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# DENSE SIGMA-COMPACT SUBSETS OF INFINITE-DIMENSIONAL MANIFOLDS[1]

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**Abstract.** In this paper four classes of separable metric infinite-dimensional manifolds are studied; those which are locally the countable infinite product of lines, those which are locally open subsets of the Hilbert cube, and those which are locally one of two dense sigma-compact subsets of the Hilbert cube. A number of homeomorphism, product, characterization, and embedding theorems are obtained concerning these manifolds.

1. **Introduction.** Let $s$ denote the countable infinite product of open intervals and regard the Hilbert cube $I^\infty$ as the product of the closures of those intervals. A *Fréchet manifold* (or *F*-manifold) is a separable metric space having an open cover by sets each homeomorphic to an open subset of $s$. This definition is justified since all separable infinite-dimensional Fréchet spaces are known to be homeomorphic to $s$ (see [7] for references). A *Hilbert cube manifold* (or *Q-manifold*) is a separable metric space having an open cover by sets each homeomorphic to an open subset of $I^\infty$.

Let $X$ be a separable metric space. A closed subset $K$ of $X$ is said to have *Property* Z *in* $X$ (more briefly we say that $K$ is a *Z-set in* $X$) provided that for any nonnull homotopically trivial open subset $U$ of $X$, $U\backslash K$ is nonnull and homotopically trivial. Following Anderson [6] we say that a subset $M$ of $X$ has the (*finite-dimensional*) *compact absorption property*, or (f-d) cap, in $X$ if (1) $M=\bigcup_{n=1}^{\infty} M_n$, where each $M_n$ is a (finite-dimensional) compact $Z$-set in $X$ such that $M_n\subset M_{n+1}$, and (2) for each $\varepsilon>0$, each integer $m>0$, and each (finite-dimensional) compact subset $K$ of $X$, there is an integer $n>0$ and an embedding $h: K\to M_n$ such that $h|K\cap M_m=\mathrm{id}$ and $d(h, \mathrm{id})<\varepsilon$. We use the convention that (f-d) cap represents two alternatives in an obvious way, one being cap and the other being f-d cap. In this sense we say that $M$ is an (f-d) cap-set for $X$ provided that $M$ has the cap (or f-d cap) in $X$.

In [6] R. D. Anderson has established the following theorem: Let $X$ be $I^\infty$ or $s$ and let $M$, $N$ be subsets of $X$, both having the (f-d) cap in $X$. Then there is a

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[1] The results of this paper have been submitted as the author's dissertation at Louisiana State University.

homeomorphism of $X$ onto itself taking $M$ onto $N$. We remark that Bessaga and Pełczyński [12] and Torunczyk [17] have also established this result by use of a device resembling the (f-d) cap.

The question of the equivalence of sets which have the (f-d) cap in an $F$-manifold was raised at a conference on infinite-dimensional topology held at Cornell University, January 5–7, 1969, and it appears as Problem 29 in the report of that meeting, *Problems in the topology of infinite-dimensional manifolds*. In §6 we give an affirmative answer to this question.

We will regard the Hilbert cube as a canonical compactification of $s$ in which $I^\infty = \prod_{i=1}^\infty I_i$ and $s = \prod_{i=1}^\infty I_i^0$, where for each $i > 0$ we have $I_i = [-1, 1]$ and $I_i^0 = (-1, 1)$. The definition of the (f-d) cap characterizes two different types of subsets of $I^\infty$. Let $\sigma$ denote the set of all points in $s$ having at most finitely many nonzero coordinates and let $\Sigma$ denote the set of all points in $s$ having at most finitely many coordinates not in $[-\frac{1}{2}, \frac{1}{2}]$.

We remark that in real Hilbert space $l^2$ there are canonical versions of $\sigma$ and $\Sigma$. The space $\sigma$ is homeomorphic to a vector subspace of $l^2$, namely, the linear span of the usual orthonormal basis. The space $\Sigma$ is homeomorphic to a vector subspace of $l^2$, namely, the linear span of the usual copy of the Hilbert cube there. It is shown in [6] that $\Sigma$ has the cap in $I^\infty$ and $\sigma$ has the f-d cap in $I^\infty$.

We will be concerned with manifolds modeled on $\Sigma$ and $\sigma$. A *$\Sigma$-manifold* (or *$\sigma$-manifold*) is a separable metric space having an open cover by sets each homeomorphic to an open subset of $\Sigma$ (or $\sigma$). In this paper we obtain a number of homeomorphism, characterization, and embedding theorems concerning $F$, $Q$, $\Sigma$, and $\sigma$-manifolds. We summarize these results in the next section.

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2. **Summary of principal results on $F$, $Q$, $\Sigma$, and $\sigma$-manifolds.** In this section we list some principal results on $F$ and $Q$-manifolds along with the principal results of this paper. The references refer to results of this paper unless other authorship is indicated.

I. *Characterization of manifolds by homotopy type.*

THEOREM 2.1 (HENDERSON [14]). *If $X$ and $Y$ are $F$-manifolds of the same homotopy type, then they are homeomorphic.*

THEOREM 2.2 (COROLLARY 10.2). *If $X$ and $Y$ are both $\sigma$-manifolds or both $\Sigma$-manifolds and are of the same homotopy type, then they are homeomorphic.*

CONJECTURE 2.1. *If $X$ and $Y$ are compact $Q$-manifolds of the same homotopy type, then they are homeomorphic.*

II. *Open embedding theorems.*

THEOREM 2.3 (HENDERSON [14]). *If $X$ is an $F$-manifold, then there is an open embedding $h: X \to s$.*

THEOREM 2.4 (THEOREM 10.3). *If $X$ is a $\sigma$ (or $\Sigma$)-manifold, then $X$ can be embedded as an open subset of $\sigma$ (or $\Sigma$).*

III. *Product theorems.*

THEOREM 2.5 (WEST [18]). *If $K$ is any countable locally-finite simplicial complex, then $|K|\times s$ is an F-manifold and $|K|\times I^\infty$ is a Q-manifold.*

THEOREM 2.6 (COROLLARY 8.4). *If $K$ is any countable locally-finite simplicial complex, then $|K|\times\sigma$ is a $\sigma$-manifold and $|K|\times\Sigma$ is a $\Sigma$-manifold.*

THEOREM 2.7 (COROLLARY 10.3). *If $X$ is a $\sigma$-manifold and $Y$ is a $\Sigma$-manifold of the same homotopy type, then $X\times I^\infty$ and $Y$ are homeomorphic.*

IV. *Factor theorems.*

THEOREM 2.8 (HENDERSON [14]). *If $X$ is an F-manifold, then there is a countable locally-finite simplicial complex $K$ such that $X$ and $|K|\times s$ are homeomorphic.*

THEOREM 2.9 (COROLLARY 10.4). *If $X$ is a $\sigma$-manifold (or $\Sigma$-manifold), then there is a countable locally-finite simplicial complex $K$ such that $X$ is homeomorphic to $|K|\times\sigma$ (or $|K|\times\Sigma$).*

CONJECTURE 2.2. *If $X$ is any Q-manifold, then there is a countable locally-finite simplicial complex $K$ such that $X$ and $|K|\times I^\infty$ are homeomorphic.*

THEOREM 2.10 (ANDERSON AND SCHORI [9]). *If $X$ is an F-manifold, then $X$, $X\times s$, and $X\times I^\infty$ are all homeomorphic.*

THEOREM 2.11 (ANDERSON AND SCHORI [9]). *If $X$ is a Q-manifold, then $X$ and $X\times I^\infty$ are homeomorphic.*

THEOREM 2.12 (THEOREM 8.3 AND THEOREM 10.2). *If $X$ is a $\sigma$-manifold and $I^n$ is any n-cell, then $X$, $X\times\sigma$, and $X\times I^n$ are all homeomorphic.*

THEOREM 2.13 (COROLLARY 8.5 AND THEOREM 10.2). *If $X$ is any $\Sigma$-manifold, then $X$, $X\times\Sigma$, and $X\times I^\infty$ are all homeomorphic.*

V. *Relationships between $F$, $Q$, $\Sigma$, and $\sigma$-manifolds.*

THEOREM 2.14 (THEOREM 10.2 AND COROLLARY 10.1). *If $X$ is a $\sigma$ (or $\Sigma$)-manifold, then $X$ can be embedded as an f-d cap (or cap)-set for an F-manifold and also for a Q-manifold.*

THEOREM 2.15 (ANDERSON [6]). *If $X$ is $I^\infty$ or $s$ and $M$, $N$ are (f-d) cap-sets for $X$, then there is a homeomorphism of $X$ onto itself taking $M$ onto $N$.*

THEOREM 2.16 (LEMMA 5.4 AND THEOREM 2.15). *If $X$ is an F or Q-manifold and $M$ is an f-d cap (or cap)-set for $X$, then $M$ is a $\sigma$ (or $\Sigma$)-manifold.*

THEOREM 2.17 (THEOREMS 6.1 AND 6.2). *If $X$ is an F or Q-manifold and $M$, $N$ are (f-d) cap-sets for $X$, then there is a homeomorphism of $X$ onto itself taking $M$ onto $N$.*

THEOREM 2.18 (ANDERSON [6]). *If $M$ is an (f-d) cap-set for $I^\infty$, then $I^\infty\backslash M$ is homeomorphic to $s$.*

THEOREM 2.19 (COROLLARY 8.3). *If $X$ is any Q-manifold and $M$ is an (f-d) cap-set for $X$, then $X\backslash M$ is an F-manifold which is of the same homotopy type as $X$.*

VI. *Subsets and supersets of F, Q, Σ, and σ-manifolds.* If $X$ is any space and $\mathscr{U}$ is any cover of $X$, then a function $f\colon X\to X$ is said to be *limited by* $\mathscr{U}$ provided that for each $x\in X$, there is a member of $\mathscr{U}$ containing both $x$ and $f(x)$.

A subset $K$ of a space $X$ is said to be *strongly negligible* provided that given any open cover $\mathscr{U}$ of $X$, there is a homeomorphism of $X$ onto $X\backslash K$ which is limited by $\mathscr{U}$.

THEOREM 2.20 (ANDERSON, HENDERSON, AND WEST [10]). *A necessary and sufficient condition that a closed subset $K$ of an F-manifold have Property Z is that $K$ be strongly negligible.*

THEOREM 2.21 (THEOREMS 6.9 AND 10.2). *A necessary and sufficient condition that a closed subset $K$ of a σ or Σ-manifold have Property Z is that $K$ be strongly negligible.*

THEOREM 2.22 (THEOREM 6.7). *Let $M$ be an (f-d) cap-set for an F or Q-manifold $X$ and let $K$ be a Z-set in $X$. Then $M\backslash K$ is an (f-d) cap-set for $X$.*

THEOREM 2.23 (THEOREM 6.6). *Let $M$ be an (f-d) cap-set for an F or Q-manifold $X$ and let $K$ be a countable union of (finite-dimensional) compact Z-sets in $X$. Then $M\cup K$ is an (f-d) cap-set for $X$.*

VII. *Homeomorphism extension theorems.* If $\mathscr{U}$ is any cover of a set $X$, then $\mathrm{St}(\mathscr{U})$ is defined (as usual) to be the star of $\mathscr{U}$ and for each $n>0$ we define $\mathrm{St}^n(\mathscr{U})$ to be the $n$th star of $\mathscr{U}$.

THEOREM 2.24 (ANDERSON AND MCCHAREN [8]). *Let $X$ be an F-manifold, let $K_1$, $K_2$ be Z-sets in $X$, let $\mathscr{U}$ be an open cover of $X$, and let $h$ be a homeomorphism of $K_1$ onto $K_2$ for which there is a homotopy $H\colon K_1\times I\to X$ such that for each $x\in K_1$, $H(x,0)=h(x)$, $H(x,1)=x$, and $H(\{x\}\times I)$ is contained in some member of $\mathscr{U}$. Then $h$ can be extended to a homeomorphism of $X$ onto itself which is limited by $\mathrm{St}^4(\mathscr{U})$.*

THEOREM 2.25 (THEOREM 11.1). *Let $X$ be a σ or Σ-manifold, let $K_1$, $K_2$ be Z-sets in $X$, let $\mathscr{U}$ be an open cover of $X$, and let $h$ be a homeomorphism of $K_1$ onto $K_2$ for which there is a homotopy $H\colon K_1\times I\to X$ such that for each $x\in K_1$, $H(x,0)=h(x)$, $H(x,1)=x$, and $H(\{x\}\times I)$ is contained in some member of $\mathscr{U}$. Then $h$ can be extended to a homeomorphism of $X$ onto itself which is limited by $\mathrm{St}^{28}(\mathscr{U})$.*

VIII. *Complete extensions of Σ and σ-manifolds.*

THEOREM 2.26 (LEMMA 10.1 AND THEOREM 10.2). *Let $X$ be a σ (or Σ)-manifold and let $Y$ be a complete separable metric space containing $X$. Then there is an F-manifold $Z$ such that $X\subset Z\subset Y$ and $X$ is an f-d cap (or cap)-set for $Z$.*

IX. *Infinite deficiency.* A subset $K$ of $I^\infty$ is said to have *infinite deficiency* provided that for each of infinitely many different coordinate directions, $K$ projects onto a single interior point of $(-1, 1)$.

THEOREM 2.27 (ANDERSON [3]). *Let $X$ be $I^\infty$ or $s$ and let $K$ be a closed subset of $X$. A necessary and sufficient condition that $K$ be a Z-set in $X$ is that there exists a homeomorphism of $X$ onto itself taking $K$ onto a set having infinite deficiency.*

THEOREM 2.28 (CHAPMAN [13]). *Let $X$ be an F-manifold and let $K$ be a closed subset of $X$. A necessary and sufficient condition that $K$ be a Z-set in $X$ is that there exists a homeomorphism $h$ of $X$ onto $X \times s$ such that $\pi_s \circ h(K)$ has infinite deficiency.*

We remark that a separate proof of Theorem 2.28 is given in Corollary 7.1 of this paper.

THEOREM 2.29 (THEOREM 7.2). *Let $X$ be a Q-manifold and let $K$ be a closed subset of $X$. A necessary and sufficient condition that $K$ be a Z-set in $X$ is that there exists a homeomorphism $h$ of $X$ onto $X \times I^\infty$ such that $\pi_{I^\infty} \circ h(K)$ has infinite deficiency.*

3. **Preliminaries.** The metric we use for $I^\infty$ and $s$ is given by

$$d((x_i), (y_i)) = \left( \sum_{i=1}^{\infty} 2^{-i}(x_i - y_i)^2 \right)^{1/2},$$

where $(x_i), (y_i) \in I^\infty$. Whenever no confusion is possible we will use $d$ to denote the metric of any space under consideration.

