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SEMIPRIMARY HEREDITARY ALGEBRAS

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Abstract. Let Σ be a semiprimary k-algebra, with radical M. If Σ admits a splitting then $\dim_k \Sigma/M \leqq \dim_k \Sigma$. The residue algebra Σ/M^2 is finite (cohomological) dimensional if and only if all residue algebras are finite dimensional. If $\dim_k \Sigma = 1$ then all residue algebras are finite dimensional.

1. **Introduction.** We consider the following properties of algebras over a field k:

(p1): $\dim_k \Sigma \leqq 1$.

(p2): $\dim_k \Sigma/I$ is finite for every two sided ideal I in Σ.

(p3): $\dim_k \Sigma/M^2$ is finite, where M is the (Jacobson) radical of Σ.

(p4): Σ is a residue algebra of Ω, where gl.dim $\Omega \leqq 1$, $\dim_k \Sigma/M$ is finite and Σ/M is isomorphic to Ω/N, N being the (Jacobson) radical of Ω.

(p5): Σ is a residue algebra of Ω, where $\dim_k \Omega \leqq 1$ and Σ/M^2 is isomorphic to Ω/N^2.

For a finite dimensional k-algebra Σ, it was proved by Eilenberg, Nagao and Nakayama in [6] that (p1) $\Rightarrow$ (p2), while Jans and Nakayama proved in [7] the implications (p3) $\Rightarrow$ (p4) $\Rightarrow$ (p5). Thus for finite dimensional k-algebras one has the equivalences (p2) $\Leftrightarrow$ (p3) $\Leftrightarrow$ (p4) $\Leftrightarrow$ (p5).

The purpose of this paper is to establish the implication (p1) $\Rightarrow$ (p2), and the equivalences (p2) $\Leftrightarrow$ (p3) $\Leftrightarrow$ (p4) for semiprimary rings that are k-algebras. The equivalence (p4) $\Leftrightarrow$ (p5) can be deduced in certain particular cases as for instance if Σ/M is a finite dimensional k-algebra. To this extent we give an example of a semiprimary ring Σ for which $\dim_k \Sigma = 1$ and $\dim_k \Sigma/M = 1$.

As it turns out the passage from finite dimensional k-algebras to semiprimary ones is made possible by a lemma that seems to be of some interest in its own sake, namely:

A semiprimary k-algebra Σ that admits a splitting $\Sigma = \Delta + M$ [9] satisfies the inequality $\dim_k \Delta \leqq \dim_k \Sigma$, where Δ denotes the residue algebra Σ/M.

In [2] Auslander proved that if Δ is a finite dimensional k-algebra and $\dim_k \Sigma$ is finite, then $\dim_k \Sigma = \text{gl.dim}\, \Sigma$. He raised the problem whether it is necessary that $\dim_k \Delta = 0$ (e.g. [4] and [5]). We prove the answer to be affirmative in case that $\dim_k \Sigma/M^2$ is finite.

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2. **Hereditary algebras.** A k-algebra Σ is said to be a semiprimary k-algebra if Σ is a semiprimary ring, i.e. its (Jacobson) radical M is nilpotent and the residue ring Σ/M is a semisimple (Artinian) ring. Set $\Delta=\Sigma/M$. We say that a k-algebra Σ is an hereditary k-algebra if $\dim_k \Sigma \leqq 1$. By Σ° we denote the apposite ring to Σ. By $(\Sigma:k)<\infty$ we denote that the k-algebra Σ is finite dimensional (as a k-vector space). For the rest we write dim for $\dim_k$, and $\otimes$ for $\otimes_k$. We say that Σ admits a splitting if $\Sigma=\Delta+M$ [7], [9]. A crucial step towards our main theorem is the following lemma:

LEMMA 1. *If Σ admits a splitting, $\Sigma=\Delta+M$, then* $\dim\Delta \leqq \dim \Sigma$.

