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# TOPOLOGIES FOR $2^X$; SET-VALUED FUNCTIONS AND THEIR GRAPHS(1)

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**Abstract.** We consider the problem of topologizing $2^X$, the set of all closed subsets of a topological space $X$, in such a way as to make continuous functions from a space $Y$ into $2^X$ precisely those functions with closed graphs. We show there is at most one topology with this property, and if $X$ is a regular space, the existence of such a topology implies that $X$ is locally compact. We then define the compact-open topology for $2^X$, which has the desired property for locally compact Hausdorff $X$. The space $2^X$ with this topology is shown to be homeomorphic to a space of continuous functions with the well-known compact-open topology. Finally, some additional properties of this topology are discussed.

1. **Introduction.** Given a topological space $X$, let $2^X$ denote the set of all closed subsets of $X$ (including the empty set $\varnothing$). We will consider the problem of putting a topology on the set $2^X$ which has a natural relationship to the given topology on $X$. In particular we choose conditions for the topology on $2^X$ which will identify continuous maps from spaces $Y$ into $2^X$ in terms of the topologies of $Y$ and $X$.

Let $Y$ be a topological space, and $F\colon Y \to 2^X$. Define the *graph* of $F$ to be the set $G_F = \{(x, y) \in X \times Y \mid x \in F(y)\}$. We say a topology on $2^X$ is *admissible* if for any space $Y$ and any continuous map $F\colon Y \to 2^X$, $G_F$ is a closed subset of $X \times Y$. Conversely, we say a topology on $2^X$ is *proper* if for any space $Y$ and function $F\colon Y \to 2^X$, if $G_F$ is closed then $F$ is continuous. We will consider the problem of giving $2^X$ a topology which is both admissible and proper, i.e., one for which continuity of functions is equivalent to their having closed graphs. (Recall that for *point-valued* functions into a compact Hausdorff space $X$ it is in fact true that the properties of continuity and closed graph are equivalent.)

In §2 we consider some properties of proper and admissible topologies for $2^X$. In particular, we will show that there is at most one topology for $2^X$ which is both proper and admissible. In addition if $X$ is a regular space ($T_1$, and points and closed

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sets can be separated), the existence of a proper-admissible topology for $2^X$ implies that $X$ is locally compact. In §3 we define a topology for $2^X$ (called the compact-open topology) which for Hausdorff spaces $X$ is always proper, and is admissible if and only if $X$ is locally compact. In §4 we will show that we can weaken the restriction on $X$, and instead require that $X\times Y$ be a $k$-space, and this will imply that $F\colon Y\to 2^X$ is continuous in the compact-open topology if and only if $G_F$ is closed. We will also describe some other properties of this topology.

2. **Admissible and proper topologies.** We begin by giving a characterization of an admissible topology.

LEMMA 2.1. *A topology on $2^X$ is admissible if and only if*

$$\Omega = \{(x, A) \in X\times 2^X \mid x \in A\}$$

*is closed in $X\times 2^X$.*

**Proof.** If the topology is admissible, then $\Omega$ is closed since it is the graph of the identity from $2^X$ to itself, which is obviously continuous.

Suppose now that $\Omega$ is closed. Let $Y$ be any space and $F\colon Y\to 2^X$ be continuous. Then $G_F=(1, F)^{-1}(\Omega)$ where $(1, F)\colon X\times Y\to X\times 2^X$ is given by $(1, F)(x, y) =(x, F(y))$ and is continuous. Hence $G_F$ is closed.

If $t_1$ and $t_2$ are two topologies for the same set, and $t_1\subset t_2$, we say $t_1$ is weaker than $t_2$, and $t_2$ is stronger than $t_1$. When more than one topology is being considered at one time, we will denote by $2^X(t)$, the space $2^X$ with the topology $t$.

LEMMA 2.2. (a) *A topology stronger than an admissible topology is also admissible.*

(b) *A topology weaker than a proper topology is also proper.*

(c) *Any admissible topology is stronger than any proper topology.*

**Proof.** Parts (a) and (b) are clear. To prove (c), let $t_1$ be a proper topology and $t_2$ be an admissible topology. Then by Lemma 2.1, $\{(x, A) \mid x \in A\}$ is closed in $X\times 2^X(t_2)$, which implies $1\colon 2^X(t_2)\to 2^X(t_1)$ is continuous. Thus $t_1\subset t_2$.