Let $Z$ denote the set of positive integers and for each $i \in Z$ let $\tau_i$ be the projection of $I^\infty$ onto $I_i$. For each $\alpha \subset Z$ define $I^\alpha = \prod_{i\in\alpha} I_i$, $s^\alpha = \prod_{i\in\alpha} I_i^0$, and let $\tau_\alpha$ be the projection of $I^\infty$ onto $I^\alpha$. For each $i \in Z$ let $W_i^+ = \tau_i^{-1}(1)$, $W_i^- = \tau_i^{-1}(-1)$, and $W_i = W_i^+ \cup W_i^-$. We call $W_i^+$ and $W_i^-$ the *endslices* of $I^\infty$ in the $i$-direction.

A subset of $I^\infty$ of the form $\prod_{i=1}^\infty J_i$ is called a *basic closed set in $I^\infty$* provided that $J_i$ is a closed subinterval of $I_i$, for each $i > 0$, and $J_i = I_i$ for all but finitely many $i$. The interior in $I^\infty$ of a basic closed set is called a *basic open set in $I^\infty$*. A *basic open set in $s$* is the intersection of a basic open set in $I^\infty$ with $s$. We call $s$ the *pseudo-interior* of $I^\infty$ and $B(I^\infty) = I^\infty \backslash s$ the *pseudo-boundary* of $I^\infty$.

A *core* is a set $C = \prod_{i=1}^\infty J_i$, where for each $i > 0$, $J_i$ is a closed interval contained in $I_i^0$. A *basic core set structured over a core* $C = \prod_{i=1} J_i$ is defined as the set of all points $(x_i) \in s$ such that $x_i \in J_i$, for all but finitely many $i$. We know from [4] that each basic core set is an apparent boundary of $I^\infty$ and from [6] that each apparent boundary of $I^\infty$ is a cap-set for $I^\infty$, where $M \subset I^\infty$ is an *apparent boundary* if there is a homeomorphism $h: I^\infty \to I^\infty$ satisfying $h(M) = B(I^\infty)$.

A homeomorphism $h$ of $I^\infty$ onto itself is said to be a $\beta^*$-homeomorphism provided that $h(s) = s$.

If $\{f_i\}_{i=1}^\infty$ is a sequence of homeomorphisms of a space $X$ onto itself for which the sequence $\{f_i \circ f_{i-1} \circ \cdots \circ f_1\}$ converges pointwise to a homeomorphism $f$ of $X$ onto itself, then we call $f$ the *infinite left product* of $\{f_i\}_{i=1}^\infty$ and write $f = L \prod_{i=1}^\infty f_i$.

We list below four convergence procedures that we will need to insure the existence of an infinite left product of homeomorphisms. The first of these is Lemma 2.1 of [3] and the second is Theorem 2 of [10]. The third and fourth are easy consequences of the apparatus used in [10] to establish the second.

CONVERGENCE PROCEDURE A. *For each homeomorphism $g$ of a compact metric space $X$ onto itself and each $\varepsilon > 0$ let*

$$\eta(g, \varepsilon) = \text{g.l.b.}\,\{d(g(x), g(y)) \mid d(x, y) \geqq \varepsilon\}.$$

*If $\{f_i\}_{i=1}^{\infty}$ is a sequence of homeomorphisms of $X$ onto itself such that*

$$d(f_i, \text{id}) < \min((3^{-i}), (3^{-i})\cdot\eta(f_{i-1}\circ\cdots\circ f_1, 2^{-i})),$$

*for all $i > 1$, then $f = L\prod_{i=1}^{\infty} f_i$ exists.*

CONVERGENCE PROCEDURE B. *Let $\mathcal{U}$ be a countable star-finite open cover of any space $X$. (By star-finite cover we mean a cover such that the closure of each member of the cover intersects the closure of only finitely many other members of the cover.) There exists an ordering $\{U_i\}_{i=1}^{\infty}$ of the elements of $\mathcal{U}$ such that for any sequence $\{f_i\}_{i=1}^{\infty}$ of homeomorphisms of $X$ onto itself, where $f_i$ is the identity on $X\backslash U_i$ for all $i > 0$, $f = L\prod_{i=1}^{\infty} f_i$ exists. Moreover, we can assign a positive integer $n_i$ to each $U_i$, independent of the choice of such $\{f_i\}_{i=1}^{\infty}$, such that $n_i \leqq n_{i+1}$ and*

$$f(U_i) = (f_{n_i}\circ f_{n_i-1}\circ\cdots\circ f_1)(U_i),$$

*for all $i > 0$.*

CONVERGENCE PROCEDURE C. *Let $\mathcal{U}$ be a countable star-finite open cover of any space $X$ and let $\{U_i\}_{i=1}^{\infty}$, $\{n_i\}_{i=1}^{\infty}$ be as in Convergence Procedure B. If $Y$ is any space and $\{f_i\}_{i=1}^{\infty}$ is any sequence of homeomorphisms of $X\times Y$ onto itself such that $f_i$ is the identity on $(X\backslash U_i)\times Y$, for all $i > 0$, then $f = L\prod_{i=1}^{\infty} f_i$ exists. Moreover, we have*

$$f(U_i\times Y) = (f_{n_i}\circ\cdots\circ f_1)(U_i\times Y),$$

*for all $i > 0$.*

CONVERGENCE PROCEDURE D. *Let $\mathcal{U}$ be a countable star-finite open cover of any space $X$ and let $\{U_i\}_{i=1}^{\infty}$ be as in Convergence Procedure B. If $\{f_i\}_{i=1}^{\infty}$ is any sequence of homeomorphisms of $X$ onto itself such that $f_1 | X\backslash U_1 = \text{id}$ and $f_{i+1} | X\backslash f_i\circ\cdots\circ f_1(U_{i+1}) = \text{id}$, for each $i > 0$, then $f = L\prod_{i=1}^{\infty} f_i$ exists.*

4. **Some basic results concerning (f-d) cap-sets.** The following result can be found in [6].

LEMMA 4.1. *Let $X$ be a metric space, let $M$ be an (f-d) cap-set for $X$, and let $f$ be a homeomorphism of $X$ onto a metric space $Y$. Then $f(M)$ is an (f-d) cap-set for $Y$.*

The next result shows that (*f-d*) cap-sets in $I^\infty$ are "maximal." We will use this result to prove a corresponding property for $F$ and $Q$-manifolds in §6. We will give no proof, since it is essentially Theorem 1 of [17]. We remark that Lemma 4.2 follows from Theorem II of [4] in the case that $M$ is a cap-set for $I^\infty$.

LEMMA 4.2. *Let $M \subset N \subset I^\infty$, where $M$ is an (f-d) cap-set for $I^\infty$ and $N$ is a countable union of (finite-dimensional) Z-sets in $I^\infty$. Then $N$ is an (f-d) cap-set for $I^\infty$.*

Our next lemma is a generalization of Anderson's results on the equivalence of (f-d) cap-sets in $s$ and $I^\infty$ [6]. We only sketch the proof.

LEMMA 4.3. *Let $M$ and $N$ be (f-d) cap-sets for $I^\infty$, let $K \subset I^\infty \backslash (M \cup N)$ be a Z-set in $I^\infty$, and let $\varepsilon > 0$ be given. Then there is a homeomorphism $h$ of $I^\infty$ onto itself such that $h(M) = N$, $h|K = \mathrm{id}$, and $d(h, \mathrm{id}) < \varepsilon$. Moreover, if $M \cup N \subset s$, then we can additionally require that $h$ be a $\beta^*$-homeomorphism.*

**Proof.** We only treat the case in which $M \cup N \subset s$ to obtain the $\beta^*$-homeomorphism $h$. Write $M = \bigcup_{n=1}^\infty M_n$ and $N = \bigcup_{n=1}^\infty N_n$ so as to satisfy the definition. It is clear that there is an integer $n_1 > 0$ and an embedding $f_1 \colon M_1 \to N_{n_1}$ such that $d(f_1, \mathrm{id}) < \varepsilon/2$. Using Lemma 3.1 of [13], we can extend $f_1$ to a $\beta^*$-homeomorphism $h_1$ such that $h_1|K = \mathrm{id}$ and $d(h_1, \mathrm{id}) < \varepsilon/2$.

Using Convergence Procedure A let $\delta = \min((3^{-2}), (3^{-2})\eta(h_1, 2^{-2}))$ and use Lemma 4.1 to get an integer $n_2 > 1$ and an embedding $f_2 \colon N_{n_1} \to h_1(M_{n_2})$ such that $f_2|h_1(M_1) = \mathrm{id}$ and $d(f_2, \mathrm{id}) < \min(\varepsilon/4, \delta)$. Then we can extend $f_2$ to a $\beta^*$-homeomorphism $g_2$ such that $g_2|(K \cup W_1 \cup h_1(W_1)) = \mathrm{id}$ and $d(g_2, \mathrm{id}) < \min(\varepsilon/4, \delta)$. We then put $h_2 = g_2^{-1}$ to complete the second stage of the construction.

It is clear that by induction we can continue this process to obtain a sequence $\{h_i\}_{i=1}^\infty$ of $\beta^*$-homeomorphisms and a sequence $\{n_i\}_{i=0}^\infty$ of integers (where $n_0 = 1$) such that the following conditions are met.

(1) $n_0 < n_2 < n_4 < \cdots$ and $n_1 < n_3 < n_5 < \cdots$,

(2) for each $i > 1$, $h_i$ is the identity on $K \cup (\bigcup_{j=1}^{i-1} W_j) \cup (h_{i-1} \circ \cdots \circ h_1)(\bigcup_{j=1}^{i-1} W_j)$,

(3) $(h_i \circ \cdots \circ h_1)(M_{n_{i-1}}) \subset N_{n_i}$, for $i$ odd,

(4) $(h_i \circ \cdots \circ h_1)(M_{n_i}) \supset N_{n_{i-1}}$, for $i$ even,

(5) $h_i$ is the identity on $N_{n_{i-2}}$, for $i \geqq 3$ and odd,

(6) $h_i$ is the identity on $h_{i-1} \circ \cdots \circ h_1(M_{n_{i-2}})$, for $i$ even,

(7) $d(h_1, \mathrm{id}) < \varepsilon/2$ and $d(h_i, \mathrm{id}) < \min(\varepsilon/2^i, 3^{-i}, (3^{-i})\eta(h_{i-1} \circ \cdots \circ h_1, 2^{-i}))$, for $i > 1$.

Using Convergence Procedure A we find that $h = L\prod_{i=1}^\infty h_i$ is a $\beta^*$-homeomorphism which satisfies the required properties. □

A special case of the next lemma is known in a more general setting (see Corollary 2 of [17]).

LEMMA 4.4. *Let $M$ be an (f-d) cap-set for $I^\infty$ and let $K$ be a Z-set in $I^\infty$. Then $M\backslash K$ is an (f-d) cap-set for $I^\infty$.*

**Proof.** Let $M = \bigcup_{n=1}^\infty M_n$ be as in the definition and for each integer $n > 0$ let $N_n = M_n \cap \{x \in I^\infty \mid d(x, K) \geqq 1/n\}$. It is then clear that $M\backslash K = \bigcup_{n=1}^\infty N_n$. It follows from Theorems 8.1 and 9.1 of [3] that Property Z and infinite deficiency are equivalent in $I^\infty$ modulo ambient homeomorphisms. This implies that each $N_n$ is a $Z$-set in $I^\infty$.

Now let $A$ be a (finite-dimensional) compact subset of $I^\infty$, let $m>0$ be an integer, and let $\varepsilon>0$ be given. It follows from Theorem II of [4] that there is a cap-set $M'$ for $I^\infty$ in $I^\infty\backslash K$. Let $M'=\bigcup_{n=1}^\infty M_n'$ be a representation as in the definition. Without loss of generality we may assume that $A$ is a $Z$-set. Let $A\backslash N_m=\bigcup_{n=1}^\infty C_n$, where each $C_n$ is a compact set satisfying $C_n\subset C_{n+1}$. Let $n_1>0$ be chosen large enough so that there is an embedding $f_1\colon C_1\to M_{n_1}'$ which satisfies $f_1(C_1)\cap N_m=\varnothing$ and $d(f_1,\mathrm{id})<\varepsilon/4$. Using Lemma 3.1 of [13] we can extend $f_1$ to a homeomorphism $f_1'\colon I^\infty\to I^\infty$ which satisfies $f_1'|N_m=\mathrm{id}$ and $d(f_1',\mathrm{id})<\varepsilon/4$.

Let $\varepsilon'=\min(\varepsilon/8, 3^{-2}, (3^{-2})\eta(f_1', 2^{-2}))$ and choose $n_2>n_1$ large enough so that there is an embedding $f_2\colon f_1'(C_2)\to M_{n_2}'$ which satisfies $f_2(f_1'(C_2))\cap N_m=\varnothing$, $f_2|f_1'(C_1)=\mathrm{id}$, and $d(f_2,\mathrm{id})<\varepsilon'$. Once more use Lemma 3.1 of [13] to extend $f_2$ to a homeomorphism $f_2'\colon I^\infty\to I^\infty$ which satisfies $f_2'|N_m=\mathrm{id}$ and $d(f_2',\mathrm{id})<\varepsilon'$.

Using Convergence Procedure A we can continue this process in an obvious fashion to obtain a sequence $\{f_i'\}_{i=1}$ of homeomorphisms of $I^\infty$ onto itself for which $f=L\prod_{i=1}f_i'$ exists and satisfies $f(A\backslash N_m)\subset M'$, $f|N_m=\mathrm{id}$, and $d(f,\mathrm{id})<\varepsilon/2$.

Let $\delta=d(f(A),K)$, which is a positive number, and let $g\colon f(A)\to M_n$ be an embedding, for some $n>0$, such that $g|A\cap N_m=\mathrm{id}$ and $d(g,\mathrm{id})<\min(\varepsilon/2,\delta)$. Then $g\circ f\colon A\to M_n$ is an embedding for which $g\circ f(A)\cap K=\varnothing$ and $d(g\circ f,\mathrm{id})<\varepsilon$. It is clear that there is an $n'\geqq n$ such that $g\circ f(A)\subset N_{n'}$, which completes the verification of the definition. □

COROLLARY 4.1. *Let $M$ be an (f-d) cap-set for $s$ and let $K$ be a $Z$-set in $s$. Then $M\backslash K$ is an (f-d) cap-set for $s$.*

**Proof.** We know from Theorem 8.2 of [3] that $\bar{K}$, the closure of $K$ in $I^\infty$, is a $Z$-set in $I^\infty$. Now any (f-d) cap-set for $s$ is an (f-d) cap-set for $I^\infty$, and any subset of $s$ which is an (f-d) cap-set for $I^\infty$ is an (f-d) cap-set for $s$. Then $M$ is an (f-d) cap-set for $I^\infty$ and from Lemma 4.4 we find that $M\backslash\bar{K}=M\backslash K$ is an (f-d) cap-set for $s$. □

5. **Property Z in manifolds.** In this section we prove some results concerning Property Z in manifolds that will be useful in later sections. We also exhibit a technique (Theorem 5.1) by which we can relate $F$ and $Q$-manifolds.

