1 \# \sigma_1) \# \alpha_2 = (\alpha_1 \# \alpha_2) \# \sigma_1$,

(5) $\alpha_1 \# (\sigma_1 + \sigma_2) = (\alpha_1 \# \sigma_1) \# \sigma_2$, and

(6) if $\alpha_1 \in \mathscr{S}^+(M, \nu_\varphi)$, then $\alpha_1 \# \sigma_1 \in \mathscr{S}^+(M, \nu_\varphi)$.

LEMMA 1. *Let $\alpha_i = (F_i, V_i, f_i) \in \mathscr{S}^+(M, \nu_\varphi)$ and suppose $\alpha_1 \# \alpha_2 = (F_3, F_3, f_3)$. Then there exists a triple $\Gamma = (H, W^{n+1}, h)$ such that*

(a) *$H: S^{n+k} \times [3, 4] \to T(\nu_\varphi)$ is a map which is transverse regular on M with $W = H^{-1}(M)$ 1-connected and $h = H | W : W \to M$;*

(b) *$H | S^{n+k} \times l = F_l \times l$, $l = 3, 4$;*

(c) *$\alpha_4 = (F_4, V_4, f_4) \in \mathscr{S}^+(M, \nu_\varphi)$ where $V_4 = F_4^{-1}(M)$ and $f_4 = F_4 | V_4$; and*

(d) *$H_i(W, V_4) = 0$ for $i \geqq t+1$ if $n \not\equiv 0$ (mod 4).*

Proof. We shall define inductively a sequence $\Gamma_i = (H_i, W_i, h_i)$, for $0 \leqq i \leqq [n/2] - 1$, such that

(1) $H_i: S^{n+k} \times [0, i+1] \to T(\nu_\varphi)$ is a map which is transverse regular on M with $W_i = H_i^{-1}(M)$ 1-connected and $h_i = H_i | W_i$;

(2) $H_i | S^{n+k} \times [0, i] = H_{i-1}$;

(3) $\partial W_i = V_3 \times 0 \cup N_i$ with N_i 1-connected;

(4) $(h_i)_* : H_t W_i \to H_t M$ is an isomorphism for $0 \leqq t \leqq i$;

(5) if $j: V_3 \times 0 \to W_i$ is the natural inclusion, then $j_* : H_t(V_3 \times 0) \to H_t W_i$ is an isomorphism for $t > i+1$; if $t = i+1$, j_* is one-to-one and $H_{i+1} W_i = j_* H_{i+1}(V_3 \times 0) \oplus G$, where G is a torsion-free group which is zero if $H_i M$ had no torsion and $(h_i)_*(G) = 0$.

Define $H_0 : S^{n+k} \times [0, 1] \to T(\nu_\varphi)$ by $H_0(x, t) = F_3(x)$. Then H_0 clearly determines a triple $\Gamma_0 = (H_0, W_0, h_0)$ which satisfies (1)–(5). Suppose $\Gamma_{i-1} = (H_{i-1}, W_{i-1}, h_{i-1})$

has been defined for $1 \leqq i \leqq [n/2]-1$ satisfying (1)–(5). Our object will be to add handles to W_{i-1} along N_{i-1} to make h_{i-1} i-connected.

Let $j'\colon N_{i-1} \to W_{i-1}$ be the natural inclusion and consider the exact sequence

$$\begin{array}{ccccc} \cdots \longrightarrow H_{i+1}(W_{i-1}, N_{i-1}) & \longrightarrow & H_i N_{i-1} & \xrightarrow{j'_*} & H_i W_{i-1} = j_* H_i(V_3 \times 0) \oplus G \to \cdots \\ \wr\wr & & \downarrow & \swarrow (h_{i-1})_* & \\ H^{n-i}(W_{i-1}, V_3 \times 0) & & H_i M & & \end{array}$$

Since $i \leqq [n/2]-1$, the universal coefficient theorem for cohomology and (5) imply that j'_* is an isomorphism. Let $K_i = \text{kernel of } (f_3)_*\colon H_i V_3 \to H_i M$. K_i is a direct summand of $H_i V_3$ because $V_3 = V_1 \# V_2$ and $(f_l)_*\colon H_i V_l \to H_i M$, $l=1, 2$, is an isomorphism. Therefore, if $K = \text{kernel of } (h_{i-1})_*\colon H_i W_{i-1} \to H_i M$, then K is a direct summand of $H_i W_{i-1}$ and $K = j_*(K_i) \oplus G$. The diagram

$$\begin{array}{ccc} H_i(V_3 \times 0) & \xrightarrow{j_*} & H_i W_{i-1} \\ (f_3 \times 0)_* \searrow & & \swarrow (h_{i-1})_* \\ & H_i M & \end{array}$$