Proof. If $\dim \Sigma=\infty$ we are done. Otherwise $\dim \Sigma$ is finite, and we may assume that $\dim \Sigma=t<\infty$. By [4] we have the equality $\text{gl.dim}\, \Sigma \otimes \Delta^\circ=\dim \Sigma=t<\infty$. Since Δ is a semisimple ring, M is a projective right Δ-module. From the natural isomorphism of $M \otimes \Delta^\circ$ with $M \otimes_\Delta (\Delta \otimes \Delta^\circ)$ it follows that $M \otimes \Delta^\circ$ is a projective right $\Delta \otimes \Delta^\circ$-module. Hence via the natural embedding of $\Delta \otimes \Delta^\circ$ into $\Sigma \otimes \Delta^\circ$, $\Sigma \otimes \Delta^\circ$ becomes a projective right $\Delta \otimes \Delta^\circ$-module. Denote by f the natural embedding of $\Delta \otimes \Delta^\circ$ into $\Sigma \otimes \Delta^\circ$, and denote by g the canonical epimorphism of $\Sigma \otimes \Delta^\circ$ onto $\Delta \otimes \Delta^\circ$, then $g \circ f$ is the identity map on $\Delta \otimes \Delta^\circ$.

For any left $\Delta \otimes \Delta^\circ$-module A we set $A_f=(\Sigma \otimes \Delta^\circ) \otimes_{(\Delta \otimes \Delta^\circ)} A$.

For any left $\Sigma \otimes \Delta^\circ$-module B we set $B_g=(\Delta \otimes \Delta^\circ) \otimes_{(\Sigma \otimes \Delta^\circ)} B$.

There results a $\Delta \otimes \Delta^\circ$ isomorphism from A onto $(A_f)_g$, for every left $\Delta \otimes \Delta^\circ$-module A.

Let A be a left $\Delta \otimes \Delta^\circ$-module, and let

$$0 \to L \to P_{t-1} \to \cdots \to P_0 \to A \to 0$$

be an exact sequence of left $\Delta \otimes \Delta^\circ$-modules, where $P_0, \ldots, P_{t-1}$ are projective $\Delta \otimes \Delta^\circ$-modules. We claim that either L is a projective $\Delta \otimes \Delta^\circ$-module, or else $L=0$.

Since $\Sigma \otimes \Delta^\circ$ is a projective right $\Delta \otimes \Delta^\circ$-module, there results an exact sequence of left $\Sigma \otimes \Delta^\circ$-modules:

$$0 \to L_f \to (P_{t-1})_f \to \cdots \to (P_0)_f \to A_f \to 0$$

where $(P_i)_f=(\Sigma \otimes \Delta^\circ) \otimes_{(\Delta \otimes \Delta^\circ)} P_i$, for $i=0, \ldots, (t-1)$. Thus $(P_0)_f, \ldots, (P_{t-1})_f$ are $\Sigma \otimes \Delta^\circ$-projective. Since $\text{l.gl.dim}\, \Sigma \otimes \Delta^\circ=\dim \Sigma=t$, it follows that either L_f is a projective $\Sigma \otimes \Delta^\circ$-module, or else $L_f=0$. Hence L is $\Delta \otimes \Delta^\circ$ isomorphic to the $\Delta \otimes \Delta^\circ$-module $(L_f)_g=(\Delta \otimes \Delta^\circ) \otimes_{(\Sigma \otimes \Delta^\circ)} L_f$. Therefore $L=0$ or else L is a projective $\Delta \otimes \Delta^\circ$-module.

Therefore for every left $\Delta \otimes \Delta^\circ$-module A we have $\text{l.p.dim}_{\Delta \otimes \Delta^\circ} A \leqq t$, thus $\text{l.gl.dim}\, \Delta \otimes \Delta^\circ \leqq t$. Since by [4] we have the equality $\dim \Delta=\text{l.gl.dim}\, \Delta \otimes \Delta^\circ$ we may conclude that the inequality $\dim \Delta \leqq \dim \Sigma$ holds.

Recall that all residue rings of a semiprimary ring Σ are of finite global dimension iff Σ is a residue ring of a semiprimary ring Ω for which $\text{gl.dim}\, \Omega \leqq 1$, and this is the case iff gl.dim Σ/M^2 is finite [9]. Under each of these equivalent conditions Σ admits a splitting $\Sigma=\Delta+M$.

The splitting of Σ is inherited by every residue ring Σ_1 of Σ, $\Sigma_1=\Delta_1+M_1$. Furthermore, Δ_1 is (up to an isomorphism) a direct factor of Δ.