From Lemma 2.2(c) we obtain the uniqueness of a topology which is both proper and admissible.

THEOREM 2.3. *For any topological space $X$, $2^X$ can have at most one topology which is both proper and admissible. Such a topology is necessarily the strongest proper topology on $2^X$ and the weakest admissible topology on $2^X$. (By "strongest" or "weakest" we mean "containing" or "contained in" all other such topologies.)*

We are thus naturally led to consider the question of when $2^X$ can have a strongest proper topology or a weakest admissible topology. Such topologies must necessarily be unique. The first question is easily settled.

PROPOSITION 2.4. *For any $X$, $2^X$ has a strongest proper topology.*

**Proof.** Let $\{t_\alpha\}$ be the family of all proper topologies. Let $t$ be the topology which has the set $\bigcup_\alpha t_\alpha$ as a subbasis. Since this is stronger than any proper topology, we need only show that it is proper.

Let $Y$ be any space, and $F$: $Y \to 2^X$ be such that $G_F$ is closed. Since any subbasic $V \in t$ is an open set for some proper topology, $F^{-1}(V)$ must be open and hence $F$ is continuous.

On the question of a weakest admissible topology, we first state the following proposition whose proof is clear.

PROPOSITION 2.5. *If $2^X$ has a weakest admissible topology, it must be the intersection of all admissible topologies.*

We now prove the main result of this section.

THEOREM 2.6. *If $X$ is a regular space and $2^X$ has a weakest admissible topology then $X$ is locally compact.*

**Proof.** Let $t$ be the weakest admissible topology on $2^X$. Let $x \in X$. We will show that $x$ has a relatively compact neighborhood.

Since $t$ is admissible, there is an open set $V \subset X$ and a $\mathscr{U} \in t$ such that $x \in V$, $\varnothing \in \mathscr{U}$ and $V \times \mathscr{U} \subset \tilde{\Omega}$, the complement in $X \times 2^X$ of the set $\Omega$ defined in Lemma 2.1. We will show that $\bar{V}$, the closure of $V$, is compact.

Let $\mathscr{A}$ be an open cover of $\bar{V}$, and let $\mathscr{A}^* = \mathscr{A} \cup \{X - \bar{V}\}$, an open cover of $X$. Let $\{F_\alpha\}$ be the collection of all closed subsets of $X$ each of which is contained in some member of $\mathscr{A}^*$, and let the sets $\{A \in 2^X \mid A \subset X - F_\alpha\}$ be a subbasis for a topology $t'$ on $2^X$. We will show now that $t'$ is admissible.

Suppose $y \in X$, $B \in 2^X$ and $y \notin B$. There is an $A^* \in \mathscr{A}^*$ such that $y \in A^*$. Since $X$ is regular, there is an open set $W \subset X$ such that

$$y \in W \subset \bar{W} \subset \tilde{B} \cap A^*.$$

$\bar{W} \subset A^*$ implies $\bar{W} \in \{F_\alpha\}$, and $\bar{W} \subset \tilde{B}$ implies that $B \subset X - \bar{W}$. Thus

$$Z = W \times \{A \mid A \subset X - \bar{W}\}$$

is an open set in $X \times 2^X(t')$, $(y, B) \in Z$ and $Z \subset \tilde{\Omega}$, showing that $\Omega$ is closed in $X \times 2^X(t')$, and hence that $t'$ is admissible.

Since $t$ is the weakest admissible topology, we must have $t \subset t'$ and thus $\mathscr{U} \in t'$. From this it follows that there exist $F_1, \ldots, F_n$ in $\{F_\alpha\}$ such that

$$\varnothing \in \bigcap_{i=1}^{n} \{A \mid A \subset X - F_i\} = \left\{A \mid A \subset X - \bigcup_{i=1}^{n} F_i\right\} \subset \mathscr{U}.$$

We show that $V \subset \bigcup_{i=1}^n F_i$. Suppose, to the contrary, that there is a $y \in V - \bigcup F_i$. We have then that $\{y\} \in \{A \mid A \subset X - \bigcup F_i\} \subset \mathscr{U}$ and therefore $(y, \{y\}) \in V \times \mathscr{U} \subset \tilde{\Omega}$. However, this is impossible since $y \in \{y\}$, and thus we have $V \subset \bigcup F_i$. It follows that also $\bar{V} \subset \bigcup F_i$.