In Lemma 1 of [10] some techniques are given for establishing Property Z in normal, locally homotopically trivial spaces. Lemma 5.1 is essentially a restatement of this result and Lemma 5.2 is a consequence of the proof that is given there.

LEMMA 5.1. *Let $X$ be normal and locally homotopically trivial. If $A$ is a closed subset of $X$ such that each point of $A$ lies in an open set $U$ for which $A\cap U$ has Property Z in $U$, then $A$ has Property Z in $X$.*

LEMMA 5.2. *Let $X$ be normal and locally homotopically trivial and let $A$ be a closed subset of $X$. If, for each point $p\in A$ and each open set $U$ containing $p$, there is a homotopically trivial open set $V$ containing $p$ such that $V\subset U$, $V\backslash A\neq\varnothing$, and $V\backslash A$ is homotopically trivial, then $A$ has Property Z in $X$.*

In Lemma 2 of [10] it is shown that if $F$ is any $Z$-set in an $F$-manifold $X$, then any closed subset of $F$ is also a $Z$-set in $X$. Lemma 5.3 establishes a similar property for $Q$-manifolds. We remark that for $F$-manifolds all compact subsets are $Z$-sets. This follows from Theorem 2.1 and the fact that any compact subset of $s$ is a $Z$-set (see Theorem 3.5 of [2] and Theorem 9.1 of [3]).

LEMMA 5.3. *Let $F$ be a $Z$-set in any $Q$-manifold $X$. If $A$ is any closed subset of $F$, then $A$ is also a $Z$-set in $X$.*

**Proof.** Using Lemma 5.1 let $U$ be any open subset of $X$ which is homeomorphic to a basic open subset of $I^\infty$. Then there is a homeomorphism $h$ of $U$ onto $I^\infty\backslash W$, where $W$ is a finite union of endslices of $I^\infty$. Since Property Z obviously carries over to open subsets we find that $h(F\cap U)$ is a $Z$-set in $I^\infty\backslash W$.

Define $K=h(F\cap U)\cup W$, which is a compact subset of $I^\infty$. Let $V$ be any nonnull, homotopically trivial open subset of $I^\infty$ and note that $V\backslash K=(V\backslash W)\backslash h(F\cap U)$. Since $W$ obviously has Property Z in $I^\infty$ it follows that $V\backslash W$ is a nonnull, homotopically trivial open subset of $I^\infty\backslash W$, hence $V\backslash K$ is a nonnull, homotopically trivial open set. This means that $K$ is a $Z$-set in $I^\infty$.

Using Theorem 8.1 of [3] there is a homeomorphism $g$ of $I^\infty$ onto itself such that $g(K)$ has infinite deficiency. Using techniques similar to those of Theorem 9.1 of [3] it now easily follows that $g\circ h(A\cap U)$ is a $Z$-set in $g\circ h(U)$. □

The following result was pointed out to the author by William Barit.

LEMMA 5.4. *Let $X$ be an $F$ or $Q$-manifold and let $M$ be an (f-d) cap-set for $X$. If $U$ is an open subset of $X$, then $M\cap U$ is an (f-d) cap-set for $U$.*

**Proof.** Let $M=\bigcup_{n=1}^\infty M_n$ be a representation as given in the definition and for each $n>0$, let $A_n=\{x\in X \mid d(x, X\backslash U)\geqq 1/n\}$. If we put $N=M\cap U$ and $N_n=M_n\cap A_n$, for each $n>0$, then $N=\bigcup_{n=1}^\infty N_n$. It follows from Lemma 5.3 and the comments preceding Lemma 5.3 that each $N_n$ is a $Z$-set in $U$. Then $N=\bigcup_{n=1}^\infty N_n$ is clearly a representation which satisfies the definition for $N$ to be an (f-d) cap-set for $U$. □

Since (f-d) cap-sets for $s$ exist, we can use Theorem 2.1 and Lemma 5.4 to prove the existence of (f-d) cap-sets for $F$-manifolds. In Lemma 5.6 we prove that cap-sets for $Q$-manifolds exist. The existence of f-d cap-sets for $Q$-manifolds will be established in §6. We will first need a preliminary result.

LEMMA 5.5. *Let $X$, $Y$ be $Q$-manifolds and let $A$ be a $Z$-set in $X$. Then $A\times Y$ is a $Z$-set in $X\times Y$.*

**Proof.** Using Lemma 5.2 let $U$ be a nonnull, homotopically trivial open subset of $X$ and let $V$ be a nonnull, homotopically trivial open subset of $Y$. All we have to do is prove that $(U\times V)\backslash(A\times Y)$ is nonnull and homotopically trivial. But $(U\times V)\backslash(A\times Y)=(U\backslash A)\times V$ is clearly nonnull and homotopically trivial. □

LEMMA 5.6. *If $X$ is any $Q$-manifold, then $X$ has a cap-set.*

**Proof.** Using Theorem 2.11 we know that there is a homeomorphism of $X$ onto $X\times I^\infty$. It will then suffice to prove that $X\times\Sigma$ has the cap in $X\times I^\infty$. Since $\Sigma$ has the cap in $I^\infty$ we can write $\Sigma=\bigcup_{n=1}^\infty M_n$ so as to satisfy the definition. Write $X=\bigcup_{n=1}^\infty X_n$, where each $X_n$ is a compact subset of $X$ lying in the interior of $X_{n+1}$. Then $X\times\Sigma=\bigcup_{n=1}^\infty (X_n\times M_n)$ which we now show to satisfy the definition of a cap-set for $X\times I^\infty$.

It follows from Lemmas 5.3 and 5.5 that each $X_n\times M_n$ is a $Z$-set in $X\times I^\infty$. Now let $m>0$ be an integer, let $K$ be a compact subset of $X\times I^\infty$, and let $\varepsilon>0$ be given. Since $\pi_{I^\infty}(K)$ is a compact subset of $I^\infty$ there is an integer $n_1>0$ and an embedding $h_1\colon \pi_{I^\infty}(K)\to M_{n_1}$ such that $h_1|\pi_{I^\infty}(K)\cap M_m=\mathrm{id}$ and $d(h_1,\mathrm{id})<\varepsilon$. It is clear that there is an integer $n_2>0$ such that $\pi_X(K)\subset X_{n_2}$. Put $n=\max(n_1,n_2)$ and we obviously have an embedding $h$ of $K$ into $X_n\times M_n$ such that $h|K\cap(X_m\times M_m)=\mathrm{id}$ and $d(h,\mathrm{id})<\varepsilon$. □

With the existence of cap-sets in $Q$-manifolds established we can now prove a result which relates $F$ and $Q$-manifolds.

THEOREM 5.1. *If $X$ is any $F$-manifold, then $X$ can be regarded as a subset of a $Q$-manifold $Y$ such that $Y\backslash X$ is a cap-set for $Y$. On the other hand, if $Y$ is any $Q$-manifold, then there is an $F$-manifold $X\subset Y$ such that $Y\backslash X$ is a cap-set for $Y$.*

**Proof.** If $X$ is any $F$-manifold, then Theorem 2.1 gives an open embedding $h\colon X\to s$. Let $Y$ be an open subset of $I^\infty$ such that $Y\cap s=h(X)$. Then $Y$ is obviously a $Q$-manifold and it follows from Lemma 5.4, and the fact that $B(I^\infty)$ is a cap-set for $I^\infty$, that $Y\backslash h(X)$ is a cap-set for $Y$.

If $Y$ is any $Q$-manifold, it follows from Lemma 5.6 that there is a cap-set $M$ for $Y$. To show that $Y\backslash M$ is an $F$-manifold let $U$ be any open subset of $Y$ which is homeomorphic to a basic open subset of $I^\infty$. Let $h\colon U\to I^\infty$ be an embedding such that $I^\infty\backslash h(U)=W$, where $W$ is a finite union of endslices of $I^\infty$. It is obvious that $h(U\cap M)$ is a cap-set for $I^\infty$ and we know that $\Sigma$ is a cap-set for $I^\infty$. Using Lemma 4.3 there is a homeomorphism $g$ of $I^\infty$ onto itself such that $g(\Sigma)=h(U\cap M)$ and $g|W=\mathrm{id}$. This means that $h(U\backslash M)$ is homeomorphic $I^\infty\backslash(\Sigma\cup W)$. But Theorem 2.18 implies that $I^\infty\backslash(\Sigma\cup W)$ is homeomorphic to $s$. □

COROLLARY 5.1. *Let $Y$ be any $Q$-manifold and let $M$ be an (f-d) cap-set for $Y$. Then $Y\backslash M$ is an $F$-manifold.*

**Proof.** For $M$ a cap-set the proof is given above. For $M$ an f-d cap-set we use the outline given above. □

In Theorem 8.2 of [3] it is shown that if $F$ is a $Z$-set in $s$, then $\bar{F}$ (closure taken in $I^\infty$) is a $Z$-set in $I^\infty$. The next result generalizes this to $Q$-manifolds.

LEMMA 5.7. *Let $X$ be a $Q$-manifold and let $M$ be a cap-set for $X$. If $F$ is a $Z$-set in $X\backslash M$, then $\bar{F}$ (closure taken in $X$) is a $Z$-set in $X$.*

**Proof.** Using Lemma 5.1 let $U$ be an open subset of $X$ and let $h\colon U\to I^\infty$ be an embedding such that $I^\infty\backslash h(U)=W$, where $W$ is a finite union of endslices of $I^\infty$.

Using the proof of Theorem 5.1 we can additionally require that $h(U \cap M) = B(I^\infty)\backslash W$. Then $h(U\backslash M) = s$ and hence $K = h(F \cap U)$ is a $Z$-set in $s$. This means that $\bar{K}$ is a $Z$-set in $I^\infty$. Then $h(\bar{F} \cap U) = \bar{K} \cap h(U)$ is a $Z$-set in $h(U)$, which is what we needed. □

The next result will be used in proving an open embedding theorem for $\sigma$ and $\Sigma$-manifolds.

LEMMA 5.8. *Let $X$ be an $F$-manifold, let $M$ be an (f-d) cap-set for $X$, and let $F$ be a closed subset of $X$ contained in $X\backslash M$. Then $F$ is a $Z$-set in $X$.*

**Proof.** Using Lemma 5.1 it is sufficient to prove that for any open subset $U$ of $X$ which is homeomorphic to $s$, $U \cap F$ is a $Z$-set in $U$. Thus, our problem is reduced to the following: if $N$ is an (f-d) cap-set for $s$ and $K$ is a closed subset of $s$ contained in $s\backslash N$, then $K$ is a $Z$-set in $s$.

To show that $K$ has Property Z in $s$ let $V$ be a nonnull, homotopically trivial open subset of $s$. Let $f: S^{n-1} \to V\backslash K$ be continuous and let $g: B^n \to V$ be a continuous extension of $f$, where $B^n$ is an $n$-ball and $S^{n-1}$ is the boundary of $B^n$. If we think of a canonical version of $N$ (i.e. $N = \sigma$ or $N = \Sigma$) then it is clear that there is a continuous function $h: g(B^n) \to s$ such that $h(g(B^n)\backslash g(S^{n-1})) \subset N$, $h|g(S^{n-1}) = \mathrm{id}$, and $d(h, \mathrm{id}) < \varepsilon$, where $\varepsilon = d(g(B^n), s\backslash V)$. Note that $h(g(B^n)) \cap K = \varnothing$, $h(g(B^n)) \subset V$, and $h \circ g$ is a continuous extension of $f$. This proves that $V\backslash K$ is homotopically trivial. Since $V \cap N \neq \varnothing$ we have $V\backslash K \neq \varnothing$. □

The following two results will be used in the last section of the paper.

LEMMA 5.9. *Let $X$ be any $F$-manifold, let $F$ be a $Z$-set in $X$, and for each $n > 0$ let $F_n$ be a $Z$-set in $X$. Then $F \cap (X\backslash\bigcup_{n=1}^\infty F_n)$ is a $Z$-set in $X\backslash\bigcup_{n=1}^\infty F_n$.*

**Proof.** Using Lemma 5.2 let $U$ be any nonnull, homotopically trivial open subset of $X$. Then $U \cap (X\backslash\bigcup_{n=1}^\infty F_n) = U\backslash\bigcup_{n=1}^\infty (F_n \cap U)$, and using the negligibility result of [4] we know that $U\backslash\bigcup_{n=1}^\infty (F_n \cap U)$ is homeomorphic to $U$. Thus $U \cap (X\backslash\bigcup_{n=1}^\infty F_n)$ is homotopically trivial. We then have $U \cap (X\backslash\bigcup_{n=1}^\infty F_n)\backslash F$ nonnull and homotopically trivial by the same argument. Lemma 5.2 then implies that $F \cap (X\backslash\bigcup_{n=1}^\infty F_n)$ is a $Z$-set in $X\backslash\bigcup_{n=1}^\infty F_n$. □

LEMMA 5.10. *Let $X$ be an $F$-manifold, let $M \subset X$ be an (f-d) cap-set for $X$, and let $F$ be a $Z$-set in $M$. Then $\bar{F}$ (closure in $X$) is a $Z$-set in $X$.*

**Proof.** Using Lemma 5.1 it is clear that our problem reduces to the following: If $N$ is an (f-d) cap-set for $s$ and $K$ is a $Z$-set in $N$, then $\bar{K}$ (closure in $s$) is a $Z$-set in $s$.

Let $U$ be a nonnull, homotopically trivial open subset of $s$, let $f: S^{n-1} \to U\backslash\bar{K}$ be a continuous function, and let $g: B^n \to U$ be a continuous extension of $f$. Using the idea contained in the proof of Lemma 5.8 there is a continuous function $h: B^n \to s$ such that $h(g(B^n)\backslash g(S^{n-1})) \subset N$, $h|g(S^{n-1}) = \mathrm{id}$, and $h \circ g(B^n) \subset U$. This argument has already shown that $U \cap N$ is homotopically trivial.