We are now ready to state and prove our main theorem that establishes the equivalences (p2) $\Leftrightarrow$ (p3) $\Leftrightarrow$ (p4) for semiprimary k-algebras.

THEOREM 1. *The following are equivalent*:

(a) $\dim \Delta<\infty$ *and* gl.dim $\Sigma/M^2<\infty$.

(b) $\dim \Sigma/I<\infty$ *for every two sided ideal I in Σ.*

(c) $\dim \Sigma/M^2<\infty$.

Proof. (a) $\Rightarrow$ (b): From gl.dim $\Sigma/M^2<\infty$ it follows by [9] that gl.dim $\Sigma/I<\infty$ for every two sided ideal I in Σ. Set $\Delta_1=(\Sigma/I)/(I+M/I)$ then Δ_1 is a direct factor of Δ, hence $\dim \Delta_1\leqq\dim \Delta$. Combining the equality $\dim \Sigma/I=$ l.gl.dim $\Sigma/I\otimes\Delta_1^\circ$ [4] with the inequality l.gl.dim $\Sigma/I\otimes\Delta_1^\circ\leqq$ l.gl.dim $\Sigma/I+\dim \Delta_1^\circ$ [5] it results that $\dim \Sigma/I$ is finite.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a): Since gl.dim $\Sigma/M^2\leqq\dim \Sigma/M^2$ [5], then gl.dim $\Sigma/M^2<\infty$. Hence by [9] Σ/M^2 admits a splitting, and thus Lemma 1 implies the inequality $\dim \Delta\leqq \dim \Sigma/M^2$, therefore $\dim \Delta$ is finite.

Observe that under each of the equivalent conditions in Theorem 1, Σ is a residue of a semiprimary k-algebra Ω with radical N, such that Ω/N is isomorphic with Δ, and gl.dim $\Omega\leqq 1$. This is an immediate consequence of Theorem 1 applied to $\Omega(\Delta, N)$ [9]. It is worth noticing that Σ admits a splitting, $\Sigma=\Delta+M$.

As for $\dim \Omega$, from $\dim \Omega=$ l.gl.dim $\Omega\otimes\Delta^\circ$ it follows that $\dim \Delta\leqq\dim \Omega\leqq \dim \Delta+1$.

In the next section we will bring some examples showing that it is possible that $\dim \Omega=\dim \Delta+1$, but it is also possible that the equality $\dim \Omega=\dim \Delta$ will hold.

Consider the case where k is the center of Σ. One can easily construct examples in which Σ is a residue ring of a semiprimary hereditary ring Ω with radical N, such that Ω/N is isomorphic with Δ, but Ω is not a k-algebra, i.e., not every semiprimary hereditary ring—of which Σ is a residue ring—is a k-algebra [10, Example 1].

Notice that if Δ is a finite dimensional k-algebra then $\dim \Delta=0$. One verifies that if $M\neq 0$ then $\dim \Omega=1$. Furthermore, if Ω is any semiprimary hereditary ring with radical N of which Σ is a residue ring, such that Ω/N is isomorphic with Δ, then Ω admits a splitting, $\Omega=\Delta+A+N^2$. Therefore, if one insists on Ω/N^2 being isomorphic to Σ/M^2 it follows that up to an isomorphism Ω is uniquely determined. This establishes the equivalence (p4) $\Leftrightarrow$ (p5) in case Δ is a finite dimensional k-algebra. Also in this case we have $\dim \Omega/I=$ gl.dim Ω/I for every two sided ideal I in Ω. In particular from [10] it results that $\dim \Omega/I\leqq\dim \Omega/N^2$, whenever $I\subset N^2$.

We do not know if this last inequality holds without the assumption $\dim \Delta = 0$.

Our next aim is to prove the implication (p1) $\Rightarrow$ (p2) for semiprimary k-algebras. Recall that the validity of this implication for finite dimensional k-algebras is based on the equality $\dim \Sigma = \text{gl.dim}\, \Sigma$, which is a consequence of $\dim \Delta = 0$ under these circumstances (e.g. [6]). For semiprimary k-algebras we have by [5] the inequality $\text{gl.dim}\, \Sigma \leqq \dim \Sigma$. Furthermore, if $\text{gl.dim}\, \Sigma/M^2$ is finite then by [9] Σ admits a splitting $\Sigma = \Delta + M$. We proceed with a sequence of corollaries to get the desired implication.