shows that $(h_{i-1})_*$ is onto $H_i M$. It follows that every element of K can be realized as an imbedded sphere in N_{i-1} with trivial normal bundle. One can now add handles to W_{i-1} along N_{i-1} to kill K as in the case of the usual Browder-Novikov theory. In fact, since $2k \geqq n+1$, the handles can be attached in $S^{n+k} \times i$ so that the method of [4] and [9] can be used to obtain $\Gamma_i = (H_i, W_i, h_i)$. Γ_i will satisfy (1)–(4) trivially. The proof of Theorem 2.1 in [1] shows how (5) can be satisfied also. This finishes the inductive definition of Γ_i.

Let $t = [n/2]$. Assume that $n \not\equiv 0 \pmod 4$. By hypothesis $H_t M = 0$ and $H_{t-1} M$ is torsion-free, and so (5) implies that $(h_{t-1} | N_{t-1})_*\colon H_l N_{t-1} \to H_l M$ is an isomorphism for $0 \leqq l \leqq [n/2]$. Define $H\colon S^{n+k} \times [3, 4] \to T(\nu_\varphi)$ by $H(x, u) = H_{t-1}(x, t(u-3))$ and set $W = H^{-1}(M)$, $h = H | W$. Then $\Gamma = (H, W, h)$ satisfies (a)–(d) in Lemma 1 and the lemma is proved in this case. If $n \equiv 0 \pmod 4$, then there is no obstruction to doing surgery on N_{t-1} in the middle dimension and one can define Γ_t satisfying (1)–(4) very much like the other Γ_i. Therefore we can get a Γ satisfying (a)–(c) in this case also. This completes the proof of Lemma 1.

We can now define an operation $+$ in $\mathscr{S}^+(M, \nu_\varphi)$ as follows: If $\alpha_i \in \mathscr{S}^+(M, \nu_\varphi)$, then we let $\alpha_1 + \alpha_2 = \alpha_4$, where α_4 is defined as in Lemma 1(c).

LEMMA 2. *$+$ is a well defined associative and commutative operation.*

Proof. Suppose that $\Gamma' = (H', W', h')$ and $\alpha'_4 = (F'_4, V'_4, f'_4)$ are triples which satisfy (a)–(d) in Lemma 1. $+$ will be well defined once we show that $\alpha'_4 = \alpha_4$. Define $H''\colon S^{n+k} \times [2, 4] \to T(\nu_\varphi)$ by $H''(x, u) = H'(x, -u+6)$ for $u \in [2, 3]$ and $H''(x, u) = H(x, u)$ for $u \in [3, 4]$. Let $W'' = (H'')^{-1}(M)$ and $h'' = H'' | W''$. It suffices to show that we can make W'' into an h-cobordism via framed surgery in $S^{n+k} \times$

(2, 4). To be precise, we are looking for a map $P\colon S^{n+k}\times[2,4]\times[0,1]\to T(\nu_\varphi)$ satisfying

(1) P is transverse regular on M;

(2) $P|S^{n+k}\times[2,4]\times 0=H''\times 0$, $P|S^{n+k}\times 2\times u=F_4'\times 2\times u$, $P|S^{n+k}\times 4\times u=F_4\times 4\times u$, for $u\in[0,1]$; and

(3) $U=P^{-1}(M)\cap S^{n+k}\times[2,4]\times 1$ is an h-cobordism.

If $n\equiv 0 \pmod 4$, then there is no obstruction to doing surgery on W'', even in the middle dimensions. If $n\not\equiv 0 \pmod 4$, then it follows from Lemma 1(d) (using our hypothesis on the homology of M in the middle dimensions) that we only have to do surgery on W'' in dimension $\leqq[n/2]$. In any case, it is therefore possible to define P inductively, similar to the definition of the Γ_i in the proof of Lemma 1. We shall omit the details and leave it to the reader to translate the construction for the Γ_i so that it is applicable in this situation.

Finally, the fact that + is associative and commutative follows from the fact that # has these properties. This finishes the proof of Lemma 2.