To complete the proof, for $i = 1, \ldots, n$ let $V_i \in \mathscr{A}^*$ be such that $F_i \subset V_i$. Therefore $\bar{V} \subset \bigcup_{i=1}^n V_i$, and we have produced a finite subcover of $\mathscr{A}^*$. We also have a finite subcover of $\mathscr{A}$ since we can omit $X - \bar{V}$ if it occurs among the $V_i$.

COROLLARY 2.7. *Let $X$ be a regular space. If the intersection of all admissible topologies for $2^X$ is admissible, then $X$ is locally compact.*

COROLLARY 2.8. *Let $X$ be a regular space. If $2^X$ has a topology which is both proper and admissible, then $X$ is locally compact.*

In the next section we will show that converses of Theorem 2.6 and the corollaries are true. That is, we will show that if $X$ is a locally compact Hausdorff space, then it has a proper-admissible topology, and hence a weakest admissible topology. (It is clearly regular.)

The problem remains as to whether the hypothesis on $X$ can be weakened, say, to Hausdorff. Theorem 2.6 says that if $X$ is a regular space which is not locally compact (e.g. $X=Q$, the space of rationals in the real line), then $2^X$ has no weakest admissible topology (and thus no proper-admissible topology). An open question is whether a nonregular Hausdorff space can have a weakest admissible topology.

Before proceeding to the next section, we will study an interesting consequence of the proof of Theorem 2.6. If $X$ is regular, and $\mathscr{A}$ is a cover of $X$ by open sets, then $\mathscr{A}$ gives rise to an admissible topology, $t_{\mathscr{A}}$ on $2^X$ as follows. Let $\{F_\alpha\}$ denote the set of closed subsets of $X$ each of which is contained in some member of $\mathscr{A}$. Let $t_{\mathscr{A}}$ be the topology on $2^X$ whose subbasis consists of all sets of the form

$$\{A \in 2^X \mid A \subset X-F_\alpha\}.$$

PROPOSITION 2.9. (a) *Let $X$ be a regular space. For each open cover $\mathscr{A}$ of $X$, $t_{\mathscr{A}}$ is an admissible topology on $2^X$.*

(b) *If $\mathscr{A}$ and $\mathscr{B}$ are open covers, and $\mathscr{A}$ is a refinement of $\mathscr{B}$, then $t_{\mathscr{A}} \subset t_{\mathscr{B}}$.*

(c) *If $\mathscr{A}$ is an open cover which is closed under finite unions, then the sets $\{A \in 2^X \mid A \subset X-F_\alpha\}$ form a basis for $t_{\mathscr{A}}$.*

(d) *Let $X$ be a normal space. If $\mathscr{B}$ is an open cover, and $\mathscr{A}$ is an open cover consisting of all elements of $\mathscr{B}$ and finite unions of elements of $\mathscr{B}$, then $t_{\mathscr{A}}=t_{\mathscr{B}}$ and hence a basis for $t_{\mathscr{B}}$ is formed by the sets given in* (c).

**Proof.** The proof of (a) is contained in the proof of Theorem 2.6. Part (b) follows from the fact that if $\mathscr{A}$ is a refinement of $\mathscr{B}$ then we have $\{F_\alpha\} \subset \{F_\beta\}$ where these are the respective collections used in the construction of $t_{\mathscr{A}}$ and $t_{\mathscr{B}}$. Part (c) follows from the fact that

$$\bigcap_{i=1}^{n} \{A \mid A \subset X-F_i\} = \left\{A \mid A \subset X-\bigcup_{i=1}^{n} F_i\right\}.$$

For part (d), since $\mathscr{B}$ is a refinement of $\mathscr{A}$, we have $t_{\mathscr{B}} \subset t_{\mathscr{A}}$. That $t_{\mathscr{A}} \subset t_{\mathscr{B}}$ follow from the fact that in a normal space if $F$ is closed and $F \subset A_1 \cup \cdots \cup A_n$, where the $A_i$ are open, then there exist closed sets $F_i$ such that $F_i \subset A_i$ and $F=F_1 \cup \cdots \cup F_n$.

