A VARIATIONAL METHOD FOR FUNCTIONS OF BOUNDED BOUNDARY ROTATION

BY

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Abstract. Let f be a function analytic in the unit disc, properly normalized, with bounded boundary rotation. There exists a Stieltjes integral representation for $1 + zf''(z)/f'(z)$. From this representation, and in view of a known variational formula for functions of positive real part, a variational formula is derived for functions of the form $q(z) = 1 + zf''(z)/f'(z)$. This formula is for functions of arbitrary boundary rotation and does not assume the functions to be univalent.

A new proof for the radius of convexity for functions of bounded boundary rotation is given. The extremal function for $\mathrm{Re}\,\{F(f'(z))\}$ is derived. Examples of univalent functions with arbitrary boundary rotation are given and estimates for the radius in which $\mathrm{Re}\,\{f'(z)\} > 0$ are computed.

The coefficient problem is solved for a_4 for all values of the boundary rotation and without the assumption of univalency.

1. **Introduction.** Variational techniques are frequently useful in solving extremal problems for a class of analytic functions. Notable recent examples are Robertson [8] and Sakaguchi [12] for the class of functions with positive real part, Hummel [2] for the class of starlike functions, and Schiffer and Tammi [13] for the class of schlicht functions of bounded boundary rotation.

In this paper we shall first establish variational formulas useful in solving extremal problems for the class V_k of functions f, properly normalized, with bounded boundary rotation. These formulas are for functions of arbitrary boundary rotation and do not assume the functions to be univalent. Secondly, we shall apply the formulas to several extremal problems for the class V_k. Finally, we shall establish an upper bound for the coefficient of z^4.

2. **Preliminary remarks.** We say $f \in V_k$ if $f(z)$ is regular in the unit disc E, $|z| < 1$, $f'(z) \neq 0$ in E, f is normalized so that $f(0) = 0$, $f'(0) = 1$, and if for some real number $k \geq 2$,

$$(2.1) \qquad \int_0^{2\pi} \left| \mathrm{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| d\theta \leq k\pi, \qquad z = re^{i\theta}, 0 \leq r < 1.$$

It is convenient to let

(2.2) $$q(z) = 1 + zf''(z)/f'(z)$$

and denote the class of functions $\{q \mid q(z) = 1 + zf''(z)/f'(z), f \in V_k\}$ by Q_k.

Paatero [5] showed that every function $q \in Q_k$ can be given by the Stieltjes integral representation

(2.3) $$q(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + e^{i\phi}z}{1 - e^{i\phi}z} \, d\psi(\phi)$$

where

$$\int_0^{2\pi} d\psi(\phi) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\psi(\phi)| \leqq k\pi,$$

$\psi(\phi)$ being a function of bounded variation on $[0, 2\pi]$.

A systematic approach to extremum problems within the class V_k is due to Lehto [4]. His method was to vary the integrator function $\psi(\phi)$ in (3.3) over all step functions subject to the constraints of (2.3). His work formed the basis for much of the work done by Pinchuk [7], Schiffer and Tammi [13], [14], and Tammi [15].

The approach in this paper is different from all of those cited above. In developing our variational formulas we shall not vary the integrator function but instead we shall vary the integrand in (2.3). To implement this idea we observe that the integrand $(1 + ze^{i\phi})/(1 - ze^{i\phi})$ is a member of the class P, of all regular functions $p(z)$ of positive real part in E normalized so that $p(0) = 1$. We then apply the variational formula due to Robertson [8] for the class P.

3. A variational formula for Q_k.

THEOREM 1. *If $q \in Q_k$ then $q^*(z) = q(z) + \delta q(z)$ is a variational formula for the class of functions $q \in Q_k$ where*

(3.1) $$\frac{\delta q(z)}{\rho} = \frac{\varepsilon z}{a - z}\left[q'(z) - \frac{q(a) - q(z)}{a - z}\right] + \frac{\bar{\varepsilon}z}{1 - \bar{a}z}\left[zq'(z) + \frac{q(z) + (q(a))^-}{1 - \bar{a}z}\right] + o(1)$$

where ε is an arbitrary complex number such that $|\varepsilon| = 1$, a is an arbitrary complex number such that $|a| < 1$ and ρ is an arbitrary small positive real number. (Here and throughout the paper ( )⁻ indicates the complex conjugate.)