LEMMA 3. *Let $\alpha_i\in\mathscr{S}^+(M,\nu_\varphi)$ and $\sigma_i\in\theta_f^{n+k,n}$. Then*

$$(\alpha_1\#(\sigma_1+\sigma_2))+\alpha_2 = (\alpha_1\#\sigma_1)+(\alpha_2\#\sigma_2).$$

Proof. This lemma is an easy consequence of the observation that

$$(\alpha_1\#(\sigma_1+\sigma_2))\#\alpha_2 = (\alpha_1\#\sigma_1)\#(\alpha_2\#\sigma_2).$$

Next, let $\alpha=(F,V,f)\in\mathscr{T}^+(M,\nu_\varphi)$. Define

$$\psi_0'\colon\mathscr{T}^+(M,\nu_\varphi)\to\pi_{n+k}T(\nu_\varphi)\quad\text{and}\quad\psi_0\colon\mathscr{S}^+(M,\nu_\varphi)\to\pi_{n+k}T(\nu_\varphi)$$

by $\psi_0'(\alpha)=[F]$ and $\psi_0=\psi_0'|\mathscr{S}^+(M,\nu_\varphi)$.

LEMMA 4. *ψ_0' and ψ_0 are well defined. ψ_0 is additive.*

Proof. Since ψ_0', ψ_0 are clearly well defined, it suffices to show that ψ_0 is additive. Let $\alpha_i\in\mathscr{S}(^+M,\nu_\varphi)$. It follows from the definitions that $\psi_0(\alpha_1+\alpha_2)=\psi_0'(\alpha_1\#\alpha_2)$ $=\psi_0'(\alpha_1)+\psi_0'(\alpha_2)=\psi_0(\alpha_1)+\psi_0(\alpha_2)$. Therefore ψ_0 is additive and the lemma is proved.

We conclude this section with the definition of two more maps. First, note that we can identify $H_{n+k}T(\nu_\varphi)$ with the integers $\boldsymbol{Z}$ in such a way that 1 corresponds to $[M]\in H_nM^n$ via the Thom isomorphism. Let

$$\deg\colon\pi_{n+k}T(\nu_\varphi)\to H_{n+k}T(\nu_\varphi)$$

be the Hurewicz homomorphism and define an additive map $\deg\colon\mathscr{S}^+(M,\nu_\varphi)\to\boldsymbol{Z}$ by $\deg(F,V,f)=\deg f$. Then we have a commutative diagram

$$\begin{array}{ccc}\mathscr{S}^+(M,\nu_\varphi) & \xrightarrow{\psi_0} & \pi_{n+k}T(\nu_\varphi)\\ \deg\searrow & & \swarrow\deg\\ & \boldsymbol{Z} = H_{n+k}T(\nu_\varphi). & \end{array}$$

3. **The group** $\theta(M, \nu_\varphi)$. Let $\alpha, \beta \in \mathscr{S}^+(M, \nu_\varphi)$ and set $\varepsilon=(T_\varphi, M, \text{identity})$. Define $\alpha \sim_e \beta$ if $\alpha+r\varepsilon=\beta+s\varepsilon$ for some nonnegative integers r and s. Obviously, $\sim_e$ is an equivalence relation and we can define $\theta(M, \nu_\varphi)=\mathscr{S}^+(M, \nu_\varphi)/\sim_e$. We shall write $[\alpha]$ for the equivalence class in $\theta(M, \nu_\varphi)$ determined by $\alpha \in \mathscr{S}^+(M, \nu_\varphi)$.

Define an operation $+$ in $\theta(M, \nu_\varphi)$ by $[\alpha]+[\beta]=[\alpha+\beta]$. It is easy to check that $+$ is a well defined associative and commutative operation. $[\varepsilon]$ acts as a zero element. The projection $\mathscr{S}^+(M, \nu_\varphi) \to \theta(M, \nu_\varphi)$ is additive.

At this point the only thing which keeps $\theta(M, \nu_\varphi)$ from being a group is that we do not know whether every element has an inverse. We shall return to this question shortly.

Define P_n as usual by

$$\begin{aligned} P_n &= 0 && \text{if } n \text{ is odd,} \\ &= Z_2 && \text{if } n \equiv 2 \pmod 4, \\ &= Z && \text{if } n \equiv 0 \pmod 4. \end{aligned}$$

We shall think of P_n as the set of framed cobordism classes $[U^n, \mathscr{F}]$ of pairs $(U^n, \mathscr{F})$, where $(U, \partial U) \subseteq (D^{n+l}, S^{n+l-1})$, $l \geqq 3$, $\mathscr{F}$ is a framing of the normal disk bundle of U, and ∂U is a homotopy $(n-1)$-sphere (see [10]).