We remark here that if $X$ is any Hausdorff space, and if we take the open cover $\mathscr{A}=\{X\}$, then the collection $\{F_\alpha\}$ consists of all closed subsets of $X$ and $t_{\mathscr{A}}$ is the well-known *upper semifinite* topology. See Michael [7, p. 179] and Ponomarev [9] (where it is called the $\kappa$ topology). Notice that for any open cover $\mathscr{B}$, $t_{\mathscr{B}} \subset t_{\mathscr{A}}$. Also, since for a regular space the weakest admissible topology $t$ is weaker than $t_{\mathscr{A}}$, this topology $t$ will not be Hausdorff.

In the next section we shall consider a topology for $2^X$ which for locally compact Hausdorff $X$ can be described as that derived from the cover of $X$ by relatively compact open sets. It therefore must be admissible. In fact, we will show it is also proper, and thus is the weakest admissible topology.

3. **Compact-open topology: $2^X$ as a function space.** Let $X$ be a topological space, and let $\{C_\alpha\}$ be the set of all *compact* subsets of $X$. We define the *compact-open topology* on $2^X$ to be the topology with the basis consisting of sets of the form $\{A \in 2^X \mid A \subset X - C_\alpha\}$.

THEOREM 3.1. *Let $X$ be a Hausdorff space. Then the compact-open topology on $2^X$ is admissible if and only if $X$ is locally compact.*

**Proof.** Suppose $X$ is locally compact and let $\mathscr{C}$ be the open cover of $X$ consisting of all relatively compact open sets. It follows that the compact-open topology is the topology $t_{\mathscr{C}}$, and since $X$ is regular, Proposition 2.9(a) shows that $t_{\mathscr{C}}$ is admissible.

Now suppose the compact-open topology is admissible and let $U \subset X$ be open and let $x \in U$. We will find a relatively compact open set $V \subset X$ such that

$$x \in V \subset \bar{V} \subset U.$$

Since $\Omega$ is closed in $X \times 2^X$, we can find an open set $V \subset X$ and a basic open set $\mathscr{W} = \{A \in 2^X \mid A \subset X - C\}$, $C$ compact, so that

$$(x, X - U) \in V \times \mathscr{W} \subset \tilde{\Omega}.$$

It follows that $V \subset C$ and thus we have

$$x \in V \subset \bar{V} \subset C \subset U,$$

and the proof is complete.

We will also show that for any space $X$, the compact-open topology on $2^X$ is proper. Before doing this, however, we will discuss a perhaps more familiar space which is homeomorphic to $2^X$ with the compact-open topology.

We let $\mathscr{S}$ denote the space having two points, 0 and 1, for which the open sets are $\varnothing$, $\mathscr{S}$, and $\{0\}$. (This is known as the Sierpinski space, see [2, p. 63].) Consider the *set* of all continuous functions from $X$ into $\mathscr{S}$, denoted $\mathscr{S}^X$. For each $f \in \mathscr{S}^X$, there corresponds a closed set $\Gamma(f) \in 2^X$ given by $\Gamma(f) = f^{-1}(1)$. Similarly, for each closed set $A \in 2^X$, we can define $\Phi(A) \in \mathscr{S}^X$ by

$$\begin{aligned} \Phi(A)(x) &= 1 \quad \text{if } x \in A, \\ &= 0 \quad \text{if } x \notin A. \end{aligned}$$

Clearly, $\Gamma(\Phi(A)) = A$ and $\Phi(\Gamma(f)) = f$. Thus we have a bijection of the set $\mathscr{S}^X$ onto the set $2^X$. Now consider $2^X$ as a topological space with the compact-open topology defined earlier, and consider $\mathscr{S}^X$ with the compact-open topology as defined for spaces of continuous functions (see, e.g., [2, p. 257]).

LEMMA 3.2. *If $\mathscr{S}^X$ and $2^X$ have the respective compact-open topologies, the bijection $\Gamma: \mathscr{S}^X \to 2^X$ is a homeomorphism.*

**Proof.** We need only verify that $\Gamma(\{f \mid f(C) \subset \{0\}\}) = \{A \mid A \subset X - C\}$ for any compact $C \subset X$. But

$$\Gamma(\{f \mid f(C) \subset \{0\}\}) = \{A \mid \Phi(A)(C) \subset \{0\}\} = \{A \mid A \cap C = \varnothing\},$$

which concludes the proof.

Thus the compact-open topology on $2^X$ is in fact the compact-open topology on a space of continuous functions. We will use this connection to show that the compact-open topology on $2^X$ is proper.