It is clear that there is an $r$, $0<r<1$, such that $h\circ g(\{x\in B^n \mid r\leqq \|x\|\leqq 1\}) \cap \bar{K}=\varnothing$. Put $S_r^{n-1}=\{x\in B^n \mid \|x\|=r\}$ and use the fact that $K$ has Property Z in $N$ to extend the map $h\circ g|S_r^{n-1}: S_r^{n-1}\to (U\cap N)\backslash K$ to a continuous function $\varphi: B_r^n \to (U\cap N)\backslash K$, where $B_r^n=\{x\in B^n \mid \|x\|\leqq r\}$. Piecing these together we then have a continuous function $\psi: B^n\to U\backslash\bar{K}$ which extends $f$. □

6. **Properties of (f-d) cap-sets in** $F$ **and** $Q$**-manifolds.** The main results of this section are Theorems 6.1 and 6.2, in which we establish the equivalence of (f-d) cap-sets in $F$ and $Q$-manifolds. We also use similar techniques to obtain a number of other results.

We first establish the equivalence of (f-d) cap-sets in $F$-manifolds.

THEOREM 6.1. *Let $X$ be an $F$-manifold, let $\mathscr{U}$ be an open cover of $X$, and let $M$, $N$ be (f-d) cap-sets for $X$. Then there is a homeomorphism $h$ of $X$ onto itself such that $h(M)=N$ and $h$ is limited by $\mathscr{U}$.*

**Proof.** Using Theorem 2.1 we may, without loss of generality, assume that $X$ is an open subset of $s$. Applying Theorem 1 of [10] there is a star-finite open cover $\{U_i\}_{i=1}^\infty$ of $X$ such that for each $i>0$, $U_i$ is a basic open subset of $s$ and there is a $V_i\in\mathscr{U}$ such that $d(U_i, s\backslash V_i)=\varepsilon_i>0$. Assume that $\{U_i\}_{i=1}^\infty$ is ordered as in Convergence Procedure B.

We may regard each Cl $(U_i)$ (closure taken in $s$) as a copy of $s$ together with a finite union of "endslices." Since $M\cap U_1$ and $N\cap U_1$ are (f-d) cap-sets for $U_1$ we can use Lemma 4.3 to get a homeomorphism $h_1$ of $X$ onto itself such that $h_1(M\cap U_1)=N\cap U_1$, $h_1|X\backslash U_1=\text{id}$, and $d(h_1,\text{id})<\varepsilon_1/n_1$, where $\{n_i\}_{i=1}^\infty$ is the sequence of Convergence Procedure B that is associated with $\{U_i\}_{i=1}^\infty$.

Proceeding by induction assume that homeomorphisms $\{h_i\}_{i=1}^j$ of $X$ onto itself have been defined so that $h_i|X\backslash U_i=\text{id}$, for $1\leqq i\leqq j$, $h_{i+1}(h_i\circ\cdots\circ h_1(M)\cap U_i)=N\cap U_i$, for $1\leqq i<j$, and $d(h_i,\text{id})<(1/n_i)\cdot\min(\varepsilon_1,\ldots,\varepsilon_i)$, for $1\leqq i\leqq j$. Then let $h_{j+1}$ be a homeomorphism of $X$ onto itself such that $h_{j+1}|X\backslash U_{j+1}=\text{id}$, $h_{j+1}(h_j\circ\cdots\circ h_1(M)\cap U_{j+1})=N\cap U_{j+1}$, and $d(h_{j+1},\text{id})<(1/n_{j+1})\cdot\min(\varepsilon_1,\ldots,\varepsilon_{j+1})$.

We can use Convergence Procedure B to conclude that $h=L\prod_{i=1}^\infty h_i$ is a homeomorphism of $X$ onto itself such that $h(M)=N$. Moreover we note that if $x\in U_j\backslash\bigcup_{i=1}^{j-1}U_i$, then $d(h(x),x)<\varepsilon_j$. This implies that $h$ is limited by $\mathscr{U}$. □

We now prove the corresponding result for $Q$-manifolds. The following lemma takes the place of Theorem 1 of [10] in this case.

LEMMA 6.1. *Let $X$ be a $Q$-manifold and let $\mathscr{U}$ be an open cover of $X$. Then there is a star-finite open cover $\{U_i\}_{i=1}^\infty$ of $X$ which refines $\mathscr{U}$ and for which each $U_i$ is homeomorphic to a basic open subset of $I^\infty$.*

**Proof.** Since $X$ is a separable, locally compact metric space, it can be written as the union of an increasing sequence $\{X_n\}_{n=1}^\infty$ of compacta, each lying in the interior of its successor.

It is clear that there is a collection $\{U_i^1\}_{i=1}^{j_1}$ of open subsets of $X$, each homeomorphic to a basic open subset of $I^\infty$, such that $\{U_i^1\}_{i=1}^{j_1}$ covers $X_1$, and for each $i$, $U_i^1 \subset X_2^0$ (the interior of $X_2$). For each $n>1$ let $\{U_i^n\}_{i=1}^{j_n}$ be an open cover of $X_n \backslash X_{n-1}^0$, where each $U_i^n$ is homeomorphic to a basic open subset of $I^\infty$, such that for each $i$, $U_i^n \subset X_{n+1}^0 \backslash X_{n-2}$ (or $U_i^2 \subset X_3^0$ for $n=2$). Then put $\{U_i\}_{i=1}^\infty = \{U_i^n \mid 1 \leqq i \leqq j_n, n \geqq 1\}$ for our required star-finite open cover of $X$. □

Since we do not have an open embedding theorem for $Q$-manifolds we will first need a preliminary result for open subsets of $I^\infty$. With the aid of Lemma 6.1 a proof of the following result can be given that is almost identical to the proof of Theorem 6.1.

Lemma 6.2. *Let $G$ be an open subset of $I^\infty$, let $\mathscr{U}$ be an open cover of $G$, and let $M$, $N$ be (f-d) cap-sets for $G$. Then there is a homeomorphism $h$ of $G$ onto itself such that $h(M)=N$ and $h$ is limited by $\mathscr{U}$.*

Theorem 6.2. *Let $X$ be a $Q$-manifold, let $\mathscr{U}$ be an open cover of $X$, and let $M$, $N$ be (f-d) cap-sets for $X$. Then there is a homeomorphism $h$ of $X$ onto itself such that $h(M)=N$ and $h$ is limited by $\mathscr{U}$.*

**Proof.** Using Lemma 6.1 it is clear that there is a star-finite open cover $\{U_i\}_{i=1}^\infty$ of $X$ such that for each $i>0$, $U_i$ is homeomorphic to an open subset of $I^\infty$ and there is a $V_i \in \mathscr{U}$ such that $d(U_i, X\backslash V_i)=\varepsilon_i>0$.

Using the notation of Theorem 6.1 we need a sequence $\{h_i\}_{i=1}^\infty$ of homeomorphisms of $X$ onto itself such that for each $i>0$, $h_i|X\backslash U_i=\mathrm{id}$, $h_1(M \cap U_1)=N \cap U_1$, $h_{i+1}(h_i \circ \cdots \circ h_1(M) \cap U_i)=N \cap U_i$, and $d(h_i, \mathrm{id})<(1/n_i)\cdot \min(\varepsilon_1, \ldots, \varepsilon_i)$. All of these except the last can be automatically met. The last condition can clearly be met by choosing an appropriate cover of each $U_i$, transferring it over to an open cover in $I^\infty$, and applying Lemma 6.2. Then $h=L\prod_{i=1}^\infty h_i$ is our required homeomorphism. □

Although it is obvious that any (f-d) cap-set for $s$ is also an (f-d) cap-set for $I^\infty$, the corresponding property for $Q$-manifolds is not quite so obvious. This is our next theorem.

Theorem 6.3. *Let $X$ be a $Q$-manifold, let $M \subset X$ be a cap-set for $X$, and let $N$ be an (f-d) cap-set for the $F$-manifold $X\backslash M$. Then $N$ is an (f-d) cap-set for $X$.*

**Proof.** Using the proof of Lemma 5.6 we know that $X \times B(I^\infty)$ is a cap-set for $X \times I^\infty$. Combining the fact that $X$ is homeomorphic to $X \times I^\infty$ with Theorem 6.2, there is a homeomorphism $h$ of $X$ onto $X \times I^\infty$ such that $h(M)=X \times B(I^\infty)$.

Let $N=\bigcup_{n=1}^\infty N_n$ be a representation as given in the definition, let $m>0$ be given, let $K$ be a (finite-dimensional) compact subset of $X$, and let $\varepsilon>0$ be given. Using $h$ to transfer the problem to $X \times I^\infty$, there is an embedding $f\colon K \to X\backslash M$ such that $f|K \cap N_m=\mathrm{id}$ and $d(f, \mathrm{id})<\varepsilon/2$. Using the fact that $N$ is an (f-d) cap-set for $X\backslash M$ there is an integer $n>0$ and an embedding $g\colon f(K) \to N_n$ such that $g|K \cap N_m=\mathrm{id}$ and $d(g, \mathrm{id})<\varepsilon/2$. Then $g \circ f$ is our required embedding. □

The next result shows that we can piece together certain portions of (f-d) cap-sets to obtain (f-d) cap-sets.

THEOREM 6.4. *Let $X$ be an $F$ or $Q$-manifold and let $U$ be an open subset of $X$. Then the following are true*:

(1) *If $M$ is an (f-d) cap-set for $U$, then there is an (f-d) cap-set $N$ for $X$ such that $N \cap U = M$.*

(2) *If $M$, $N$ are (f-d) cap-sets for $X$, then $(M \backslash U) \cup (N \cap U)$ is an (f-d) cap-set for $X$.*

**Proof.** For the proof of (1) let $N'$ be an (f-d) cap-set for $X$. (From Theorem 6.3 we now know that f-d cap-sets exist in $Q$-manifolds.) Using Theorems 6.1 and 6.2 we can obtain a homeomorphism $h$ of $X$ onto itself such that $h(N' \cap U) = M$ and, by a proper choice of a cover of $U$, $h|X \backslash U = \text{id}$. Then $N = h(N')$ satisfies $N \cap U = M$.

For the proof of (2) let $h$ be a homeomorphism of $X$ onto itself such that $h(M \cap U) = N \cap U$ and $h|X \backslash U = \text{id}$. Then $h(M) = (M \backslash U) \cup (N \cap U)$ is an (f-d) cap-set for $X$. □

In the next theorem we prove that (f-d) cap-sets can be determined locally.

THEOREM 6.5. *Let $X$ be an $F$ or $Q$-manifold, let $\{U_i\}_{i=1}^{\infty}$ be an open cover of $X$, and for each $i > 0$ let $M_i$ be an (f-d) cap-set for $U_i$. Then $M = \bigcup_{i=1}^{\infty} M_i$ is an (f-d) cap-set for $X$.*

**Proof.** We only sketch the proof for $Q$-manifolds. The proof for $F$-manifolds is similar. Let $\{V_i\}_{i=1}^{\infty}$ be a star-finite open cover of $X$ which is a refinement of $\{U_i\}_{i=1}^{\infty}$ and which is ordered as in Convergence Procedure B. Let $N$ have the (f-d) cap in $X$ and use the techniques of Theorem 6.1 to inductively define a sequence $\{h_i\}_{i=1}^{\infty}$ of homeomorphisms of $X$ onto itself for which $h_1(M \cap V_1) = N \cap V_1$, $h_{i+1}(h_i \circ \cdots \circ h_1(M \cap V_{i+1})) = N \cap h_i \circ \cdots \circ h_1(V_{i+1})$, for all $i > 0$, $h_1|X \backslash V_1 = \text{id}$, and $h_{i+1}|X \backslash h_i \circ \cdots \circ h_1(V_{i+1}) = \text{id}$, for all $i > 0$. We remark that in order to achieve the first and second conditions we have to make use of Lemma 4.1. Then Convergence Procedure D assures us that $h = L\prod_{i=1}^{\infty} h_i$ is a homeomorphism of $X$ onto itself which obviously satisfies $h(M) = N$. □

We can easily use this result to obtain a generalization of Lemma 4.2 to $F$ and $Q$-manifolds.

THEOREM 6.6. *Let $X$ be an $F$ or $Q$-manifold, let $M$ be an (f-d) cap-set for $X$, and let $K$ be a countable union of (finite-dimensional) compact $Z$-sets in $X$. Then $M \cup K$ is an (f-d) cap-set for $X$.*

**Proof.** In case $X$ is an $F$-manifold we use Theorem 2.1 to get an open embedding $h: X \to s$. Using Theorem 6.4 there is an (f-d) cap-set $N$ for $s$ such that $N \cap h(X) = h(M)$. By Lemma 4.2 we know that $N \cup h(K)$ is an (f-d) cap-set for $s$, hence $(N \cup h(K)) \cap h(X) = h(M) \cup h(K)$ is an (f-d) cap-set for $h(X)$.

If $X$ is a $Q$-manifold let $U$ be any open subset of $X$ which is homeomorphic to a basic open subset of $I^{\infty}$. Then there is a homeomorphism $f$ of $U$ onto $I^{\infty} \backslash W$, where

$W$ is a finite union of endslices. It is clear that $f(M \cap U)$ is an (f-d) cap-set for $I^\infty$ and $f(K \cap U)$ is a countable union of (finite-dimensional) compact $Z$-sets in $I^\infty$. From Theorem 4.2 we know that $f(M \cap U) \cup f(K \cap U)$ is an (f-d) cap-set for $I^\infty$, hence for $f(U)$. Thus Theorem 6.5 implies that $M \cup K$ is an (f-d) cap-set for $X$. □

We can also use Theorem 6.5 to prove that we can remove certain portions of an (f-d) cap-set and still have an (f-d) cap-set.

THEOREM 6.7. *Let $X$ be an $F$ or $Q$-manifold, let $M$ be an* (*f-d*) *cap-set for $X$, and let $F$ be a $Z$-set in $X$. Then $M \backslash F$ is an* (*f-d*) *cap-set for $X$.*

**Proof.** In case $X$ is an $F$-manifold let $U$ be an open subset of $X$ which is homeomorphic to $s$. Then $F \cap U$ is a $Z$-set in $U$. Using Corollary 4.1 we find that $(M \cap U)\backslash(F \cap U)$ has the (f-d) cap in $U$. Then Theorem 6.5 implies that $M \backslash F$ is an (f-d) cap-set for $X$.

If $X$ is a $Q$-manifold let $U$ be an open subset of $X$ for which there is a homeomorphism $h$ of $U$ onto $I^\infty \backslash W$, where $W$ is a finite union of endslices. We observed in Lemma 5.3 that $f(F \cap U) \cup W$ is a compact $Z$-set in $I^\infty$. Using Lemma 4.4 we find that $h(M \cap U)\backslash[h(F \cap U) \cup W] = h(M \cap U)\backslash h(F \cap U)$ is an (f-d) cap-set for $I^\infty$, and hence for $h(U)$. Once more Theorem 6.5 implies that $M \backslash F$ is an (f-d) cap-set for $X$. □

In our next theorem we show how an (f-d) cap-set of a product relates to the (f-d) cap-sets of the factors.