Proof. We begin with Robertson's formula [8] which may be written $p^*(z) = p(z) + \delta p(z)$ where

(3.2) $$\frac{\delta p(z)}{\rho} = \frac{\varepsilon z}{a - z}\left[p'(z) - \frac{p(a) - p(z)}{a - z}\right] + \frac{\bar{\varepsilon}z}{1 - \bar{a}z}\left[zp'(z) + \frac{p(z) + (p(a))^-}{1 - \bar{a}z}\right] + o(1).$$

We note that the above form is not quite the form which appears in Robertson's original paper but by redefining ε and ρ in his original formula one can see the above form is an equivalent formula. Next we let

$$q^*(z) = \frac{1}{2\pi} \int_0^{2\pi} p^*(z, \phi) \, d\psi(\phi)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} p(z, \phi) + \rho \delta p(z, \phi) \, d\psi(\phi)$$

(3.3)

$$= q(z) + \frac{\rho}{2\pi} \int_0^{2\pi} \frac{\varepsilon z}{a-z} \left[p'(z, \phi) - \frac{p(a, \phi) - p(z, \phi)}{a-z} \right] d\psi(\phi)$$

$$+ \frac{\rho}{2\pi} \int_0^{2\pi} \frac{\bar{\varepsilon} z}{1 - \bar{a}z} \left[zp'(z, \phi) + \frac{p(z, \phi) + (p(a, \phi))^-}{1 - \bar{a}z} \right] d\psi(\phi) + o(\rho).$$

Integrating we have

(3.4)

$$q^*(z) = q(z) + \rho \left\{ \frac{\varepsilon z}{a-z} \left[q'(z) - \frac{q(a) - q(z)}{a-z} \right] \right.$$

$$\left. + \frac{\bar{\varepsilon} z}{1 - \bar{a}z} \left[zq'(z) + \frac{q(z) + (q(a))^-}{1 - \bar{a}z} \right] \right\} + o(\rho)$$

which is the desired formula. It remains to show that $q^* \in Q_k$ given that $q \in Q_k$. To see this we consider

(3.5)

$$\int_0^{2\pi} |\operatorname{Re} q^*(z)| \, d\theta = \int_{\theta=0}^{2\pi} \left| \operatorname{Re} \frac{1}{2\pi} \int_{\phi=0}^{2\pi} p^*(z, \phi) \, d\psi(\phi) \right| d\theta$$

$$\leqq \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} |\operatorname{Re} p^*(z, \phi)| \, |d\psi(\phi)| \, d\theta.$$

But, $p^*(z, \phi)$ has a positive real part so the right-hand side of (3.5) becomes

(3.6)

$$\frac{1}{2\pi} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \operatorname{Re} \{p^*(z, \phi)\} |d\psi(\phi)| \, d\theta$$

$$= \int_{\phi=0}^{2\pi} \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \operatorname{Re} \{p^*(z, \phi)\} \, d\theta |d\psi(\phi)|.$$

But $\operatorname{Re} p^*(z, \phi)$ is harmonic in E thus

(3.7)

$$\frac{1}{2\pi} \int_{\theta=0}^{2\pi} \operatorname{Re} p^*(z, \phi) \, d\theta = p^*(0, \phi) = 1$$

and we have

(3.8)

$$\int_0^{2\pi} |\operatorname{Re} q^*(z)| \, d\theta \leqq \int_{\phi=0}^{2\pi} |d\psi(\phi)| \leqq k\pi.$$

Thus the same variational formula obtained by Robertson for functions p with positive real part in E and normalized so that $p(0)=1$ has now been extended to the larger class of functions q regular in E, normalized so that $q(0)=1$ and having the property that, for some real number $k \geqq 2$, $\int_0^{2\pi} |\operatorname{Re} q(re^{i\theta})| \, d\theta \leqq k\pi$ for $0 \leqq r < 1$. Note that when $k=2$, we have

(3.9)

$$2\pi \geqq \int_0^{2\pi} |\operatorname{Re} q(re^{i\theta})| \, d\theta \geqq \int_0^{2\pi} \operatorname{Re} q(re^{i\theta}) \, d\theta = 2\pi.$$

Thus $|\operatorname{Re} q(z)| = \operatorname{Re} q(z)$ and q has a positive real part in E in the case $k=2$. In this case V_k reduces to the class of functions f regular and convex in E.

Due to the formal resemblance of the variational formula for Q_k to the formula for P we may state the following corollaries.