Suppose that $\alpha_i=(F_i, V_i, f_i) \in \mathscr{S}^+(M, \nu_\varphi)$ and that $\psi_0(\alpha_1)=\psi_0(\alpha_2)$. Then there is a map $H: S^{n+k} \times [1, 2] \to T(\nu_\varphi)$ which is transverse regular on M such that $H|S^{n+k} \times i=F_i \times i$. Set $W^{n+1}=H^{-1}(M)$ and $h=H|W$. Let us try to make W into an h-cobordism just as in Lemma 2. The only problem occurs when we try to do surgery in the middle dimension $[(n+1)/2]$. This difficulty was circumvented in Lemma 2 by our conditions on M and n. But it follows from by now standard techniques that the obstruction to doing this surgery is a well defined element $\gamma(\alpha_1, \alpha_2) \in P_{n+1}$. In fact, we may assume that W is diffeomorphic to $V_2 \times [1, 2)$ $\pm U^{n+1}$, where $\pm$ denotes the boundary connected sum along $V_2 \times 1$ and $(U, \partial U)$ $\subseteq (S^{n+k} \times [1, 2], S^{n+k} \times 1)$ is a $([(n+1)/2]-1)$-connected π-manifold with ∂U a homotopy sphere and a framing $\mathscr{F}$ of its normal disk bundle which is induced by H (see [8, p. 20]). Then $\gamma(\alpha_1, \alpha_2)=[U, \mathscr{F}]=\gamma(U, \mathscr{F})$ (see [10, §4.5] for a definition of $\gamma(U, \mathscr{F})$).

Let $(U_1, \mathscr{F}_1)$ be a disjoint copy of $(U, \mathscr{F})$ so that $(U_1, \partial U_1) \subseteq (S^{n+k} \times [1, 2]$, $S^{n+k} \times 1)$. Define $W \pm -U_1$ in a natural manner, where we take the boundary connected sum along $V_1 \subseteq \partial W$. It is easy to obtain a map $H_1: S^{n+k} \times [1, 2] \to T(\nu_\varphi)$ which is transverse regular on M such that $H_1^{-1}(M)=W \pm -U_1$ and $H_1|S^{n+k} \times 2$ $=F_2 \times 2$. (H_1 is gotten by a construction similar to the one found in the definition of the connected sum of an element of $\mathscr{T}^+(M, \nu_\varphi)$ with a framed homotopy sphere.) If we let $F_3 \times 1=H_3|S^{n+k} \times 1$, $V_3=F_3^{-1}(M)$, and $f_3=F_3|V_3$, then we can also assume that $(F_3, V_3, f_3)=\alpha_1 \# -(\partial U_1, \mathscr{F}_1|\partial U_1)$. But there is no longer any obstruction to making $W \pm -U_1$ into an h-cobordism since $\gamma((U, \mathscr{F}) \pm -(U_1, \mathscr{F}_1))=\gamma(U, \mathscr{F})$ $-\gamma(U_1, \mathscr{F}_1)=0$ (see [10, §4.5]).

We summarize this discussion in a lemma. Let $\partial_1: P_{n+1} \to \theta_f^{n+k,n}$ be given by $\partial_1([U, \mathscr{F}]) = [\partial U, \mathscr{F}|\partial U]$. ∂_1 is a homomorphism. This was essentially proved in [10].

Lemma 5. *Let* $\alpha_i \in \mathscr{S}^+(M, \nu_\varphi)$ *and suppose that* $\psi_0(\alpha_1) = \psi_0(\alpha_2)$. *Then*

$$\alpha_1 \# \partial_1(\gamma(\alpha_1, \alpha_2)) = \alpha_2.$$

Next, let

$$\pi^0_{n+k}T(\nu_\varphi) = \text{kernel of deg: } \pi_{n+k}T(\nu_\varphi) \to H_{n+k}T(\nu_\varphi),$$

and define $\psi: \theta(M, \nu_\varphi) \to \pi^0_{n+k}T(\nu_\varphi)$ by $\psi([\alpha]) = \psi_0(\alpha) - (\deg \alpha)\psi_0(\varepsilon)$.

Lemma 6. ψ *is a well defined additive map.*

Proof. Suppose that $[\alpha] = [\beta]$. Then $\alpha + r\varepsilon = \beta + s\varepsilon$ for some nonnegative integers r and s. Hence $\psi_0(\alpha) + r\psi_0(\varepsilon) = \psi_0(\alpha + r\varepsilon) = \psi_0(\beta + s\varepsilon) = \psi_0(\beta) + s\psi_0(\varepsilon)$ and $\deg \alpha + r = \deg(\alpha + r\varepsilon) = \deg(\beta + s\varepsilon) = \deg \beta + s$. It follows that

$$\begin{aligned}\psi_0(\alpha) - (\deg \alpha)\psi_0(\varepsilon) &= \psi_0(\alpha) - s\psi_0(\varepsilon) + (s - \deg \alpha)\psi_0(\varepsilon)\\ &= \psi_0(\beta) - r\psi_0(\varepsilon) + (s - \deg \alpha)\psi_0(\varepsilon)\\ &= \psi_0(\beta) - (\deg \beta)\psi_0(\varepsilon) + (\deg \beta - \deg \alpha + s - r)\psi_0(\varepsilon)\\ &= \psi_0(\beta) - (\deg \beta)\psi_0(\varepsilon),\end{aligned}$$

and so ψ is well defined. Clearly $\psi(0) = 0$.