COROLLARY 1. *If* Σ *admits a splitting*, $\Sigma = \Delta + M$, *then* $\dim \Delta \leqq \dim \Sigma/I$ *for every two sided ideal* I *in* Σ *that is contained in the radical.*

Proof. Since the splitting is inherited by all residue rings of Σ, and since $I \subset M$ implies that $(\Sigma/I)/(I + M/I)$ is isomorphic with Δ, then we have applying Lemma 1: $\dim \Delta \leqq \dim \Sigma/I$.

COROLLARY 2. *If* $M^2 = 0$ *then* $\dim \Delta \leqq \dim \Sigma$.

Proof. If $\dim \Sigma = \infty$ we are done. Otherwise $\dim \Sigma$ is finite, hence $\text{gl.dim}\, \Sigma$ is finite. Therefore Σ admits a splitting, $\Sigma = \Delta + M$, and the result follows from Lemma 1.

COROLLARY 3. *If* $\dim \Sigma = 1$ *then* $\dim \Delta \leqq 1$.

Proof. The proof is an immediate consequence of Lemma 1 which is applicable in this case, since $\text{gl.dim}\, \Sigma \leqq \dim \Sigma = 1$ implies the splitting of Σ.

It seems interesting to notice that one can prove that $\dim \Delta$ is finite by observing that $\Delta \otimes \Delta^\circ$ is a residue ring of the hereditary ring $\Sigma \otimes \Delta^\circ$ by the nilpotent two sided ideal $M \otimes \Delta^\circ$ (e.g. [6]).

As a consequence there results the implication (p1) $\Rightarrow$ (p2).

COROLLARY 4. *If* $\dim \Sigma = 1$ *then* $\dim \Sigma/I$ *is finite for every two sided ideal* I *in* Σ.

Proof. This is an immediate consequence of Theorem 1 since by Corollary 3 $\dim \Delta \leqq 1$, and since $\text{gl.dim}\, \Sigma \leqq \dim \Sigma$ implies that $\text{gl.dim}\, \Sigma/M^2$ is finite.

3. **Examples.** In this section we will bring some examples of k-algebras, all the residue algebras of which have finite cohomological dimension. We will be mainly concerned with the inequalities $\dim \Delta \leqq \dim \Sigma \leqq \dim \Delta + \text{gl.dim}\, \Sigma$, and with the equality $\dim \Sigma = \text{gl.dim}\, \Sigma$ without $(\Delta : k)$ being finite.

Let $k(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be the field of rational functions in $n+m$ variables over the field k. We will identify k $(k(x_1, \ldots, x_n), k(y_1, \ldots, y_m))$ with its natural embedding in $k(x_1, \ldots, x_n, y_1, \ldots, y_m)$.

EXAMPLE 1. Let Σ be the k-subalgebra of the 2×2 matrix algebra over the field of rational functions in one variable over a field k, $k(x)$. A matrix σ belongs to Σ iff σ is of the form

$$\begin{Vmatrix} a & 0 \\ b & c \end{Vmatrix},$$

where a is an element of k, and b, c are elements of $k(x)$.

Obviously Σ is a left Artinian hereditary ring with radical M of square zero, and $\dim \Delta = 1$. We claim that $\dim \Sigma = 1$. It will suffice to show that $\text{l.gl.dim}\, \Sigma \otimes k(x) = 1$. Identify $\Sigma \otimes k(x)$ with a subring of the 2×2 matrix algebra over $k(x) \otimes k(x)$, namely: σ' belongs to $\Sigma \otimes k(x)$ iff σ' is of the form:

$$\begin{Vmatrix} \alpha & 0 \\ \beta & \gamma \end{Vmatrix},$$

where α belongs to $k \otimes k(x)$ (which is isomorphic to $k(x)$), and β, γ belong to $k(x) \otimes k(x)$.