COROLLARY 1. *If $F(u)$ is analytic in the u-plane and if $q \in Q_k$ then on $|z|=r<1$*

$$(3.10) \qquad \min_{q \in Q_k} \min_{|z|=r} \operatorname{Re}\{F(q(z))\} = \min_{|z|=r} \operatorname{Re}\{F(q_0(z))\}$$

where

$$q_0(z) = \sum_{i=1}^{2} \lambda_i \left(\frac{1+\varepsilon_i z}{1-\varepsilon_i z}\right) \qquad with \sum_{i=1}^{2} \lambda_i = 1, \sum_{i=1}^{2} |\lambda_i| \leq \frac{k}{2}, |\varepsilon_i| = 1.$$

COROLLARY 2. *If $F(u, v)$ is analytic in the u-plane and in the v-plane and if $q \in Q_k$ then on $|z|=r<1$*

$$(3.11) \qquad \min_{q \in Q_k} \min_{|z|=r} \operatorname{Re}\{F(q(z), zq'(z))\} = \min_{|z|=r} \operatorname{Re}\{F(q_1(z), zq_1'(z))\}$$

where

$$q_1(z) = \sum_{i=1}^{4} \lambda_i \left(\frac{1+\varepsilon_i z}{1-\varepsilon_i z}\right) \qquad with \sum_{i=1}^{4} \lambda_i = 1, \sum_{i=1}^{4} |\lambda_i| \leq \frac{k}{2}, |\varepsilon_i| = 1.$$

The proofs of these corollaries and other similar results follow directly from the work done by Robertson [9] in proving the corresponding results for the class P. One notices, however, in the special case of the class P, i.e., $k=2$, it is possible to reduce the number of summands n in the extremal functions by $n/2$. This is not possible in the general case of the class V_k, $k>2$, as we shall see in the next result.

We next consider a result which was first shown by Paatero [6] and subsequently by another method by Robertson [10]. Our method, however, is different than either of theirs and illustrates well the contents of this section.

THEOREM 2. *The radius of convexity for the class V_k is*

$$\frac{k-(k^2-4)^{1/2}}{2} = \frac{2}{k+(k^2-4)^{1/2}}.$$

The bound is sharp.

Proof. The radius of convexity for V_k is the largest number r, $0<r<1$, such that for all $f \in V_k$, $\operatorname{Re}\{1+zf''(z)/f'(z)\} \geq 0$ for all z, $|z| \leq r$. Equivalently $\operatorname{Re}\{q(z)\} \geq 0, q(z) = 1+zf''(z)/f'(z)$. By Corollary 1 $\min_{q \in Q_k} \min_{|z|=r} \operatorname{Re}\{q(z)\}$ occurs for a function

$$q(z) = \sum_{i=1}^{2} \lambda_i \left(\frac{1+\varepsilon_i z}{1-\varepsilon_i z}\right), \qquad \lambda_1+\lambda_2 = 1, \lambda_1-\lambda_2 \leq \frac{k}{2}, |\varepsilon_i| = 1.$$

But since $(1-r)/(1+r) \leq \operatorname{Re}\{(1+\varepsilon_i z)/(1-\varepsilon_i z)\} \leq (1+r)/(1-r)$ for any ε_i, it follows that the minimum for the real part of a function of this type can be found by letting

the value $(1-r)/(1+r)$ be multiplied by λ_1 and $(1+r)/(1-r)$ by λ_2. Clearly $\lambda_1 = (k+2)/4$, $\lambda_2 = (k-2)/4$ whence the extremal function q is given by

$$(3.12) \qquad q(z) = \frac{k+2}{4}\left(\frac{1-z}{1+z}\right) - \frac{k-2}{4}\left(\frac{1+z}{1-z}\right) = \frac{1-kz+z^2}{1-z^2}.$$

But the real part of this function is positive if and only if $(1-kr+r^2) > 0$ whence $r \leq 2/(k+(k^2-4)^{1/2})$. Thus we have not only established the result but demonstrated a situation where the number of summands in Corollary 1 cannot be reduced to one.

4. A variational formula for functions in V_k. We turn now to the corresponding variational formulas for $f \in V_k$. Beginning with the formula in Theorem 1 we let $1 + zf''(z)/f'(z) = q(z)$ and $1 + zf^{*\prime\prime}(z)/f^{*\prime}(z) = q^*(z)$ and apply the variational formula for $q \in Q_k$. Thus we have

$$1 + \frac{zf^{*\prime\prime}(z)}{f^{*\prime}(z)} = q(z) + \delta q(z)$$

$$(4.1) \qquad = q(z) + \rho\left\{\frac{\varepsilon z}{a-z}\left[q'(z) - \frac{q(a)-q(z)}{a-z}\right]\right.$$

$$\left. + \frac{\bar{\varepsilon} z}{1-\bar{a}z}\left[zq'(z) + \frac{q(z)+(q(a))^-}{1-\bar{a}z}\right]\right\} + o(\rho)$$