THEOREM 6.8. *Let $X$, $Y$ both be $F$-manifolds or both be $Q$-manifolds and let $M \subset X$, $N \subset Y$ be* (*f-d*) *cap-sets. Then $M \times N$ is an* (*f-d*) *cap-set for $X \times Y$.*

**Proof.** First assume that $X$ and $Y$ are $F$-manifolds. Let $f_1 \colon X \to s^{\alpha_1}$ and $f_2 \colon X \to s^{\alpha_2}$ be open embeddings, where $\alpha_1$ is the set of odd positive integers and $\alpha_2 = Z\backslash\alpha_1$. Then $f_1(X) \times f_2(Y)$ is an open subset of $s = s^{\alpha_1} \times s^{\alpha_2}$. We know that there is an (f-d) cap-set $P_1$ for $s^{\alpha_1}$ such that $P_1 \cap f_1(X) = f_1(M)$ and an (f-d) cap-set $P_2$ for $s^{\alpha_2}$ such that $P_2 \cap f_2(Y) = f_2(N)$.

It is obvious that there exist canonical (f-d) cap-sets $R_1$ for $s^{\alpha_1}$ and $R_2$ for $s^{\alpha_2}$ such that $R_1 \times R_2$ is an (f-d) cap-set for $s$. This implies that $P_1 \times P_2$ is an (f-d) cap-set for $s$ and thus $(P_1 \times P_2) \cap (f_1(X) \times f_2(Y)) = f_1(M) \times f_2(N)$ is an (f-d) cap-set for $f_1(X) \times f_2(Y)$.

Now assume that $X$ and $Y$ are $Q$-manifolds. Using Lemmas 5.3 and 5.5 we find that if $K_1 \subset X$ is a compact $Z$-set in $X$ and $K_2 \subset Y$ is a compact $Z$-set in $Y$, then $K_1 \times K_2$ is a $Z$-set in $X \times Y$. If we use the techniques involved in the proof of Lemma 5.6 it follows that if $M$ and $N$ are cap-sets, then $M \times N$ is a cap-set for $X \times Y$. Thus all we have left to do is the case in which $M$ and $N$ are f-d cap-sets.

Let $M'$ be a cap-set for $X$ and let $N'$ be a cap-set for $Y$. It clearly follows from the techniques involved in the proof of Lemma 5.6 that $X \times N'$ and $M' \times Y$ are cap-sets for $X \times Y$. Using Theorem 6.6 we find that $(X \times N') \cup (M' \times Y)$ is a cap-set for $X \times Y$.

Now choose f-d cap-sets $M''$ for $X\backslash M'$ and $N''$ for $Y\backslash N'$. Using what we proved above we know that $M''\times N''$ is an f-d cap-set for $(X\backslash M')\times(Y\backslash N')$. Since $(X\times Y)\backslash[(X\times N')\cup(M'\times Y)]=(X\backslash M')\times(Y\backslash N')$ it follows from Theorem 6.3 that $M''\times N''$ is an f-d cap-set for $X\times Y$. Then we can clearly use Theorem 6.2 to relate this to $M$ and $N$. $\square$

We can use Theorem 6.7 to characterize $Z$-sets in (f-d) cap-sets.

THEOREM 6.9. *Let $X$ be an F-manifold and let $M$ be an (f-d) cap-set for $X$. A necessary and sufficient condition that a closed subset $K$ of $M$ has Property Z in $M$ is that $K$ be strongly negligible in $M$.*

**Proof.** If $K$ is strongly negligible in $M$, then we can use the proof of Theorem 2.20 to prove that $K$ is a $Z$-set in $M$.

Thus assume that $K$ is a $Z$-set in $M$ and let $\{U_i\}_{i=1}^{\infty}$ be an open cover of $M$. For each $i>0$ let $V_i$ be an open subset of $X$ satisfying $V_i\cap M=U_i$. Without loss of generality we may assume that $X=\bigcup_{i=1}^{\infty} V_i$.

By Lemma 5.10, $\bar{K}$ (closure in $X$) is a $Z$-set in $X$. Using Theorem 6.7 it follows that $M\backslash\bar{K}$ is an (f-d) cap-set for $X$. By Theorem 6.1 there exists a homeomorphism $h$ of $X$ onto itself such that $h(M)=M\backslash\bar{K}$ and $h$ is limited by $\{V_i\}_{i=1}^{\infty}$. Thus $K$ is strongly negligible in $M$. $\square$

7. **Infinite-deficiency in manifolds.** In [13] a proof of the following theorem is given.

THEOREM 7.1. *Let $X$ be an F-manifold and let $F$ be a closed subset of $X$. A necessary and sufficient condition that $F$ have Property Z in $X$ is that there exists a homeomorphism $h$ of $X$ onto $X\times s$ such that $\pi_s(h(F))$ has infinite deficiency.*

The technique involved in the proof of Theorem 7.1 is to use Theorem 2.10 and then to modify the copy of $F$ in $X\times s$ so that its projection in $s$ has infinite deficiency.

In our next theorem we give a similar characterization of Property Z in $Q$-manifolds. As a corollary we get an easy proof of Theorem 7.1.

THEOREM 7.2. *Let $X$ be a $Q$-manifold and let $F$ be a closed subset of $X\times I^{\infty}$. A necessary and sufficient condition that $F$ have Property Z in $X\times I^{\infty}$ is that there exists a homeomorphism $h$ of $X\times I^{\infty}$ onto itself taking $F$ onto a set whose projection in $I^{\infty}$ has infinite deficiency. Moreover, we can construct $h$ so that $h(X\times B(I^{\infty}))=X\times B(I^{\infty})$.*

**Proof.** The step from infinite deficiency to Property Z is easy and is similar to the proof of Theorem 9.1 of [3]. Thus assume that $F$ is a $Z$-set in $X\times I^{\infty}$. Since $X\times B(I^{\infty})$ is a cap-set for $X\times I^{\infty}$, it follows from Theorem 6.7 that $(X\times B(I^{\infty}))\backslash F$ is also a cap-set for $X\times I^{\infty}$. Using Theorem 6.2 there is a homeomorphism $f_1$ of $X\times I^{\infty}$ onto itself such that $f_1((X\times B(I^{\infty}))\backslash F)=X\times B(I^{\infty})$. This gives us

$$f_1(F)\cap(X\times B(I^{\infty})) = \varnothing.$$

Let $j>0$ be an integer and for each $x \in X \times \prod_{i \neq j} I_i$ let $U_x \times (\delta_x, 1]$ be an open subset of $(X \times \prod_{i \neq j} I_i) \times I_j$ containing $(x, 1)$ such that $(U_x \times (\delta_x, 1]) \cap f_1(F) = \varnothing$. Note that $\{U_x \mid x \in X \times \prod_{i \neq j} I_i\}$ is an open cover of $X \times \prod_{i \neq j} I_i$. Thus we can use Lemma 6.1 to get a star-finite open refinement $\{U_i\}_{i=1}^{\infty}$ of $\{U_x \mid x \in X \times \prod_{i \neq j} I_i\}$ which covers $X \times \prod_{i \neq j} I_i$. Moreover we may assume that the $U_i$'s are indexed as in Convergence Procedure B. It is clear that for each $i>0$ there is a number $\delta_i > 0$ such that $\frac{1}{2} \leqq \delta_i < 1$ and $(U_i \times (\delta_i, 1]) \cap f_1(F) = \varnothing$.

Now construct a closed cover $\{C_i\}_{i=1}^{\infty}$ of $X \times \prod_{i \neq j} I_i$ such that $C_i \subset U_i$, for all $i>0$. For each $i>0$ let $\varphi_i : X \times \prod_{i \neq j} I_i \to [0, 1]$ be a continuous function such that $\varphi_i(x) = 1$ for $x \in C_i$ and $\varphi_i(x) = 0$ for $x \in X \times \prod_{i \neq j} I_i \backslash U_i$. For each $i>0$ we now construct a homeomorphism $g_i$ of $(X \times \prod_{i \neq j} I_i) \times I_j$ onto itself which slides points linearly in the $I_j$-direction as follows: if $x \in X \times \prod_{i \neq j} I_i$, then $\{x\} \times I_j$ is taken linearly onto itself such that $g_i(x, \delta_i) = (x, \delta_i(1-\varphi_i(x)) + (\frac{1}{2})\varphi_i(x))$. Applying Convergence Procedure C to $\{g_i\}_{i=1}^{\infty}$ let $g = L\prod_{i=1}^{\infty} g_i$, which is a homeomorphism of $X \times I^{\infty}$ onto itself satisfying $g \circ f_1(F) \subset (X \times \prod_{i \neq j} I_i) \times [-1, \frac{1}{2}]$. Using similar techniques we can construct a homeomorphism $g'$ of $X \times I^{\infty}$ onto itself which slides points only in the $I_j$-direction such that $g' \circ g \circ f_1(F) \subset X \times \prod_{i \neq j} I_i \times [-\frac{1}{2}, \frac{1}{2}]$. Then put $g^j = g' \circ g$. We note that $g^j$ is a homeomorphism of $X \times I^{\infty}$ onto itself which satisfies $g^j \circ f_1(F) \subset X \times \prod_{i \neq j} I_i \times [-\frac{1}{2}, \frac{1}{2}]$, $\pi_X \circ g^j(x, t) = x$, and

$$\tau_i \circ \pi_{I^{\infty}} \circ g^j(x, t) = \tau_i(t),$$

for all $i \neq j$ and $(x, t) \in X \times I^{\infty}$.

Exercising some control on the "size" of the $g^j$'s and using Theorem 4.2 of [7], it is now clear that $f_2 = L\prod_{j=1}^{\infty} g^j$ gives a homeomorphism $f_2$ of $X \times I^{\infty}$ onto itself such that $f_2(f_1(F)) \subset X \times \prod_{i=1}^{\infty} [-\frac{1}{2}, \frac{1}{2}]$ and $f_2(X \times B(I^{\infty})) = X \times B(I^{\infty})$. Using Theorem 3.5 of [2] there is a $\beta^*$-homeomorphism $\psi: I^{\infty} \to I^{\infty}$ such that $\psi(\prod_{i=1}^{\infty} [-\frac{1}{2}, \frac{1}{2}])$ has infinite deficiency. Then $f_3 = \mathrm{id} \times \psi$ gives a homeomorphism of $X \times I^{\infty}$ onto itself such that $\pi_{I^{\infty}} \circ f_3(X \times \prod_{i=1}^{\infty} [-\frac{1}{2}, \frac{1}{2}])$ has infinite deficiency and $f_3(X \times B(I^{\infty})) = X \times B(I^{\infty})$.

Now $f_3 \circ f_2 \circ f_1$ is a homeomorphism of $X \times I^{\infty}$ onto itself such that $\pi_{I^{\infty}} \circ f_3 \circ f_2 \circ f_1(F)$ has infinite deficiency. We also note that

$$f_3 \circ f_2 \circ f_1[F \cap (X \times B(I^{\infty}))] \subset X \times s$$

and is a countable union of compact sets. Using Corollary 5.7 of [2] for pushing a $\sigma$-compact subset of $s$ to $B(I^{\infty})$ there is a homeomorphism $f_4$ of $X \times I^{\infty}$ onto itself such that $\pi_{I^{\infty}} \circ f_4 \circ f_3 \circ f_2 \circ f_1(F)$ has infinite deficiency and $f_4 \circ f_3 \circ f_2 \circ f_1(X \times B(I^{\infty})) = X \times B(I^{\infty})$. Then $h = f_4 \circ f_3 \circ f_2 \circ f_1$ satisfies our requirements. □

COROLLARY 7.1. *If $X$ is an $F$-manifold and $F$ is a $Z$-set in $X \times s$, then there is a homeomorphism $h$ of $X \times s$ onto itself such that $\pi_s \circ h(F)$ has infinite deficiency.*

**Proof.** Using Theorem 5.1 there is a $Q$-manifold $Y$ containing $X \times s$ such that $Y \backslash (X \times s)$ is a cap-set for $Y$. This means that there is a homeomorphism $f$ of $Y$ onto

$Y\times I^{\infty}$ such that $f(X\times s)=Y\times s$. Then $f(F)$ is a $Z$-set in $Y\times s$. By Lemma 5.7 we know that $\mathrm{Cl}\,(f(F))$ (closure taken in $Y\times I^{\infty}$) is a Z-set in $Y\times I^{\infty}$. Thus there is a homeomorphism $g$ of $Y\times I^{\infty}$ onto itself such that $g(Y\times s)=Y\times s$ and $g(\mathrm{Cl}\,(f(F)))$ has infinite deficiency. We can write $s=s_1\times s_2$, where $s_1$, $s_2$ are copies of $s$ and $g(\mathrm{Cl}\,(f(F)))$ is infinitely deficient in the $s_2$ factor. Then the following diagram, with appropriate restrictions on the homeomorphisms involved, gives our desired homeomorphism.

$$X\times s \xrightarrow{\quad f \quad} Y\times s \xrightarrow{\quad g \quad} Y\times s = (Y\times s_1)\times s_2 \xrightarrow{f^{-1}x\text{ id}} X\times s. \quad \square$$

8. **Product and factor theorems.** The first result we establish is a relationship between cap-sets and f-d cap-sets in $F$-manifolds. For its proof we will require two preliminary lemmas.

The first lemma gives sufficient conditions in order that a subset of $I^{\infty}$ be a cap-set for $I^{\infty}$. These conditions are also necessary, and in [5] Anderson used them for a definition of cap-set.

LEMMA 8.1. *Let $M\subset I^{\infty}$ be a countable union of Z-sets in $I^{\infty}$ which satisfies the property that for each $\varepsilon>0$ and each pair $A$, $B$ of compact subsets of $I^{\infty}$, with $A\subset B\cap M$, there is an embedding $h\colon B\to M$ such that $h|A=\mathrm{id}$ and $d(h,\mathrm{id})<\varepsilon$. Then $M$ is a cap-set for $I^{\infty}$.*

**Proof.** Let $N$ be a cap-set for $I^{\infty}$ and write $N=\bigcup_{n=1}^{\infty} N_n$, as in the definition. Then there is an embedding $f_1\colon N_1\to M$. Since $f_1(N_1)$ obviously has Property Z in $I^{\infty}$ we can use Lemma 2.1 of [13] to extend $f_1$ to a homeomorphism $g_1$ of $I^{\infty}$ onto itself.