Let $[\alpha], [\beta] \in \theta(M, \nu_\varphi)$. Then $\psi([\alpha] + [\beta]) = \psi([\alpha + \beta]) = \psi_0(\alpha + \beta) - (\deg(\alpha + \beta))\psi_0(\varepsilon) = \psi_0(\alpha) + \psi_0(\beta) - (\deg \alpha)\psi_0(\varepsilon) - (\deg \beta)\psi_0(\varepsilon) = \psi([\alpha]) + \psi([\beta])$. Thus ψ is additive and Lemma 6 is proved.

Note. If $\theta(M, \nu_\varphi)$ is a group, then ψ is in fact a homomorphism.

Define $\partial_0: P_{n+1} \to \mathscr{S}^+(M, \nu_\varphi)$ by $\partial_0(\gamma) = \varepsilon \# \partial_1(\gamma)$, and let $\partial: P_{n+1} \to \theta(M, \nu_\varphi)$ be the composition of ∂_0 followed by the projection of $\mathscr{S}^+(M, \nu_\varphi)$ onto $\theta(M, \nu_\varphi)$.

Lemma 7. ∂_0 *and* ∂ *are well defined maps.* ∂ *is a homomorphism.*

Proof. ∂_0 and ∂ are well defined because $\#$ is well defined. Let $\gamma_i \in P_{n+1}$. Then Lemma 3 implies that

$$\begin{aligned}\partial(\gamma_1 + \gamma_2) &= [\partial_0(\gamma_1 + \gamma_2)] = [\varepsilon \# \partial_1(\gamma_1 + \gamma_2)] = [\varepsilon \# \partial_1(\gamma_1 + \gamma_2) + \varepsilon]\\ &= [\varepsilon \# \partial_1(\gamma_1) + \varepsilon \# \partial_1(\gamma_2)] = [\varepsilon \# \partial_1(\gamma_1)] + [\varepsilon \# \partial_1(\gamma_2)]\\ &= \partial(\gamma_1) + \partial(\gamma_2),\end{aligned}$$

i.e., ∂ is a homomorphism.

Finally, let $\mu: \pi^0_{n+k}T(\nu_\varphi) \to P_n$ be the well-known mapping which assigns to every $x \in \pi^0_{n+k}T(\nu_\varphi)$ the surgery obstruction to finding a representative $F: S^{n+k} \to T(\nu_\varphi)$ for $x + \psi_0(\varepsilon) \in \pi_{n+k}T(\nu_\varphi)$ such that F is transverse regular on M and $F|F^{-1}(M): F^{-1}(M) \to M$ is a homotopy equivalence. If $n \not\equiv 2 \pmod 4$, then $\mu = 0$. If $n \equiv 2 \pmod 4$, then our knowledge of μ is in general limited (see [3]); however, with our restrictions on the homology of M, μ is a well defined homomorphism.

Consider the diagram

$$\begin{array}{ccccccc} & \nearrow^{\partial_0} & \mathscr{S}^+(M, \nu_\varphi) & \xrightarrow{\psi_0} & \pi_{n+k}T(\nu_\varphi) & & \\ & & \downarrow & & & & \\ P_{n+1} & \xrightarrow[\partial]{} & \theta(M, \nu_\varphi) & \xrightarrow[\psi]{} & \pi^0_{n+k}T(\nu_\varphi) & \xrightarrow[\mu]{} & P_n \end{array}$$

THEOREM 1. *$\theta(M, \nu_\varphi)$ is an abelian group and the bottom row is exact.*

Proof. It is easy to see that $\psi_0\partial_0=0$. From this it follows immediately that $\psi\partial=0$, i.e. (image ∂)$\subseteq$(kernel ψ). Let $[\alpha]\in$(kernel ψ) and let $\deg\alpha=d$. Then $\psi_0(\alpha)=d\psi_0(\varepsilon)=\psi_0(d\varepsilon)$. By Lemma 5,

$$(d-1)\varepsilon+\partial_0(\gamma(d\varepsilon,\alpha)) = (d-1)\varepsilon+(\varepsilon\ \#\ \partial_1(\gamma(d\varepsilon,\alpha))) = d\varepsilon\ \#\ \partial_1(\gamma(d\varepsilon,\alpha)) = \alpha.$$

Therefore, $\partial(\gamma(d\varepsilon,\alpha))=[\alpha]$, and we have shown that (kernel ψ)$\subseteq$(image ∂). This proves that (image ∂)$=$(kernel ψ).