Suppose $Y$ is any space. For every map $\alpha: X \times Y \to \mathscr{S}$, where $\alpha$ is continuous on $X$ for each fixed $y \in Y$, there is an associated map $\alpha: Y \to \mathscr{S}^X$ given by $[\hat{\alpha}(y)](x) = \alpha(x, y)$. Conversely, given $\hat{\alpha}: Y \to \mathscr{S}^X$, we can define $\alpha: X \times Y \to \mathscr{S}$. Recall that a topology on the function space $\mathscr{S}^X$ is said to be proper if for any space $Y$ and function $\alpha: X \times Y \to \mathscr{S}$, the continuity of $\alpha$ (jointly in $x$ and $y$) implies the continuity of $\hat{\alpha}$. Similarly, a topology on $\mathscr{S}^X$ is said to be admissible if for any space $Y$ and function $\hat{\alpha}: Y \to \mathscr{S}^X$, the continuity of $\hat{\alpha}$ implies that of $\alpha$. Equivalently, $\mathscr{S}^X$ has an admissible topology if and only if the evaluation map $\omega: \mathscr{S}^X \times X \to \mathscr{S}$, given by $\omega(f, x) = f(x)$, is continuous. See [2, p. 274].

Since the sets $\mathscr{S}^X$ and $2^X$ are in a one-to-one correspondence, a topology on one can be viewed as a topology on the other, making $\Gamma$ a homeomorphism. We now show that the various definitions of admissible and proper are compatible.

LEMMA 3.3. *A topology on the set of subsets $2^X$ is admissible [proper] if and only if as a topology on the function space $\mathscr{S}^X$ it is admissible [proper].*

**Proof.** The proof follows from the observation that $\alpha: X \times Y \to \mathscr{S}$ is continuous if and only if

$$\begin{aligned}\alpha^{-1}(1) &= \{(x, y) \in X \times Y \mid [\hat{\alpha}(y)](x) = 1\} \\ &= \{(x, y) \mid x \in \Gamma(\hat{\alpha}(y))\} \\ &= G_{\Gamma \circ \hat{\alpha}}\end{aligned}$$

is closed.

We remark here that the continuity of the evaluation map $\omega$ is equivalent to the closure of the set $\Omega$ defined in §2.

COROLLARY 3.4. *The compact-open topology on $2^X$ is always proper.*

**Proof.** This follows by Lemmas 3.2 and 3.3 and the fact that the compact-open topology on $\mathscr{S}^X$ is always proper (see [2, Theorem 3.1, p. 261]).

COROLLARY 3.5. *Let $X$ be a Hausdorff space. Then the compact-open topology on $\mathscr{S}^X$ is admissible if and only if $X$ is locally compact.*

The next two results provide converses to Theorem 2.6 and its corollaries.

THEOREM 3.6. *If $X$ is a locally compact Hausdorff space, then the compact-open topology on $2^X$ is both proper and admissible, and hence is the weakest admissible topology on $2^X$.*

COROLLARY 3.7. *If $X$ is a locally compact Hausdorff space, then the intersection of all admissible topologies for $2^X$ is admissible.*

In the next section, we will exploit further the function space structure of $2^X$.

4. **Compact-open topology: Further properties.** Each space of subsets and function space considered in this section will be assumed to have the respective compact-open topology.

We now change the original problem somewhat. Given spaces $X$ *and* $Y$ we ask under what conditions can we assert that, for any function $F$: $Y \to 2^X$, the continuity of $F$ (in the compact-open topology) is equivalent to its graph $G_F$ being closed.

To give an answer to that question, we must recall the definition of a $k$-space. A Hausdorff space $Z$ is said to be a $k$-space if $U \subset X$ is open in $X$ if and only if for every compact subset $C$ of $Z$, $U \cap C$ is open in $C$. It is known (see [2, p. 248]) that all locally compact and all first countable Hausdorff spaces (and hence all metric spaces) are $k$-spaces. Also if both $X$ and $Y$ are first countable Hausdorff spaces, or one is a locally compact Hausdorff space and the other is a $k$-space then $X \times Y$ is a $k$-space (see [2, p. 263]). This is important because of the following result.

THEOREM 4.1. *If $X$ is locally compact Hausdorff or if $X \times Y$ is a $k$-space, then the continuity of $F$: $Y \to 2^X$ is equivalent to its graph $G_F$ being closed.*

**Proof.** For the case where $X$ is locally compact Hausdorff, this is Theorem 3.6. Since the compact-open topology is always proper, we always have that closed graph implies continuity. Thus we need only prove that if $X \times Y$ is a $k$-space, then continuity of $F$ implies $G_F$ is closed. This follows from [2, Corollary 3.2, p. 261] and the observation made in the proof of Lemma 3.3.

