

WEAKENING A THEOREM ON DIVIDED POWERS

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Abstract. We show that if a Hopf algebra has finite dimensional primitives and a primitive lies in arbitrarily long finite sequences of divided powers then it lies in an infinite sequence of divided powers.

Introduction. K. Newman has shown us a counter-example to [3, Theorem 2, p. 521]. The theorem states that if H is a cocommutative Hopf algebra over a perfect field k of characteristic $p > 0$ and k is the unique simple subcoalgebra of H then a primitive element 1x of H lies in a sequence of divided powers $1 = {}^0x, {}^1x, \dots, {}^{p^n+1}x$ if and only if 1x has coheight n , for $n = 0, 1, \dots, \infty$. (The sequence should be considered infinite if $n = \infty$.) The proof given in [3, p. 524] correctly shows that the existence of the sequence of divided powers implies that 1x has the desired coheight for all n . The proof there also correctly shows that for finite n if 1x has coheight n then the desired sequence of divided powers exists. The error seems to be the assertion $K^1 \cong H^1/(\bigcup \ker F^i)$ [3, p. 254, lines 23–24].

Newman's example shows that 1x may have infinite coheight—so lies in arbitrarily long finite sequences of divided powers—and yet 1x lies in no infinite sequence of divided powers.

We show here that with the further assumption that the primitives of H are finite dimensional then 1x having infinite coheight implies that 1x lies in an infinite sequence of divided powers. This result is important because it plays a key role in the proof of Jacobson's conjecture in [1]. We explain this in more detail at the end of the present paper.

1. Suppose C is a cocommutative coalgebra with a unique simple subcoalgebra which is one dimensional. We identify this simple subcoalgebra with k and so consider $k \subset C$. Then the primitive elements of C , $P(C) \equiv \{c \in C \mid \Delta c = 1 \otimes c + c \otimes 1\}$.

HEYNEMAN'S THEOREM. *If $\dim P(C) < \infty$ then C satisfies the minimum condition and descending chain condition for subcoalgebras.*

Proof. For a vector space U the coalgebra $Sh(U)$ is defined in [4, p. 244, p. 254]. By [4, p. 254, 12.1.1] there is a finite-dimensional space U and an injective coalgebra

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map $F: C \rightarrow Sh(U)$. In [4, p. 261] $B(U)$ is defined as the maximal cocommutative subcoalgebra of $Sh(U)$. Since C is cocommutative $\text{Im } F \subset B(U)$ and we consider F as a coalgebra injection $C \xrightarrow{F} B(U)$. Since F is injective by applying F to the coalgebras in a family of subcoalgebras of C we see that it suffices to show that $B(U)$ has the minimal condition on subcoalgebras. Similarly for the descending chain condition.

By [4, p. 278, Example-Exercise] the linear dual $\text{Hom}_k(B(U), k)$ is a power-series ring in $\dim U$ variables; hence, is Noetherian. By [4, p. 16, 1.4.3] if D is a subcoalgebra of $B(U)$ then $D^\perp = \{f \in \text{Hom}_k(B(U), k) \mid f(D) = 0\}$ is an ideal in $\text{Hom}_k(B(U), k)$. Thus the maximal condition and ascending chain condition for ideals in $\text{Hom}_k(B(U), k)$ implies the desired conditions for $B(U)$. Q.E.D.

THEOREM 2. *Let k be a perfect field of characteristic $p > 0$ and H a cocommutative Hopf algebra over k where k is the unique simple subcoalgebra of H . Suppose 1x is a primitive element of H .*

(i) *If $1 = {}^0x, {}^1x, {}^2x, \dots, {}^{p^{n+1}-1}x$ is a sequence of divided powers in H then 1x has coheight n , for $n = 0, 1, \dots, \infty$.*

(ii) *If 1x has coheight n then there is a sequence of divided powers in H . $1 = {}^0x, {}^1x, \dots, {}^{p^{n+1}-1}x$, for $n = 0, 1, \dots$*

(iii) *If $\dim P(H) < \infty$ and 1x has infinite coheight then there is an infinite sequence of divided powers in H , $\{{}^i x\}_{i=0}^\infty$.*

Proof. In [3, p. 524, proof of Theorem 2] statements 1 and 2 are correctly proved. In [3, p. 520, Theorem 1] the V map $V: H \rightarrow H$ is defined. By [3, p. 521, Theorem 1] $V^n(V) = V \cdots V(H)$ (n -times) is a sub-Hopf algebra of H , thus a subcoalgebra. Assuming $\dim P(H) < \infty$ the descending chain $H \supset V(H) \supset V^2(H) \cdots$ must stabilize say at $V^n(H)$. Thus $VV^n(H) = V^n(H)$. Since 1x has infinite coheight it follows that ${}^1x \in V^n(H)$. Thus $1 = {}^0x, {}^1x$ may be extended to an infinite sequence of divided powers lying in $V^n(H)$ by [3, p. 522, Lemma 7]. Q.E.D.

The statement just above [1, p. 285, 3.3.4] that $(PH)_\infty = \{x \in PH \mid \text{there is an infinite sequence of divided powers lying over } x\}$ is false (by Newman's example) unless we have $\dim PH < \infty$, so Theorem 2 (this paper) applies. Thus the proof given for [1, p. 285, 3.3.4] is incorrect unless one assumes that *both* PH and PJ are finite dimensional. The use made of 3.3.4 is [1, p. 290, 3 lines above 3.5] in the proof of 3.5.3, Jacobson's conjecture, where *both* PH and PJ are finite dimensional. (Actually it can be shown by techniques developed by Newman that 3.3.4 is correct as it stands.)

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