Using the above properties of $M$ we can inductively obtain a sequence $\{g_i\}_{i=1}^{\infty}$ of homeomorphisms of $I^{\infty}$ onto itself such that

(1) for each $i>0$, $g_i\circ\cdots\circ g_1(N_i)\subset M$,

(2) for each $i>0$, $g_{i+1}|g_i\circ\cdots\circ g_1(N_i)=\mathrm{id}$, and

(3) for each $i>1$, $d(g_i,\mathrm{id})<\min\,((3^{-i}), (3^{-i})\cdot\eta(g_{i-1}\circ\cdots\circ g_1, 2^{-i}))$.

Applying Convergence Procedure A we find that $g=L\prod_{i=1}^{\infty} g_i$ gives a homeomorphism of $I^{\infty}$ onto itself such that $g(N)\subset M$. Since $g(N)$ is a cap-set for $I^{\infty}$ it follows from Lemma 4.2 that $M$ is also. $\square$

LEMMA 8.2. *For each $n>0$ define*

$$M_n = \{x\in I^{\infty} \mid \tau_i(x) = 0, \textit{for } i \geqq 2n-1 \textit{ and odd}\}.$$

*Then $M=\bigcup_{n=1}^{\infty} M_n$ has the cap in $I^{\infty}$.*

**Proof.** We will prove that $M$ has the properties of Lemma 8.1. Thus let $A$ be a compact subset of $M$ and let $\varepsilon>0$ be given. It will clearly be sufficient to prove the existence of an embedding $h\colon I^{\infty}\to M$ satisfying $h|A=\mathrm{id}$ and $d(h,\mathrm{id})<\varepsilon$.

Let $\mathscr{U}$ be an open cover of $I^{\infty}\backslash A$ such that for elements $U\in\mathscr{U}$, $\mathrm{diam}\,(U)$ "becomes small" as $d(U,A)$ "becomes small." Then let $\{U_i\}_{i=1}^{\infty}$ be a star-finite refinement of $\mathscr{U}$ such that each $U_i$ is a basic open set in $I^\infty$ and assume that $\{U_i\}_{i=1}^\infty$ is ordered as in Convergence Procedure B. Choose an odd integer $m$ such that $(\sum_{i=m}^\infty 2^{2-i})^{1/2} < \varepsilon$. We will construct $h$ so that $h$ does not affect the first $m-1$ coordinates of any point.

Using the simplified geometry of the basic open sets $U_i$, we can rotate coordinates in an obvious manner to obtain embeddings $h_i: I^\infty \to I^\infty$ such that for each $i>0$, $h_i(U_i) \subset M$, $h_i | I^\infty \backslash U_i = \mathrm{id}$, and $h_i$ does not affect the first $m-1$ coordinates of any point. Each point $x$ of $I^\infty$ is moved only a finite number of times in the sequence $h_1(x)$, $h_2 \circ h_1(x)$, $h_3 \circ h_2 \circ h_1(x), \ldots$. Thus $h(x) = \lim_{i\to\infty} h_i \circ \cdots \circ h_1(x)$ defines an embedding $I^\infty$ into $M$ such that $h|A = \mathrm{id}$ and $d(h, \mathrm{id}) < \varepsilon$. □

THEOREM 8.1. *Let $X$ be an F-manifold and let $M \subset X$ be an f-d cap-set for $X$. Then $M \times I^\infty$ is a cap-set for $X \times I^\infty$.*

**Proof.** We will use Theorem 6.5 by showing that if $U$ is an open subset of $X$ which is homeomorphic to $s$, then $(M \cap U) \times I^\infty$ has the cap in $U \times I^\infty$.

We know that there is a homeomorphism $f$ of $U$ onto $s$ satisfying $f(M \cap s) = \sigma$. Thus it will suffice to prove that $\sigma \times I^\infty$ has the cap in $s \times I^\infty$. Let $I_1^\infty = \prod \{I_i \mid i \text{ odd}\}$ and $I_2^\infty = \prod \{I_i \mid i \text{ even}\}$. Let $A$ be the set of all elements of $I_1^\infty$ having at most finitely many nonzero coordinates. Using Lemma 4.2 we know that $A$ has the f-d cap in $I_1^\infty$. Lemma 8.2 shows that $A \times I_2^\infty$ has the cap on $I_1^\infty \times I_2^\infty$. There is a homeomorphism $g$ of $I_1^\infty$ onto itself such that $g(A) = \sigma_1$, where $\sigma_1$ is the set of all points in $s_1 = \prod \{I_i^0 \mid i \text{ odd}\}$ having at most finitely many nonzero coordinates. Thus $\sigma_1 \times I_2^\infty$ has the cap in $I_1^\infty \times I_2^\infty$ and hence in $s_1 \times I_2^\infty$. This is what we wished to prove. □

COROLLARY 8.1. *Let $X$ be an F-manifold, let $M \subset X$ be an f-d cap-set for $X$, and let $N \subset X$ be a cap-set for $X$. Then $M \times I^\infty$ is homeomorphic to $N$.*

**Proof.** This follows immediately from Theorems 6.1 and 8.1. □

We know from Theorem 2.8 that given any $F$-manifold $X$, there is a countable locally-finite simplicial complex $K$ such that $X$ is homeomorphic to $|K| \times s$. We use this in our next theorem.

THEOREM 8.2. *Let $X$ be an F-manifold and let $K$ be a complex as above. If $M \subset X$ is a cap-set (or f-d cap-set) for $X$, then $M$ is homeomorphic to $|K| \times \Sigma$ (or $|K| \times \sigma$).*

**Proof.** All that we need to do is prove that $|K| \times \sigma$ has the f-d cap in $|K| \times s$, since the proof that $|K| \times \Sigma$ has the cap in $|K| \times s$ is easy and resembles the proof of Lemma 5.6.

Write $|K| = \bigcup_{n=1}^\infty |K_n|$, where $|K_n|$ is a finite complex contained in the interior of $|K_{n+1}|$, and write $\sigma = \bigcup_{n=1}^\infty M_n$ so as to satisfy the definition of an f-d cap-set for $s$. We will show that the representation $|K| \times \sigma = \bigcup_{n=1}^\infty (|K_n| \times M_n)$ satisfies the definition for $|K| \times \sigma$ to have the f-d cap in $|K| \times s$. To this end let $m > 0$ be an integer, let $F$ be a finite-dimensional compact subset of $|K| \times s$, and let $\varepsilon > 0$ be given.

If $C$ is any compact subset of $s$, then there is a homeomorphism of $s$ onto

$s \times [0, 1)$ taking $C$ into $s \times \{0\}$. This means that there is an isotopy $G: s \times I \to s$ such that $G_0 = \mathrm{id}$, $d(x, G_t(x)) < \varepsilon/3$, for all $t \in I$ and $x \in s$, and $G_t(s) \cap M_m = \varnothing$, for all $t > 0$. For each $x \in F$ define $f(x) = G_t \circ \pi_s(x)$, where

$$t = d(x, |K_m| \times M_m)/(1 + d(x, |K_m| \times M_m)).$$

Then $f: F \to s$ is a continuous function such that $d(f, \pi_s) < \varepsilon/3$ and $f(F \backslash (|K_m| \times M_m)) \cap M_m = \varnothing$.

In [8] it is shown that if $X$ is any $F$-manifold, $A$ is a topologically complete separable metric space, $\mathscr{U}$ is an open cover of $X$, and $g: A \to X$ is a continuous function, then there is an embedding $\tilde{g}: A \to X$ such that $\tilde{g}$ is $\mathscr{U}$-close to $g$ (i.e. for each $a \in A$ there exists an element of $\mathscr{U}$ containing both $\tilde{g}(a)$ and $g(a)$). If we let $X = s \backslash M_m$ and choose an appropriate cover of $X$, then we can use this result to obtain a continuous function $\tilde{f}: F \to s$ such that $\tilde{f} \mid |K_m| \times M_m = \pi_s \mid |K_m| \times M_m$, $\tilde{f} | F \backslash (|K_m| \times M_m)$ is an embedding, and $d(\tilde{f}, f) < \varepsilon/3$.

Note that $\tilde{f}(F)$ is a finite-dimensional compact subset of $s$. Thus there exists an integer $n > 0$ and an embedding $g: \tilde{f}(F) \to M_n$ such that $g | \tilde{f}(F) \cap M_m = \mathrm{id}$ and $d(g, \mathrm{id}) < \varepsilon/3$. Now let $n' \geq n$ be an integer such that $\pi_{|K|}(F) \subset |K_{n'}|$. Define $h: F \to |K_{n'}| \times M_{n'}$ by $h(x) = (\pi_{|K|}(x), g \circ \tilde{f}(x))$, for all $x \in F$. Then $h$ is an embedding, $h | F \cap (|K_m| \times M_m) = \mathrm{id}$, and $d(h, \mathrm{id}) < \varepsilon$, as we wanted. □

COROLLARY 8.2. *Let $X$ be an $F$ or $Q$-manifold, let $M \subset X$ have the f-d cap, and let $N \subset X$ have the cap. Then $X$, $M$, and $N$ are all of the same homotopy type.*

**Proof.** For $X$ an $F$-manifold the proof follows from Theorem 8.2. If $X$ is a $Q$-manifold we have already shown (Lemma 5.6 and Theorem 6.2) that $N$ is homeomorphic to $X \times \Sigma$, and hence has the same homotopy type as $X$. Using Corollary 5.1 we find that $M$ has the same homotopy type as $N$. □

COROLLARY 8.3. *Let $X$ be a $Q$-manifold and let $M \subset X$ have the (f-d) cap. Then $X \backslash M$ is an $F$-manifold of the same homotopy type as $X$.*

**Proof.** If $N$ is an (f-d) cap-set for $X \backslash M$, then Corollary 8.2 implies that $N$ and $X \backslash M$ have the same homotopy type. From Theorem 6.2 it follows that $N$ is homeomorphic to $M$, and from Corollary 8.2 it follows that $M$ and $X$ have the same homotopy type. □

From Theorem 2.5 we know that if $K$ is any countable locally-finite simplicial complex, then $|K| \times s$ is an $F$-manifold. We use this to prove a corresponding property for $\sigma$ and $\Sigma$-manifolds.

COROLLARY 8.4. *If $K$ is any countable locally-finite simplicial complex, then $|K| \times \sigma$ is a $\sigma$-manifold and $|K| \times \Sigma$ is a $\Sigma$-manifold.*

**Proof.** From Theorem 8.2 we know that $|K| \times \sigma$ is an f-d cap-set for $|K| \times s$ and $|K| \times \Sigma$ is a cap-set for $|K| \times s$. Then Lemma 5.4 implies that $|K| \times \sigma$ is a $\sigma$-manifold and $|K| \times \Sigma$ is a $\Sigma$-manifold. □

We can easily establish some factor theorems for (f-d) cap-sets in $F$-manifolds.

THEOREM 8.3. *Let $X$ be an $F$-manifold, let $M$ be an f-d cap-set for $X$, and let $I^n$ be any n-cell. Then $M$, $M\times\sigma$, and $M\times I^n$ are all homeomorphic.*

**Proof.** Using Theorem 6.8 it follows that $M\times\sigma$ is an f-d cap-set for $X\times s$. This implies that $M$ and $M\times\sigma$ are homeomorphic.

Since $X$ and $X\times I^\infty$ are homeomorphic it follows that $X$ and $X\times I^n$ are also homeomorphic. Thus to show that $M$ and $M\times I^n$ are homeomorphic all we have to do is prove that $M\times I^n$ is an f-d cap-set for $X\times I^n$. Using Theorem 6.5 it is clearly sufficient to prove that if $U$ is any open subset of $X$ which is homeomorphic to $s$, then $(M\cap U)\times I^n$ is an f-d cap-set for $U\times I^n$. Thus the problem reduces to the following: Show that $\sigma\times I^n$ is an f-d cap-set for $s\times I^n$.

Let $s_1=\prod\{I_i^0 \mid i>n\}$ and $\sigma_1=\{x\in s_1 \mid x$ has at most finitely many nonzero coordinates$\}$. Then Lemma 4.2 implies $\sigma_1\times I^n$ is an f-d cap-set for $I^\infty=I_1^\infty\times I^n$, where $I_1^\infty=\prod\{I_i \mid i>n\}$. Thus $\sigma_1\times I^n$ is an f-d cap-set for $s_1\times I^n$, which is what we wished to prove. $\square$

COROLLARY 8.5. *Let $X$ be an $F$-manifold and let $M$ be a cap-set for $X$. Then $M$, $M\times\Sigma$, and $M\times I^\infty$ are all homeomorphic.*

**Proof.** It follows from Corollary 8.1 that $M$ is homeomorphic to $N\times I^\infty$, where $N$ is an f-d cap-set for $X$. Thus $M$ and $M\times I^\infty$ are homeomorphic. It then follows that $\Sigma$ and $\sigma\times I^\infty$ are homeomorphic. Thus (with "$\simeq$" meaning "is homeomorphic to")

$$M\times\Sigma \simeq (N\times I^\infty)\times(\sigma\times I^\infty) \simeq (N\times\sigma)\times I^\infty \simeq N\times I^\infty \simeq M. \quad \square$$

9. **Permuting sets which have the (f-d) cap.** The following result is a generalization of Lemma 4.3. Since its proof is similar, except that we are handling several subsets of $I^\infty$ having the (f-d) cap at one time, we omit its proof.

LEMMA 9.1. *Let $\{M_i\}_{i=1}^n$ ($n$ finite or $\infty$) be a collection of disjoint subsets of $I^\infty$ which have the (f-d) cap in $I^\infty$ and let $\{N_i\}_{i=1}^n$ be a collection of disjoint subsets of $I^\infty$ which have the (f-d) cap in $I^\infty$. If $K\subset B(I^\infty)\backslash[\bigcup_{i=1}^n (M_i\cup N_i)]$ is compact, then there is a homeomorphism $h$ of $I^\infty$ onto itself such that $h|K=\mathrm{id}$ and $h(M_i)=N_i$, for all $i$. If $\bigcup_{i=1}^n (M_i\cup N_i)\subset s$, then we may additionally require that $h$ be a $\beta^*$-homeomorphism.*

We can easily apply the techniques of §6 to prove the following theorem.

THEOREM 9.1. *Let $\{M_i\}_{i=1}^n$ ($n$ finite or $\infty$) be a collection of disjoint subsets of an $F$ or $Q$-manifold $X$ which have the (f-d) cap in $X$ and let $\{N_i\}_{i=1}^n$ be a collection of disjoint subsets of $X$ which have the (f-d) cap in $X$. Then there is a homeomorphism $h$ of $X$ onto itself such that $h(M_i)=N_i$, for all $i$.*

In our next theorem we prove that any finite collection of $n$ disjoint (f-d) cap-sets for an $F$ or $Q$-manifold can be permuted with a period $n$ homeomorphism.