Theorem 4.1 tells us that, with the right conditions on $X$ or $X \times Y$, any continuous function $F$: $Y \to 2^X$ gives rise to a closed subset of $X \times Y$. Conversely, any closed subset $A \subset X \times Y$ gives rise to a continuous function $F$: $Y \to 2^X$, defined by $F(y) = \{x \mid (x, y) \in A\}$, having the property that $G_F = A$. Thus we have a bijection $G$ between the set of continuous functions from $Y$ into $2^X$, denoted $(2^X)^Y$, and the set of closed subsets of $X \times Y$, $2^{X \times Y}$, defined by $G(F) = G_F$.

THEOREM 4.2. *Let $X$ be a locally compact Hausdorff space, or let $X \times Y$ be a $k$-space. Then the function $G$: $(2^X)^Y \to 2^{X \times Y}$, which assigns to each continuous $F$ its graph $G_F$, is defined and is a homeomorphism.*

**Proof.** This follows from Lemma 3.2 and [2, Theorem 5.3, p. 265].

COROLLARY 4.3. *If $X\times Y$ is a k-space, then $(2^X)^Y$ is homeomorphic to $(2^Y)^X$. In fact the homeomorphism is given by mapping the function $F\colon Y\to 2^X$ to the function $F^*\colon X\to 2^Y$ given by $F^*(x)=\{y \mid x\in F(y)\}$.*

COROLLARY 4.4. *Let $F\colon Y\to 2^X$ and let $F^*(x)=\{y \mid x\in F(y)\}$ for all $x\in X$. If $X$ is locally compact Hausdorff, then $F$ is continuous implies $F^*\colon X\to 2^Y$ is defined and is continuous. If $X\times Y$ is a k-space, then $F$ is continuous if and only if $F^*$ is defined and is continuous. (Note that by $F^*$ is defined we mean $F^*(x)$ is closed for all $x\in X$.)*

Thus, under the proper conditions, the study of continuous closed-set valued functions from $Y$ to $X$ is equivalent to the study of closed subsets of $X\times Y$. Conversely, consideration of closed subsets of $X\times Y$ leads to the study of continuous closed-set valued functions from one factor into the other.

The remaining results of this section describe some further continuity properties of the compact-open topology on $2^X$. The following is obvious.

PROPOSITION 4.5. *Let $X$ be a Hausdorff space. Then*

(a) *the map $i\colon X\to 2^X$ given by $i(x)=\{x\}$ is continuous, and*

(b) *if $f\colon Y\to X$ is continuous, the map $F\colon Y\to 2^X$ given by $F(y)=\{f(y)\}$ is continuous.*

THEOREM 4.6. *Let $X$ and $Y$ be any spaces. Then the map $H\colon Y^X\times 2^Y\to 2^X$ given by $H(f, B)=f^{-1}(B)$ is continuous in $f$ for fixed $B$ and in $B$ for fixed $f$. If $Y$ is a locally compact Hausdorff space, then $H$ is continuous jointly in $f$ and $B$.*

**Proof.** The theorem is true for the map $T\colon Y^X\times \mathcal{S}^Y\to \mathcal{S}^X$ given by $T(f, g)=g\circ f$ [2, p. 259]. Noting that if $g\in \mathcal{S}^Y$ is such that $B=g^{-1}(1)$, then $(g\circ f)^{-1}(1)=f^{-1}(g^{-1}(1))=f^{-1}(B)$, the proof follows from Lemma 3.2.

We will denote by $H_B\colon Y^X\to 2^X$ the map $H_B(f)=H(f, B)$ and by $H_f\colon 2^Y\to 2^X$ the map $H_f(B)=H(f, B)$.