Let J be a left ideal in $\Sigma \otimes k(x)$, then one readily verifies that J is of one of the following two types:

Type 1. Every element in J is of the form

$$\begin{Vmatrix} 0 & 0 \\ \beta & \gamma \end{Vmatrix}.$$

Type 2. J is a direct sum of two subideals J_1 and J_2 where every element of J_1 is of the form

$$\begin{Vmatrix} 0 & 0 \\ 0 & \gamma \end{Vmatrix}$$

and a matrix σ' belongs to J_2 iff it is of the form

$$\begin{Vmatrix} \alpha & 0 \\ \beta & 0 \end{Vmatrix}.$$

Since $\dim \mathrm{k}(x) = 1$ it follows from [3, Theorem 5.4, p. 14] that in either case J is a projective left $\Sigma \otimes k(x)$-module. Hence by [3, Theorem 5.4, p. 14] it follows that $\text{l.gl.dim}\, \Sigma \otimes k(x) = 1$.

A similar treatment, using the fact that $k(x_1, \ldots, x_n) \otimes k(x_1, \ldots, x_n)$ is a Neotherian ring—where $k(x_1, \ldots, x_n)$ is the field of rational functions in n variables over k—gives:

Example 1*. Let Σ be the k-subalgebra of the 2×2 matrix algebra over $k(x_1, \ldots, x_n)$. A matrix σ belongs to Σ iff σ is of the form

$$\begin{Vmatrix} a & 0 \\ b & c \end{Vmatrix},$$

where a is an element of k, and b, c are elements of $k(x_1, \ldots, x_n)$.

Σ is a left Artinian ring with radical M of square zero, $\dim \Delta = n$ [5], and $\text{gl.dim}\, \Sigma = 1$. Finally, by the remark made above, we have by checking $\dim \Sigma$ via $\text{l.gl.dim}\, \Sigma \otimes k(x_1, \ldots, x_n)$ that $\dim \Sigma = n$.

Example 2. By taking successive rings of triangular matrices of the ring Σ that was constructed in Example 1 (1*) we obtain a left Artinian ring $\Lambda = T_{n_t}(\cdots T_{n_1}(\Sigma) \cdots)$. By [5] it follows that

$$\dim \Lambda = \text{gl.dim}\, \Lambda = t+1 \qquad (\dim \Lambda = t+n, \text{gl.dim}\, \Lambda = t+1).$$

Furthermore, if N is the radical of Λ then $\dim \Lambda/N = \dim \Delta$, since Λ/N is isomorphic to a direct product of $n_1 \cdots n_t$ copies of Δ.

Summarizing we have:

PROPOSITION 1. *For every pair of positive integers n, s there exists a k-algebra Σ for which* $\text{gl.dim}\, \Sigma = s$, $\dim \Delta = n$, *and* $\dim \Sigma < \text{gl.dim}\, \Sigma + \dim \Delta$.

Taking $n=1$ there will result a k-algebra Σ for which $\dim \Sigma = \text{gl.dim}\, \Sigma < \infty$, such that $\dim \Delta = 1$.

EXAMPLE 3. Let Σ be the k-subalgebra of the 2×2 matrix algebra over $k(x, y)$—the field of rational functions in two variables over the field k. A matrix σ belongs to Σ iff σ is of the form

$$\begin{Vmatrix} a & 0 \\ b & c \end{Vmatrix},$$

where a belongs to $k(y)$, b belongs to $k(x, y)$, and c belongs to $k(x)$.

Σ is a semiprimary ring with radical of square zero. Obviously $\dim \Delta = 1$, $\text{gl.dim}\, \Sigma = 1$, and it is an easy exercise to check that $\dim \Sigma = 2$.

EXAMPLE 3*. Take Σ to be the k-subalgebra of the 2×2 matrix algebra over $k(x_1, \ldots, x_n, y_1, \ldots, y_n)$—the field of rational functions in $2n$ variables over the field k. A matrix σ belongs to Σ iff σ is of the form

$$\begin{Vmatrix} a & 0 \\ b & c \end{Vmatrix},$$

where a belongs to $k(y_1, \ldots, y_n)$, b belongs to $k(x_1, \ldots, x_n, y_1, \ldots, y_n)$, and c belongs to $k(x_1, \ldots, x_n)$.