THEOREM 9.2. *Let $\{M_i\}_{i=1}^n$ be a finite collection of disjoint subsets of an F or Q-manifold X, with each $M_i$ having the (f-d) cap in X. Then there is a period n homeomorphism h of X onto itself such that $h(M_i)=M_{i+1}$, for $1 \leqq i \leqq n-1$, and $h(M_n)=M_1$.*

**Proof.** We only treat the case in which $X$ is an $F$-manifold and each $M_i$ is a cap-set for $X$. The other cases are similar.

Let $\{[a_i, b_i]\times[c_i, d_i]\}_{i=1}^n$ be a collection of disjoint rectangles in $(-1, 1)\times(-1, 1)$ and let $f$ be a homeomorphism of $(-1, 1)\times(-1, 1)$ onto itself of period $n$ such that $f([a_i, b_i]\times[c_i, d_i])=[a_{i+1}, b_{i+1}]\times[c_{i+1}, d_{i+1}]$, for $1 \leqq i \leqq n-1$, and

$$f([a_n, b_n]\times[c_n, d_n]) = [a_1, b_1]\times[c_1, d_1].$$

For $1 \leqq i \leqq n$ let $C_i$ be the subset of $s$ which consists of all points $x \in s$ such that $\tau_{2j-1}(x) \in [a_i, b_i]$ and $\tau_{2j}(x) \in [c_i, d_i]$, for all $j>0$.

For $1 \leqq i \leqq n$ let $N_i$ be the basic core set in $s$ structured over $C_i$. Then $\{N_i\}_{i=1}^n$ is a collection of disjoint cap-sets for $s$. Let $g$ be the homeomorphism of $s$ onto itself such that $\tau_{\{2j-1,2j\}} \circ g(x)=f(\tau_{\{2j-1,2j\}}(x))$, for all $j>0$. Then $g$ is a homeomorphism of $s$ onto itself of period $n$ such that $g(N_i)=N_{i+1}$, for $1 \leqq i \leqq n-1$, and $g(N_n)=N_1$.

Now let $M$ be a cap-set for $X$ and use Theorem 6.8 to conclude that $\{M\times N_i\}_{i=1}^n$ is a collection of disjoint cap-sets for $X\times s$. Also $\mathrm{id}\times g\colon X\times s \to X\times s$ gives a homeomorphism of period $n$ which permutes the collection $\{M\times N_i\}_{i=1}^n$. Since $X$ is homeomorphic to $X\times s$ this means that there is a collection $\{N_i'\}_{i=1}^n$ of disjoint cap-sets for $X$ and a period $n$ homeomorphism $h'$ of $X$ onto itself which permutes $\{N_i'\}_{i=1}^n$.

Using Theorem 9.1 there is a homeomorphism $\varphi$ of $X$ onto itself such that $\varphi(M_i)=N_i'$, for $1 \leqq i \leqq n$. Then $h=\varphi^{-1} \circ h' \circ \varphi$ satisfies our required properties. □

Using similar devices we can also permute a countably infinite collection of disjoint (f-d) cap-sets for an $F$ or $Q$-manifold.

THEOREM 9.3. *Let $\{M_i\}_{i=-\infty}^{\infty}$ be a collection of disjoint subsets of an F or Q-manifold X, each $M_i$ having the (f-d) cap in X. Then there is a homeomorphism h of X onto itself such that $h(M_i)=M_{i+1}$, for all i.*

**Proof.** Once more we only treat the case in which $X$ is an $F$-manifold and each $M_i$ is a cap-set for $X$.

Let $\{[a_i, b_i]\}_{i=1}^{\infty}$ be a collection of disjoint subintervals of $(-1, 1)$ such that $b_i < a_{i+1}$, for all $i$, and let $f$ be a homeomorphism of $(-1, 1)$ onto itself such that $f([a_i, b_i])=[a_{i+1}, b_{i+1}]$, for all $i$. For each $i$ let $C_i$ be the subset of $s$ consisting of all points $x \in s$ for which $\tau_j(x) \in [a_i, b_i]$, for all $j>0$. As in Theorem 9.2 let $N_i$ be the basic core set in $s$ structured over $C_i$. It is now clear that we can use techniques like those used in the proof of Theorem 9.2 to complete the proof of this theorem. □

10. **Open embedding theorems for $\sigma$ and $\Sigma$-manifolds.** The main result of this section is Theorem 10.3 in which we prove that any $\sigma$ (or $\Sigma$)-manifold can be embedded as an open subset of $\sigma$ (or $\Sigma$). Our proof makes use of Henderson's open

embedding theorem for $F$-manifolds (Theorem 2.3). Our strategy is to first embed the $\sigma$ (or $\Sigma$)-manifold as an f-d cap (or cap)-set for an $F$-manifold (Theorem 10.2), and then apply Henderson's result.

In this and the next section we will need the following well-known result which is due to Lavrent'ev (see [16, p. 235]).

THEOREM 10.1. *Each homeomorphism between subspaces $A$ and $B$ of topologically complete metric spaces $X$ and $Y$, respectively, can be extended to a homeomorphism between $G_\delta$-subsets of $X$ and $Y$.*

The following result gives a technique for finding $F$-manifolds in topologically complete metric spaces. We will also use this in the next section.

LEMMA 10.1. *Let $X$ be an $F$-manifold, let $M$ be an (f-d) cap-set for $X$, let $Y$ be a topologically complete metric space, and let $h: M \to Y$ be an embedding. Then there is a copy $X'$ of $X$ such that $h(M) \subset X' \subset Y$ and $h(M)$ is an (f-d) cap-set for $X'$.*

**Proof.** Using Theorem 10.1 we can extend $h$ to a homeomorphism $H: X'' \to H(X'')$, where $X''$ and $H(X'')$ are $G_\delta$-sets. Using Lemma 5.8 we find that $X \backslash X'' = \bigcup_{i=1}^{\infty} F_i$, where each $F_i$ is a $Z$-set in $X$. Using Theorem 1 of [4] there is a homeomorphism $h$ of $X$ onto $X''$. Then $X''$ is a copy of $X$ and it is obvious that $M$ is an (f-d) cap-set for $X''$. We set $X' = H(X'')$ and note that $h(M)$ is an (f-d) cap-set for $X'$. □

We are now ready for the major step in establishing our open embedding theorems. This will also be used in the next section.

THEOREM 10.2. *If $X$ is a $\sigma$ (or $\Sigma$)-manifold, then there is an $F$-manifold $Y$ and an embedding $h: X \to Y$ such that $h(X)$ is an f-d cap (or cap)-set for $Y$.*

**Proof.** Let $\{U_i\}_{i=1}^{\infty}$ be an open cover of $X$ such that each $U_i$ is homeomorphic to an open subset of $\sigma$ (or $\Sigma$). Without loss of generality assume that $X \subset Z$, where $Z$ is a topologically complete separable metric space. For each $i > 0$ let $V_i$ be an open subset of $Z$ such that $V_i \cap X = U_i$ and let $G_i$ be an open subset of $s$ such that $G_i \cap \sigma$ (or $G_i \cap \Sigma$) is homeomorphic to $U_i$. Then $G_i \cap \sigma$ (or $G_i \cap \Sigma$) is an f-d cap (or cap)-set for $G_i$, and making use of Lemma 10.1 there is an embedding $h_i: G_i \to V_i$ such that $U_i \subset h_i(G_i)$ and $U_i$ is an f-d cap (or cap)-set for $h_i(G_i)$.

We note that each $h_i(G_i)$ is a topologically complete metric space and hence it must be a $G_\delta$ in $V_i$. Thus for each $i > 0$ we have $V_i \backslash h_i(G_i) = \bigcup_{n=1}^{\infty} A_i^n$, with each $A_i^n$ closed in $Z$. We also note that for each $i$ and $n$, $A_i^n \cap X = \varnothing$. This means that $A_i^n \cap h_j(G_j)$ is a $Z$-set in $h_j(G_j)$, for all $n$, $i$, and $j$. Now put $G_i' = h_i(G_i) \backslash \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} A_i^n$, and from Theorem 1 of [4] we know that each $G_i'$ is homeomorphic to $h_i(G_i)$. It is also true that each $U_i$ is an f-d cap (or cap)-set for $G_i'$.

We note that for each $j > 0$, $V_j \cap (\bigcup_{i=1}^{\infty} G_i') = G_j'$. This means that $Y = \bigcup_{i=1}^{\infty} G_i'$ is an $F$-manifold. Since $X$ has the f-d cap (or cap) locally in $Y$, we find that $X$ is an f-d cap (or cap)-set for $Y$. □

We can also prove a corresponding property for $Q$-manifolds.

COROLLARY 10.1. *If $X$ is a $\sigma$ (or $\Sigma$)-manifold, then there is a $Q$-manifold $Y$ and an embedding $h: X \to Y$ such that $h(X)$ is an f-d cap (or cap)-set for $Y$.*

**Proof.** Let $Z$ be an $F$-manifold and let $f: X \to Z$ be an embedding such that $f(X)$ is an f-d cap (or cap)-set for $Z$. Using Theorem 5.1 there is a $Q$-manifold $Y$ and an embedding $g: Z \to Y$ such that $Y \backslash g(Z)$ is a cap-set for $Y$. Then Theorem 6.3 implies that $g \circ f(X)$ is an f-d cap (or cap)-set for $Y$. □

We can easily use Theorem 10.2 to classify $\sigma$ and $\Sigma$-manifolds according to homotopy type.

COROLLARY 10.2. *If $X$ and $Y$ are both $\sigma$-manifolds (or both $\Sigma$-manifolds) and are of the same homotopy type, then they are homeomorphic.*

**Proof.** Using Theorem 10.2 there are $F$-manifolds $X'$, $Y'$ and embeddings $f: X \to X'$, $g: Y \to Y'$ such that $f(X)$ is an f-d cap (or cap)-set for $X'$ and $g(Y)$ is an f-d cap (or cap)-set for $Y'$. Since $X$ and $Y$ have the same homotopy type, it follows from Corollary 8.2 that $X'$ and $Y'$ have the same homotopy type. Using Theorem 2.1 we have $X'$ homeomorphic to $Y'$. Then Theorem 6.1 implies that $f(X)$ is homeomorphic to $g(Y)$. □

We can also use these techniques to sharpen Corollary 8.1.

COROLLARY 10.3. *Let $X$ be a $\sigma$-manifold and let $Y$ be a $\Sigma$-manifold of the same homotopy type. Then $X \times I^\infty$ and $Y$ are homeomorphic.*

**Proof.** Using the techniques of the proof of Corollary 10.2 there is an $F$-manifold $Z$ and embeddings $f: X \to Z$, $g: Y \to Z$ such that $f(X)$ is an f-d cap-set for $Z$ and $g(Y)$ is a cap-set for $Z$. Then Corollary 8.1 implies that $f(X) \times I^\infty$ and $g(Y)$ are homeomorphic. □

Finally we sharpen Theorem 8.2.

COROLLARY 10.4. *Let $X$ be a $\sigma$ (or $\Sigma$)-manifold. Then there is a locally-finite simplicial complex $K$ such that $X$ is homeomorphic to $|K| \times \sigma$ (or $|K| \times \Sigma$).*

**Proof.** This is an immediate application of Theorem 10.2. □

We now use Theorem 10.2 to establish open embedding theorems for $\sigma$ and $\Sigma$-manifolds.

THEOREM 10.3. *Let $X$ be a $\sigma$ (or $\Sigma$)-manifold. Then $X$ can be embedded as an open subset of $\sigma$ (or $\Sigma$).*

**Proof.** Using Theorem 10.2 there is an embedding $h: X \to Y$, where $Y$ is an $F$-manifold and $h(X)$ is an f-d cap (or cap)-set for $Y$. Using Henderson's open embedding theorem there is an open embedding $g: Y \to s$. Then $g \circ h(X)$ is an f-d cap (or cap)-set for $g(Y)$, which is an open subset of $s$. Also $g(Y) \cap \sigma$ (or $g(Y) \cap \Sigma$) is an open subset of $\sigma$ (or $\Sigma$) and has the f-d cap (or cap) in $g(Y)$. By Theorem 6.1 we have $g \circ h(X)$ and $g(Y) \cap \sigma$ (or $g(Y) \cap \Sigma$) homeomorphic, which is what we wished to prove. □

11. **Extending homeomorphisms in $\sigma$ and $\Sigma$-manifolds.** The main result of this section is Theorem 11.1 in which we establish a homeomorphism extension theorem for $\sigma$ and $\Sigma$-manifolds. Our strategy is to use techniques similar to those used in §10 to transfer the problem to an $F$-manifold and then apply Theorem 2.24.

We will need a number of preliminary results. The first two of these follow routinely from a close examination of Lavrent'ev's theorem on p. 335 of [16].

LEMMA 11.1 *Let $X$ be a topological space, let $Y$ be a topologically complete metric space, and let $f: A \to Y$ be a continuous function, where $A \subset X$. Then there is a $G_\delta$-subset $B$ of $X$ such that $A \subset B \subset \bar{A}$ and an extension of $f$ to a continuous function $g: B \to Y$.*

LEMMA 11.2. *Let $X$ and $Y$ be topologically complete metric spaces, let $A$ be a subset of $X$, and let $f: A \to Y$ be an embedding. If $B$ is a $G_\delta$-subset of $X$ and $g: B \to Y$ is a continuous extension of $f$, then there is a $G_\delta$-subset $B'$ of $X$ such that $A \subset B' \subset \bar{A}$ and $g|B'$ is an embedding.*

The proof of our next lemma is elementary.

LEMMA 11.3. *Let $X$ be a metric space and let $Y$ be a compact metric space. Then the projection $\pi_X$ is a closed map.*

We now combine these results to show that we can extend homotopies.

LEMMA 11.4. *Let $X$ be a topologically complete metric space and let $A$ be a subset of $X$. Let $h: A \times I \to Y$, where $Y$ is a topologically complete metric space, be a homotopy such that $h|A \times \{0\}$ and $h|A \times \{1\}$ are embeddings. Then there is a $G_\delta$-subset $B$ of $X$ such that $A \subset B \subset \bar{A}$ and an extension of $h$ to a homotopy $H: B \times I \to Y$ such that $H|B \times \{0\}$ and $H|B \times \{1\}$ are embeddings.*

**Proof.** Using Lemma 11.1 there is a $G_\delta$-subset $C$ of $X \times I$ such that $A \times I \subset C \subset \mathrm{Cl}\,(A \times I)$ (closure taken in $X \times I$) and an extension of $h$ to a continuous function $f: C \to Y$. Using Lemma 11.3 we find that $\pi_X((X \times I) \backslash C) = D$ is an $F_\sigma$-subset of $X$. Then $B_1 = X \backslash D$ is a $G_\delta$-subset of $X$ such that $B_1 \times I \subset C$, which means that $g = f|B_1 \times I$ is an extension of $h$.