Next, let $x\in\pi^0_{n+k}T(\nu_\varphi)$. Suppose that $\mu(x)=0$. Then $x+\psi_0(\varepsilon)\in\pi_{n+k}T(\nu_\varphi)$ belongs to the image of ψ_0, i.e., there is an $\alpha\in\mathscr{S}^+(M,\nu_\varphi)$ with $\psi_0(\alpha)=x+\psi_0(\varepsilon)$. Hence, $\psi([\alpha])=\psi_0(\alpha)-(\deg\alpha)\psi_0(\varepsilon)=x$, so that (kernel μ)$\subseteq$(image ψ). Conversely, let $[\alpha]\in\theta(M,\nu_\varphi)$ and set $y=\psi_0(\alpha)-(\deg\alpha-1)\psi_0(\varepsilon)\in\pi_{n+k}T(\nu_\varphi)$. Then $\deg y=1$. Assume $n\equiv 2 \pmod 4$. By definition, $\mu(\psi([\alpha]))$ is the obstruction to finding a representative $F\colon S^{n+k}\to T(\nu_\varphi)$ for y such that F is transverse regular on M and $F|F^{-1}(M)\colon F^{-1}(M)\to M$ is a homotopy equivalence. But using our hypothesis on the homology of M, we see that for this particular y we can start with a representative F such that $H_t(F^{-1}(M))=0$ and $H_{t-1}(F^{-1}(M))$ is torsion-free, where $t=[n/2]$. Therefore, we shall never have to do surgery in the middle dimension; and so $\mu(\psi([\alpha]))=0$. Since $\mu=0$ when $n\not\equiv 2 \pmod 4$, we have shown that (image ψ) $=$(kernel μ).

It remains to prove that $\theta(M,\nu_\varphi)$ is an abelian group. As was observed earlier, it suffices to show that every $a\in\theta(M,\nu_\varphi)$ has an inverse. Choose a $b\in\psi^{-1}(-\psi(a))$. Such a b exists because (image ψ)$=$(kernel μ) is a subgroup of $\pi^0_{n+k}T(\nu_\varphi)$. Then $\psi(a+b)=\psi(a)+\psi(b)=\psi(a)-\psi(a)=0$. By exactness we can now find a $\gamma\in P_{n+1}$ such that $\partial(\gamma)=a+b$. Hence $0=\partial(0)=\partial(\gamma-\gamma)=\partial(\gamma)+\partial(-\gamma)=a+(b+\partial(-\gamma))$. This finishes the proof of Theorem 1.

Let us show that Theorem 1 is a generalization of a well known exact sequence. Suppose that $M^n=S^n$ and $\varphi\colon S^n\to S^{n+k}$ is the standard inclusion. Define $\lambda_0\colon\mathscr{S}^+(S^n,\nu_\varphi)\to\theta_f^{n+k,n}$ by $\lambda_0((F,V,f))=[V,\mathscr{F}_F]$, where $\mathscr{F}_F$ is the framing of V induced from the framing of S^n in S^{n+k}. (Note that V is indeed a homotopy sphere.) Clearly λ_0 is well defined. Also, $\lambda_0(\alpha+r\varepsilon)=\lambda_0(\alpha)$ for each $\alpha\in\mathscr{S}^+(S^n,\nu_\varphi)$ because if $\mathscr{F}_0$ is the standard framing of ν_φ, then $(\Sigma^n,\mathscr{F})\ \#\ (S^n,\mathscr{F}_0)=(\Sigma^n,\mathscr{F})$ for every framed homotopy sphere $(\Sigma^n,\mathscr{F})$ in S^{n+k}. (Here $\#$ denotes the operation of framed connected sum which induces the addition in $\theta_f^{n+k,n}$.) Therefore, λ_0 induces a well-defined map $\lambda\colon \theta(S^n, \nu_\varphi) \to \theta_f^{n+k,n}$ given by $\lambda([\alpha]) = \lambda_0(\alpha)$.

If $[\Sigma, \mathscr{F}] \in \theta_f^{n+k,n}$, let $f\colon \Sigma \to S^n$ be a homotopy equivalence with $\deg f = 1$. Let $g\colon \nu_\Sigma \to \nu_\varphi$ be given by $g(\mathscr{F}(y, u)) = (h(y), u) \in S^n \times D^k \subseteq S^{n+k}$ for $(y, u) \in \Sigma \times D^k$. Then g induces a map $F\colon S^{n+k} \to T(\nu_\varphi)$, and $\lambda_0((F, \Sigma, f)) = [\Sigma, \mathscr{F}]$. Thus λ_0, and hence λ is onto.