COROLLARY 4.7. *Let $X$ be a Hausdorff space. Let $f\colon Y\to X$ be continuous. Then the map $f^{-1}\colon X\to 2^Y$ is continuous.*

**Proof.** The map $H_f\circ i\colon X\to 2^X\to 2^Y$ is continuous, and $H_f\circ i(x)=H_f(\{x\})=f^{-1}(x)$.

We remark here that if $f\colon Y\to X$ and if $F\colon Y\to 2^X$ is given by $F(y)=\{f(y)\}$, then $F^*\colon X\to 2^Y$ is given by $F^*(x)=\{y \mid x\in\{f(y)\}\}=f^{-1}(x)$, provided $F^*$ is defined, i.e., provided $f^{-1}(x)$ is closed for all $x\in X$. Thus by Corollary 4.4 we get that if $X\times Y$ is a $k$-space then $f^{-1}$ is defined and is continuous if and only if $F$ is continuous. Notice that $F^*$ acts in some sense as an inverse of $F$.

THEOREM 4.8. *Let $X$ be a compact Hausdorff space and $Y$ a k-space. Then $f\colon Y\to X$ is continuous if and only if $f^{-1}\colon X\to 2^Y$ is defined and is continuous.*

**Proof.** By Corollary 4.7 and the preceding discussion, all we need prove is that if $F(y)=\{f(y)\}$ is continuous then $f$ is continuous.

Let $U \subset X$ be open. Since $X$ is compact Hausdorff, $\mathscr{V} = \{A \in 2^X \mid A \subset U\}$ is open in $2^X$, hence the continuity of $F$ implies $F^{-1}(\mathscr{V})$ is open in $Y$. But

$$F^{-1}(\mathscr{V}) = \{y \in Y \mid f(y) \in U\} = f^{-1}(U).$$

PROPOSITION 4.9. *The map*

$$\Sigma\colon 2^X \times 2^X \to 2^X,$$

*given by* $\Sigma(A, B) = A \cup B$, *is continuous.*

**Proof.** Notice that

$$\{(A, B) \in 2^X \times 2^X \mid A \cup B \subset U\} = \{A \in 2^X \mid A \subset U\} \times \{B \in 2^X \mid B \subset U\}.$$

COROLLARY 4.10. *If* $F_i\colon Y \to 2^X$ *is continuous for* $i=1, 2$, *then* $F_1 \cup F_2\colon Y \to 2^X$, *given by* $(F_1 \cup F_2)(y) = F_1(y) \cup F_2(y)$, *is continuous.*

**Proof.** Let $F_1 \times F_2\colon Y \to 2^X \times 2^X$ be the product map, then $F_1 \cap F_2 = \Sigma \circ (F_1 \times F_2)$.

LEMMA 4.11. *Let* $F_\alpha\colon Y \to 2^X$ *and let* $\bigcap F_\alpha\colon Y \to 2^X$ *be given by* $(\bigcap F_\alpha)(y) = \bigcap F_\alpha(y)$. *Then* $G_{\cap F_\alpha} = \bigcap G_{F_\alpha}$.

**Proof.**

$$\begin{aligned} G_{\cap F_\alpha} &= \{(x, y) \mid x \in \textstyle\bigcap F_\alpha(y)\} \\ &= \{(x, y) \mid x \in F_\alpha(y) \text{ for all } \alpha\} \\ &= \textstyle\bigcap \{(x, y) \mid x \in F_\alpha(y)\} \\ &= \textstyle\bigcap G_{F_\alpha}. \end{aligned}$$

PROPOSITION 4.12. *Let* $X$ *be a locally compact Hausdorff space or let* $X \times Y$ *be a k-space. Then if* $F_\alpha\colon Y \to 2^X$ *is continuous for all* $\alpha$, *then* $\bigcap F_\alpha$ *is continuous.*

**Proof.** By Theorem 4.1, $G_{F_\alpha}$ is closed for all $\alpha$. From Lemma 4.11, we conclude that $G_{\cap F_\alpha}$ is closed, hence that $\bigcap F_\alpha$ is continuous.

The last few results suggest some applications of the continuity properties of the compact-open topology.

PROPOSITION 4.13. *Let* $X$ *be a locally compact topological group. Then the map* $C\colon 2^X \times X \to 2^X$, *given by* $C(A, x) = xA$, *is continuous.*

**Proof.** $X$ is necessarily Hausdorff (see [8, p. 27]). We need only show $G_C$ is closed. But

$$\begin{aligned} G_C &= \{(A, x, y) \mid y \in xA\} \\ &= \{(A, x, y) \mid x^{-1}y \in A\} \\ &= (1, m)^{-1}(\Omega), \end{aligned}$$

where $(1, m)\colon 2^X \times X \times X \to 2^X \times X$ is given by $(1, m)(A, x, y) = (A, x^{-1}y)$ and is continuous. Thus $G_C$ is closed.