Applying Lemma 11.2 to $g|B_1 \times \{0\}$ and $g|B_1 \times \{1\}$ we find that there are $G_\delta$-subsets $B_2$ and $B_3$ of $X$ such that $A \subset B_2 \subset B_1$, $A \subset B_3 \subset B_1$, $g|B_2 \times \{0\}$ is an embedding, and $g|B_3 \times \{0\}$ is an embedding. Then put $B = B_2 \cap B_3$ and $H = g|B \times I$ to fulfill the conditions of the lemma. □

If we examine the proof of Lemma 11.4 we can easily sharpen it.

COROLLARY 11.1. *Using the notation of Lemma* 11.4 *let $\mathcal{U}$ be an open cover of $Y$ such that for each $x \in A$, $h(\{x\} \times I)$ is contained in some member of $\mathcal{U}$. Then we can construct $H$ so that for each $x \in B$, $H(\{x\} \times I)$ is contained in some member of* $\mathrm{St}\,(\mathcal{U})$.

We are now ready to prove our main result.

THEOREM 11.1. *Let $X$ be a $\Sigma$ (or $\sigma$) -manifold, let $\mathscr{U}$ be an open cover of $X$, let $A$ be a separable metric space, and let $f: A\times I\to X$ be a homotopy such that $f|A\times\{0\}$ and $f|A\times\{1\}$ are embeddings, $f(A\times\{0\})$ and $f(A\times\{1\})$ are $Z$-sets in $X$ and for each $x\in A, f(\{x\}\times I)$ is contained in some member of $\mathscr{U}$. Then the induced homeomorphism of $f(A\times\{0\})$ onto $f(A\times\{1\})$ can be extended to a homeomorphism of $X$ onto itself which is limited by* $\mathrm{St}^{28}(\mathscr{U})$.

**Proof.** Without loss of generality assume that $X\subset Y$, where $Y$ is an $F$-manifold and $X$ is an (f-d) cap-set for $Y$. Moreover we can assume that for each $U\in\mathscr{U}$, there is an open subset $V_U$ of $Y$ such that $V_U\cap X=U$ and $Y=\bigcup\{V_U \mid U\in\mathscr{U}\}$.

Let $Z$ be a topologically complete metric space containing $A$ and use Lemma 11.4 to get a $G_\delta$-subset $B$ of $Z$ such that $A\subset B\subset\bar{A}$ and an extension of $f$ to a homotopy $f_1: B\times I\to Y$ such that $f_1|B\times\{0\}$ and $f_1|B\times\{1\}$ are embeddings. Using Corollary 11.1 we can additionally assume that for each $x\in B, f_1(\{x\}\times I)$ is contained in some member of $\mathrm{St}(\mathscr{V})$, where $\mathscr{V}=\{V_U \mid U\in\mathscr{U}\}$.

Now put $F_1=\mathrm{Cl}(f(A\times\{0\}))$ and $F_2=\mathrm{Cl}(f(A\times\{1\}))$, where the closure is taken in $Y$, and note that Lemma 5.10 implies that $F_1$ and $F_2$ are $Z$-sets in $Y$.

Since $Y\times B(I^\infty)$ is a cap-set for $Y\times I^\infty$, there is a homeomorphism $h_1: Y\to Y\times I^\infty$ such that $h_1(X)=Y\times B(I^\infty)$. Using Theorem 7.2 there is a homeomorphism $h_2: Y\times I^\infty\to Y\times I^\infty$ such that $\pi_{I^\infty}\circ h_2\circ h_1(F_1\cup F_2)$ has infinite deficiency and $h_2(Y\times B(I^\infty))=Y\times B(I^\infty)$. It is obvious that by adjusting one of the coordinates in which $\pi_{I^\infty}\circ h_2\circ h_1(F_1\cup F_2)$ is deficient we can get a homeomorphism $h_3: Y\times I^\infty\to Y\times I^\infty$ such that $h_3\circ h_2\circ h_1(F_1\cup F_2)\cap h_2\circ h_1(F_1\cup F_2)=\varnothing$, $h_3(Y\times B(I^\infty))=Y\times B(I^\infty)$, and $h_3$ is limited by $h_2\circ h_1(\mathscr{V})$. Then $h=(h_2\circ h_1)^{-1}\circ h_3\circ h_2\circ h_1$ is a homeomorphism of $Y$ onto itself such that $h(X)=X$, $h(F_2)\cap F_1=\varnothing$, and $h$ is limited by $\mathscr{V}$.

Using the construction of $h_2$ we can easily get a homotopy $F: Y\times I\to Y$ such that for each $x\in Y$, $F(x,0)=x$, $F(x,1)=h(x)$, and $F(\{x\}\times I)$ is contained in some member of $\mathscr{V}$. We can clearly use $F$ to modify $f_1$ to a homotopy $f_2: B\times I\to Y$ such that $f_2|B\times\{0\}=f_1|B\times\{0\}$, $f_2|B\times\{1\}=h\circ f_1|B\times\{1\}$, and for each $x\in B$, $f_2(\{x\}\times I)$ is contained in some member of $\mathrm{St}^2(\mathscr{V})$.

Now let $g_1$ be a homeomorphism of $Y$ onto $Y\times I^\infty$ such that $\pi_{I^\infty}\circ g_1(F_1\cup h(F_2))$ has infinite deficiency. Without loss of generality we may assume that $\tau_1\circ\pi_{I^\infty}\circ g_1(F_1\cup h(F_2))=\{0\}$. Now apply Theorem 2.24 to obtain a homeomorphism $g_2$ of $Y\times I^\infty$ onto itself such that $\tau_1\circ\pi_{I^\infty}\circ g_2\circ g_1(F_1\cup h(F_2))=\{-1\}$.

Using this fact we can easily get a homotopy $G: Y\times I\to Y$ such that for $x\in Y$ and $t\in(0,1)$, $G(x,0)=G(x,1)=x$, $G(x,t)\notin F_1\cup h(F_2)$, and $G(\{x\}\times I)$ is contained in some member of $\mathscr{V}$. We can obviously use $G$ to modify $f_2$ to a homotopy $f_3: B\times I\to Y$ such that $f_3|B\times\{0\}=f_1|B\times\{0\}$, $f_3|B\times\{1\}=h\circ f_1|B\times\{1\}$, $f_3(B\times(0,1))\cap(F_1\cup h(F_2))=\varnothing$, and for each $x\in B$, $f_3(\{x\}\times I)$ is contained in some member of $\mathrm{St}^3(\mathscr{V})$.

Put $C_1=f_3(B\times\{0\})$, which is a $G_\delta$-subset of $F_1$, and $C_2=f_3(B\times\{1\})$, which is a $G_\delta$-subset of $h(F_2)$. Let $F_1\backslash C_1=\bigcup_{n=1}^\infty A_n$ and $h(F_2)\backslash C_2=\bigcup_{n=1}^\infty B_n$, where each $A_n$ and each $B_n$ is a closed subset of $Y$. It is then clear that $(\bigcup_{n=1}^\infty A_n)\cap(\bigcup_{n=1}^\infty B_n)=\varnothing$ and for each $n>0$, $A_n$, $B_n$ are $Z$-sets in $Y$.

Let $Y'=Y\backslash\bigcup_{n=1}^\infty (A_n\cup B_n)$, which is homeomorphic to $Y$ by Theorem I of [4], and let $\mathscr{V}'=\{V\cap Y' \mid V\in\mathscr{V}\}$. We note that $C_1$ and $C_2$ are $Z$-sets in $Y'$ and $f_3(B\times I)\subset Y'$. If we let $\varphi$ be the homeomorphism of $C_1$ onto $C_2$ induced by $f_3$, then there is a homotopy $\Phi: C_1\times I\to Y$ such that for each $x\in C_1$, $\Phi(x,0)=x$, $\Phi(x,1)=\varphi(x)$, and $\Phi(\{x\}\times I)$ is contained in some member of $\mathrm{St}^3(\mathscr{V}')$.

Define $\varphi_1: C_1\cup C_2\to C_1\cup C_2$ by $\varphi_1|C_1=\varphi$ and $\varphi_1|C_2=\varphi^{-1}$. Clearly there is a homotopy $\Phi_1:(C_1\cup C_2)\times I\to Y$ such that for each $x\in C_1\cup C_2$, $\Phi_1(x,0)=x$, $\Phi_1(x,1)=\varphi_1(x)$, and $\Phi_1(\{x\}\times I)$ is contained in some member of $\mathrm{St}^3(\mathscr{V}')$. Moreover, we note that $\varphi_1((C_1\cup C_2)\cap X)=(C_1\cup C_2)\cap X$.

Since $X\backslash C_1\cup C_2$ is an (f-d) cap-set for $Y'$, there is a homeomorphism $\psi_1$ of $Y'$ onto itself such that $\psi_1(X\backslash(C_1\cup C_2))=X$ and $\psi_1$ is limited by $\mathscr{V}'$. Using Theorem 2.24 there is a homeomorphism $\psi_2$ of $Y'$ onto itself which extends the homeomorphism $\psi_1\circ\varphi_1\circ\psi_1^{-1}:\psi_1(C_1\cup C_2)\to\psi_1(C_1\cup C_2)$ and is limited by $\mathrm{St}^{24}(\mathscr{V}')$. Using Theorem 6.1 there is a homeomorphism $\psi_3$ of $Y'$ onto itself such that $\psi_3(\psi_2(X))=X$, $\psi_3|\psi_1(C_1\cup C_2)=\mathrm{id}$, and $\psi_3$ is limited by $\mathscr{V}'$. Then $\psi=\psi_1^{-1}\circ\psi_3\circ\psi_2\circ\psi_1$ is a homeomorphism of $Y'$ onto itself such that $\psi(X)=X$, $\psi$ extends $\varphi$, and $\psi$ is limited by $\mathrm{St}^{27}(\mathscr{V}')$.

We note now that $g=(h^{-1}|X)\circ(\psi|X)$ is a homeomorphism of $X$ onto itself which fulfills our requirements. □

By examining the proof of Theorem 11.1 we can state a stronger result.

COROLLARY 11.2. *Let the hypotheses be as in Theorem* 11.1. *Then there is an F-manifold Z containing X and an open cover* $\mathscr{V}$ *of Z such that* $\{V\cap X \mid V\in\mathscr{V}\}=\mathscr{U}$ *and for which the induced homeomorphism of* $f(A\times\{0\})$ *onto* $f(A\times\{1\})$ *can be extended to a homeomorphism of Z onto itself which takes X onto itself and is limited by* $\mathrm{St}^{28}(\mathscr{V})$.

If we apply Theorem 11.1 to $\sigma$ or $\Sigma$ we can easily obtain the following homeomorphism extension theorem, where the homotopy takes place along straight line segments joining each point to its image.

COROLLARY 11.3. *Let* $X=\sigma$ *or* $\Sigma$, *let* $K_1$ *and* $K_2$ *be Z-sets in X, let* $\varepsilon>0$ *be given, and let h be a homeomorphism of* $K_1$ *onto* $K_2$ *such that* $d(h,\mathrm{id})<\varepsilon$. *Then h can be extended to a homeomorphism H of X onto itself such that* $d(H,\mathrm{id})<29\varepsilon$.

We remark that by exercising a bit more care the estimate $\mathrm{St}^{28}(\mathscr{U})$ in Theorem 11.1 can be improved upon, however it does not seem worth the effort. In regard to Corollary 11.3 it is possible to give a proof independent of Theorem 11.1 and obtain the estimate $2\varepsilon$, where Theorem 11.1 would be replaced by the homeomorphism extension theorem announced by Barit [11].

## References

1. R. D. Anderson, *Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **72** (1966), 515–519. MR **32** #8298.

2. ———, *Topological properties of the Hilbert cube and the infinite product of open intervals*, Trans. Amer. Math. Soc. **126** (1967), 200–216. MR **34** #5045.

3. ———, *On topological infinite deficiency*, Michigan Math. J. **14** (1967), 365–383. MR **35** #4893.

4. ———, *Strongly negligible sets in Fréchet manifolds*, Bull. Amer. Math. Soc. **75** (1969), 64–67. MR **38** #6634.

5. ———, *A characterization of apparent boundaries of the Hilbert cube*, Notices Amer. Math. Soc. **16** (1969), 429. Abstract #697–G17.

6. ———, *On sigma-compact subsets of infinite-dimensional spaces*, Trans. Amer. Math. Soc. (submitted).

7. R. D. Anderson and R. H. Bing, *A complete elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **74** (1968), 771–792. MR **37** #5847.

8. R. D. Anderson and John D. McCharen, *On extending homeomorphisms to Fréchet manifolds*, Proc. Amer. Math. Soc. **25** (1970), 283–289.

9. R. D. Anderson and R. Schori, *Factors of infinite-dimensional manifolds*, Trans. Amer. Math. Soc. **142** (1969), 315–330.

10. R. D. Anderson, David W. Henderson and James E. West, *Negligible subsets of infinite-dimensional manifolds*, Compositio Math. **21** (1969), 143–150.

11. William Barit, *Small extensions of small homeomorphisms*, Notices Amer. Math. Soc. **16** (1969), 295. Abstract #663–715.

12. C. Bessaga and A. Pełczyński, *Estimated extension theorem, homogeneous collections and skeletons, and their applications to topological classifications of linear metric spaces and convex sets*, Fund. Math. (submitted).

13. T. A. Chapman, *Infinite deficiency in Fréchet manifolds*, Trans. Amer. Math. Soc. **148** (1970), 137–146.

14. David W. Henderson, *Infinite-dimensional manifolds are open subsets of Hilbert space*, Bull. Amer. Math. Soc. **75** (1969), 759–762.

15. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Math. Series, vol. 4, Princeton Univ. Press, Princeton, N. J., 1941. MR **3**, 312.

16. K. Kuratowski, *Topologie*. Vol. 1: *Espaces métrisables, espaces complets*, 2nd ed., Monografie Mat., Tom 20, PWN, Warsaw, 1948. MR **10**, 389.

17. H. Torunczyk, *Skeletonized sets in complete metric spaces and homeomorphisms of the Hilbert cube*, Bull. Acad. Polon. Sci. Cl. III **18** (1970), 119–126.

18. James E. West, *Infinite products which are Hilbert cubes*, Trans. Amer. Math. Soc. **150** (1970), 1–25.

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