Next, let $\alpha, \beta \in \mathscr{S}^+(S^n, \nu_\varphi)$ and suppose that $\lambda_0(\alpha) = \lambda_0(\beta)$. Without loss of generality assume that $\deg \beta - \deg \alpha = r \geqq 0$. Then it is not hard to show that $\alpha + r\varepsilon = \beta$. (Observe that if $\alpha_i \in \mathscr{S}^+(S^n, \nu_\varphi)$, then $\alpha_1 + \alpha_2 = \alpha_1 \# \alpha_2$.) It follows that λ is one-to-one. But λ_0, and hence λ, is additive since the addition in $\mathscr{S}^+(S^n, \nu_\varphi)$ and $\theta_f^{n+k,n}$ both come from a connected sum operation, and so we have proved

LEMMA 8. *λ is an isomorphism.*

Now ν_φ is trivial, and so $T(\nu_\varphi) = S^{n+k} \vee S^k$ by [11]. $\pi_{n+k}T(\nu_\varphi) = \pi_{n+k}S^{n+k} \oplus \pi_{n+k}S^k$, where deg maps the first factor isomorphically onto $H_{n+k}T(\nu_\varphi) = \mathbf{Z}$. Therefore, we can identify $\pi^0_{n+k}T(\nu_\varphi)$ in a natural way with $\pi_{n+k}S^k$. (In fact, one can make this identification in the case of any imbedding $\varphi\colon S^n \to S^{n+k}$ with $2k \geqq n+3$ because ν_φ will then be trivial by [6].) It follows from Lemma 8 that the sequence

$$P_{n+1} \xrightarrow[\partial]{} \theta(S^n, \nu_\varphi) \xrightarrow[\psi]{} \pi^0_{n+k}T(\nu_\varphi) \xrightarrow[\mu]{} P_n$$

can be identified with the Milnor-Kervaire sequence

$$P_{n+1} \xrightarrow[\partial_1]{} \theta_f^{n+k,n} \xrightarrow[\psi_1]{} \pi_{n+k}S^k \xrightarrow[\mu_1]{} P_n,$$

where ψ_1 is defined via the Pontrjagin-Thom construction and μ_1, like μ, is the usual surgery obstruction.

These observations lead us to another definition of $\theta(M, \nu_\varphi)$. Briefly, it is possible to define $\theta(M, \nu_\varphi)$ to be the h-cobordism classes of (F, V, f) for which f is a homotopy equivalence. The sum of $[(F_1, V_1, f_1)]$ and $[(F_2, V_2, f_2)]$ is defined to be the class of that triple (F_3, V_3, f_3) which is obtained from $(F_1, V_1, f_1) \# (F_2, V_2, f_2) \# (T_\varphi, -M, \text{identity})$ by surgery for which f_3 is a homotopy equivalence. In order that this addition is well defined and that we get a group we have to be able to do the necessary surgery. This is why we need some conditions on n and the homology of M. Our condition, that either $n \equiv 0 \pmod 4$ or $H_tM = 0$ and $H_{t-1}M$ is torsion-free, can probably be weakened. The reason that we did not give this straightforward definition of $\theta(M, \nu_\varphi)$ at the beginning and proceeded in a roundabout fashion to define $\mathscr{S}^+(M, \nu_\varphi)$ first is that $\mathscr{S}^+(M, \nu_\varphi)$ and ψ_0 are interesting in their own right (see the next section).

Next, observe that the inclusion $i\colon S^{n+k} \subseteq S^{n+k+1}$ induces natural maps $\mathscr{T}_0^+(M, \nu_\varphi) \to \mathscr{T}_0^+(M, \nu_{i\varphi})$, $\mathscr{S}_0^+(M, \nu_\varphi) \to \mathscr{S}_0^+(M, \nu_{i\varphi})$, $\mathscr{T}^+(M, \nu_\varphi) \to \mathscr{T}^+(M, \nu_{i\varphi})$, $\mathscr{S}^+(M, \nu_\varphi) \to \mathscr{S}^+(M, \nu_{i\varphi})$, and $\theta(M, \nu_\varphi) \to \theta(M, \nu_{i\varphi})$. These "suspension" maps are clearly additive and will all be denoted by s. Let θ^k be the trivial k-disk bundle over M. Then $\nu_{i\varphi}=\nu_\varphi \oplus \theta^1$, and the following diagrams commute:

$$\begin{array}{ccccc}
 & \overset{\partial_0}{\nearrow} & \mathscr{S}^+(M,\nu_\varphi\oplus\theta^1) = \mathscr{S}^+(M,\nu_{i\varphi}) & \xrightarrow{\psi_0} & \pi_{n+k+1}T(\nu_{i\varphi}) = \pi_{n+k+1}S(T(\nu_\varphi)) \\
P_{n+1} & & \uparrow s & & \uparrow s_\# \\
 & \underset{\partial_0}{\searrow} & \mathscr{S}^+(M,\nu_\varphi) & \xrightarrow[\psi_0]{} & \pi_{n+k}T(\nu_\varphi)
\end{array}$$

$$\begin{array}{ccccc}
 & \overset{\partial}{\nearrow} & \theta(M,\nu_\varphi\oplus\theta^1) = \theta(M,\nu_{i\varphi}) & \xrightarrow{\psi} & \pi^0_{n+k+1}T(\nu_{i\varphi}) = \pi^0_{n+k+1}S(T(\nu_\varphi)) \\
P_{n+1} & & \uparrow s & & \uparrow s_\# \\
 & \underset{\partial}{\searrow} & \theta(M,\nu_\varphi) & \xrightarrow[\psi]{} & \pi^0_{n+k}T(\nu_\varphi)
\end{array}$$

($S(T(\nu_\varphi))$ is the reduced suspension of $T(\nu_\varphi)$ and $s_\#$ is the usual suspension map on homotopy.)

Finally, define

$$\theta(M) = \lim_t \theta(M, \nu_\varphi \oplus \theta^t).$$

It follows from the above remarks and Theorem 2 that $\theta(M)$ is a well-defined abelian group. In fact, $\theta(M)$ is isomorphic to $\theta(M, \nu_\varphi \oplus \theta^t)$ whenever $k+t \geqq n+3$. $\theta(M)$ is the group of manifolds which are "framed" homotopy equivalent to M.

4. **Conclusion.** We would like to conclude with some unanswered questions which arise naturally in the context of this paper:

1. $\theta(M, \nu_\varphi)$ corresponds to $\theta_f^{n+k,n}$. A natural analogue of $\theta^{n+k,n}$, the h-cobordism classes of (unframed) homotopy n-spheres in S^{n+k}, would seem to be the set, $\mathscr{S}_k(M)$, of h-cobordism classes of homotopy smoothings of M which are imbedded in S^{n+k}. The set $\mathscr{S}_k(M)$, for large k, was considered in [12] and fit into an exact sequence. There is a commutative diagram

$$\begin{array}{ccccccc}
 & \nearrow & \mathscr{S}_k(M) & \longrightarrow & [M, F_{k-1}/0_{k-1}] & \searrow & \\
P_{n+1} & & \uparrow & & \uparrow & & P_n \\
 & \underset{\partial}{\searrow} & \theta(M,\nu_\varphi) & \xrightarrow[\psi]{} & \pi^0_{n+k}T(\nu_\varphi) & \underset{\mu}{\nearrow} &
\end{array}$$

The top row was defined in [12] for large k and shown to be exact. It reduces to the Milnor-Kervaire sequence when $M^n = S^n$. Can one generalize other sequences in [5] and [10]?

2. When is an element $x \in \pi_{n+k}T(\nu_\varphi)$ in the image of ψ_0? This question is partially answered in [2] and involves the study of Browder-Novikov theory for maps of degree $d > 1$. Note how much easier it is to determine the image of ψ.

3. Let $(F, V, f) \in \mathscr{S}^+(M, \nu_\varphi)$. Is V homotopy equivalent to M? (The homotopy equivalence may have no relation to f.) This question in conjunction with question 2 has bearing on the problem of whether manifolds imbed in the metastable range. Since $f_*(f^*(a) \cap [V]) = (\deg f)(a \cap [M])$ for $a \in H^i M$, it follows that $f_*: H_i V \to H_i M$ is an isomorphism for $0 \leqq i < n$ whenever $H_i M$ is finite and the order of $H_i M$ is relatively prime to $\deg f$ for $0 < i < n$. This suggests a somewhat weaker question: Is V homotopy equivalent to M if $f_*: H_i V \to H_i M$ is an isomorphism for $0 \leqq i < n$?

4. When does a manifold M^n admit a map $f: M \to M$ of degree $d > 0$? Are there some more or less simple conditions on the homology or homotopy groups which will guarantee the existence of f? This problem fits into our context because it is related to the previous questions about $\mathscr{S}^+(M, \nu_\varphi)$ and $\pi_{n+k} T(\nu_\varphi)$.

5. Would it be useful to study manifolds M^n which have the property that $d\varepsilon = (F_d, \varphi(M^n), f_d)$? For example, $M^n = S^n$ has this property. Do products of spheres $S^l \times S^{n-l}$ behave similarly?

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