COROLLARY 4.14. *Multiplication by an element of a locally compact topological group $X$ is a homeomorphism of $2^X$ onto itself.*

In fact, Corollary 4.14 also follows from a more general result. Recall the definition $H_f(B)=f^{-1}(B)$.

PROPOSITION 4.15. *Let $f: X \to Y$ and $g: Y \to Z$ be continuous, then $H_{g\circ f}=H_f \circ H_g$. In particular, if $f$ is a homeomorphism of $X$ onto $Y$ then $H_f$ is a homeomorphism of $2^Y$ onto $2^X$.*

Now consider the space $2^{X\times X}$ as the space of all closed binary relations $R$ on $X$. We define the maps

$$M: 2^{X\times X} \to 2^X$$

by $M(R)=\{x \in X \mid \{x\}\times X \subset R\}$, and

$$E: 2^{X\times X} \times X \to 2^X$$

by $E(R, x)=\{y \in X \mid (x, y) \in R, (y, x) \in R\}$. It is easy to show that $M(R)$ and $E(R, x)$ are always closed sets, thus $M$ and $E$ are defined. The map $M$ takes $R$ to its set of "maximum" elements, while the map $E$ takes $R$ and $x$ to the "equivalence class" of $x$ with respect to $R$.

THEOREM 4.16. *If $X$ is locally compact, then both $M$ and $E$ are continuous.*

**Proof.** Let $\Omega_2=\{(R, x, y) \mid (x, y) \in R\}$, by hypothesis a closed set in $2^{X\times X}\times X\times X$. Then

$$\begin{aligned} G_M &= \{(R, x) \mid x \in M(R)\} \\ &= \{(R, x) \mid \{x\}\times X \subset R\} \\ &= \{(R, x) \mid (R, x, y) \in \Omega_2, \text{ for all } y \in X\} \\ &= (2^X \times X)-p(\tilde{\Omega}_2), \end{aligned}$$

where $p$ is the projection of $2^{X\times X}\times X\times X$ onto the first two coordinates. Hence $G_M$ is closed.

Also,

$$\begin{aligned} G_E &= \{(R, x, y) \mid y \in E(R, x)\} \\ &= \{(R, x, y) \mid (x, y) \in R, (y, x) \in R\} \\ &= \Omega_2 \cap \Omega_2', \end{aligned}$$

where $\Omega_2'=\{(R, x, y) \mid (y, x) \in R\}$ is also closed. Thus $G_E$ is closed.

5. **Discussion.** If we had viewed the space $2^X$ as a function space earlier, then Lemma 2.2, Theorem 2.3 and Proposition 2.4 could be seen as following from known results about function spaces [2, p. 275]. However, it seemed simpler to directly indicate their proofs, which follow the proofs in [2].

The proof of Theorem 2.6 follows Arens' proof of the fact that if $I^X$ has a weakest (he calls it strongest) admissible topology where $I$ is the unit interval and $X$ is *completely regular*, then $X$ is locally compact [1, Theorem 3]. Note that what we have essentially proved is that if the function space $\mathscr{S}^X$ has a weakest admissible topology where $X$ is *regular*, then $X$ is locally compact.

Notice that the compact-open topology coincides with the upper-semifinite topology (defined at the end of §2 or see [7, p. 179]) when the space $X$ is compact Hausdorff.

We note here that if $X$ is a locally compact Hausdorff space with a countable base, then by [2, Theorem 5.2, p. 65], $2^X$ will have a countable base. In addition, for any space $X$, $2^X$ with the compact-open topology is trivially compact since $2^X$ itself must belong to any open cover.

The compact-open topology for $2^X$ appears as half of the generating set for Fell's $H$-topology [4]. It was later isolated by Effros, who called it the global topology [3, p. 931]. Effros describes convergence in this topology when $X$ is a locally compact Hausdorff space. It is interesting to note that this description is equivalent to Arens' description of convergence in $Y^X$ for locally compact Hausdorff $X$ [1, Theorem 4] when one consider $2^X$ as the space $\mathscr{S}^X$. See also [5].

Finally, an equivalent version of the compact-open topology (for spaces of open subsets) was considered by Kannai [6] in connection with some problems in Mathematical Economics. Theorem 2.6 and Theorem 3.1 would seem to indicate that Kannai's methods will not work in spaces which are not locally compact.

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