Σ is a semiprimary ring, and $\text{gl.dim}\, \Sigma = 1$. Again by straightforward computations it follows that $\dim \Sigma = n+1$, and from [5] $\dim \Delta = n$.

EXAMPLE 4. By taking successive rings of triangular matrices of the ring Σ that was constructed in Example 3 (3*) we obtain a semiprimary ring $\Lambda = T_{n_t}(\cdots T_{n_1}(\Sigma)\cdots)$ with radical N. From [5] it follows that $\text{gl.dim}\ \Lambda = t+1$, and $\dim \Lambda = t+2$ ($\dim \Lambda = t+n+1$). Furthermore, since Λ/N is the direct product of $n_1 \cdots n_t$ copies of Δ, $\dim \Lambda/N = 1$ ($\dim \Lambda/N = n$).

Summarizing we obtain:

PROPOSITION 2. *For every pair of positive integers n, s there exists a k-algebra Σ for which* $\text{gl.dim}\, \Sigma = s$, $\dim \Delta = n$, *and* $\dim \Sigma = \text{gl.dim}\, \Sigma + \dim \Delta$.

Notice that in all our examples, k is the center of each of the constructed rings.

4. **Applications.** In [2] Auslander proved that if $\dim \Sigma < \infty$ and if $(\Delta : k) < \infty$ then $\dim \Sigma = \text{gl.dim}\, \Sigma$. He raised the problem whether $\dim \Sigma$ is necessarily zero. In [4] Eilenberg proved that if $\dim \Sigma < \infty$ and $(\Sigma : k) < \infty$ then $\dim \Delta = 0$ and $\dim \Sigma = \text{gl.dim}\, \Sigma$. In §3 we saw that it is possible to have $\dim \Sigma < \infty$ and $\dim \Sigma = \text{gl.dim}\, \Sigma$ without $(\Sigma : k)$ nor $(\Delta : k)$ being finite. Furthermore, $\dim \Sigma = \text{gl.dim}\, \Sigma$ may hold without $\dim \Delta$ being zero. Still we have:

PROPOSITION 3. *If* $M^2=0$, *if* $(\Delta:k)<\infty$, *and if* $\dim \Sigma$ *is finite then* $\dim \Delta=0$.

Proof. By Corollary 2 $\dim \Delta \leqq \dim \Sigma < \infty$. Since $(\Delta:k)<\infty$ we now have $\dim \Delta=0$.

In this respect it is worth stating an immediate consequence of Theorem 1, that turns out to be just an affirmative answer to the problem raised by Auslander in a particular case.

COROLLARY 5. *If* $(\Delta:k)<\infty$ *then the following are equivalent*:

(a) $\dim \Delta=0$ *and* $\mathrm{gl.dim}\, \Sigma/M^2<\infty$,

(b) $\dim \Sigma/M^2<\infty$.

Under each of these equivalent conditions $\dim \Sigma/I=\mathrm{gl.dim}\, \Sigma/I<\infty$ *for every two sided ideal I in* Σ.

Let Σ_1 (Σ_2) be a semiprimary k-algebra with radical M_1 (M_2), and set $\Delta_i=\Sigma_i/M_i$ for $i=1, 2$. Assuming that $(\Delta_i:k)<\infty$, and $\dim \Sigma_i/M_i^2<\infty$ for $i=1, 2$ it follows that $\dim \Delta_i=0$ for $i=1, 2$. Denote $\Delta=\Delta_1 \otimes \Delta_2$, and $N=M_1 \otimes \Delta_2+\Delta_1 \otimes M_2$ then it readily follows that $\dim \Omega \leqq 1$, where $\Omega=\Omega(\Delta, N)$ [9]. Furthermore, $\Sigma_1 \otimes \Sigma_2$ is a residue k-algebra of Ω, and $(\Delta:k)<\infty$. We therefore have:

THEOREM 2. *The class of semiprimary k-algebras* $\mathfrak{C}$ *is closed under tensor products. A semiprimary k-algebra* Σ *belongs to* $\mathfrak{C}$ *iff* $\dim \Sigma/M^2<\infty$ *and* $(\Delta:k)<\infty$.

Notice that this theorem is no longer valid if we replace $\dim \Sigma/M^2<\infty$ by $\mathrm{gl.dim}\, \Sigma/M^2<\infty$.

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