which can be written as

$$(4.2) \qquad \frac{f^{*\prime\prime}(z)}{f^{*\prime}(z)} = \frac{q(z)-1}{z} + \rho\left\{\frac{\varepsilon}{a-z}\left[q'(z) - \frac{q(a)-q(z)}{a-z}\right]\right.$$

$$\left. + \frac{\bar{\varepsilon}}{1-\bar{a}z}\left[zq'(z) + \frac{q(z)+(q(a))^-}{1-\bar{a}z}\right]\right\} + o(\rho),$$

$$(4.3) \qquad \frac{f^{*\prime\prime}(z)}{f^{*\prime}(z)} = \frac{f''(z)}{f'(z)} + \rho\left\{\frac{\varepsilon}{a-z}\left[q'(z) - \frac{q(a)-q(z)}{a-z}\right]\right.$$

$$\left. + \frac{\bar{\varepsilon}}{1-\bar{a}z}\left[zq'(z) + \frac{q(z)+(q(a))^-}{1-\bar{a}z}\right]\right\} + o(\rho).$$

Integrating the above (using integration by parts on the right-hand side whenever $q'(z)$ is involved) we have

$$\log f^{*\prime}(z) = \log f'(z)$$

$$(4.4) \qquad + \rho\left\{\varepsilon\left[\frac{q(z)}{a-z}\Big|_0^z - \int_0^z \frac{q(z)\,dz}{(a-z)^2} + \int_0^z \frac{q(z)\,dz}{(a-z)^2} - \int_0^z \frac{q(a)\,dz}{(a-z)^2}\right]\right.$$

$$\left. + \bar{\varepsilon}\left[\frac{zq(z)}{1-\bar{a}z}\Big|_0^z - \int_0^z \frac{q(z)\,dz}{(1-\bar{a}z)^2} + \int_0^z \frac{q(z)\,dz}{(1-\bar{a}z)^2} + \int_0^z \frac{(q(a))^-\,dz}{(1-\bar{a}z)^2}\right]\right\} + o(\rho).$$

We notice that the integrals in (4.4) may not be well defined for $|z| \geq |a|$. To circumvent this possible difficulty we temporarily assume $a \neq 0$, $|z| < |a|$. Formula (4.4) is then valid for the disc $|z| < |a|$. We have

$$\log f^{*\prime}(z) = \log f'(z)$$

(4.5)
$$+ \rho \left\{ \varepsilon \left[\frac{q(z)}{a-z} - \frac{1}{a} - \frac{q(a)}{a-z} + \frac{q(a)}{a} \right] \right.$$

$$\left. + \bar{\varepsilon} \left[\frac{zq(z)}{1-\bar{a}z} + \frac{(q(a))^-}{\bar{a}(1-\bar{a}z)} - \frac{(q(a))^-}{\bar{a}} \right] \right\} + o(\rho).$$

Letting $q(z) = 1 + (zf''(z)/f'(z))$ $q(a) = 1 + af''(a)/f'(a)$ we have

$$\log f^{*\prime}(z) = \log f'(z)$$

(4.6)
$$+ \rho \left\{ \frac{\varepsilon z}{a-z} \left[\frac{f''(z)f'(a) - f'(z)f''(a)}{f'(z)f'(a)} \right] \right.$$

$$\left. + \frac{\bar{\varepsilon}z}{1-\bar{a}z} \left[\frac{2f'(z)(f'(a))^- + zf''(z)(f'(a))^- + \bar{a}f'(z)(f''(a))^-}{(f'(a))^- f'(z)} \right] \right\} + o(\rho).$$

Exponentiating both sides of the last equation and putting all terms involving a power of ρ higher than 1 into $o(\rho)$ we arrive at

(4.7)
$$f^{*\prime}(z) = f'(z) + \rho \left\{ \frac{\varepsilon z}{(a-z)f'(a)} [f''(z)f'(a) - f'(z)f''(a)] + \frac{\bar{\varepsilon}z}{(1-\bar{a}z)(f'(a))^-} \right.$$

$$\left. \cdot [2f'(z)(f'(a))^- + zf''(z)(f'(a))^- + \bar{a}f'(z)(f''(a))^-] \right\} + o(\rho).$$

Now the right-hand side of (4.7) is analytic and single-valued for $|z| < |a|$, $a \neq 0$. However, $f^{*\prime}(z)$, in view of (4.1), is analytic in $|z| < 1$, so by the principle of analytic continuation the right-hand side of (4.7) is also analytic in $|z| < 1$. Now, by a compactness argument we may remove the restriction $a \neq 0$. Thus (4.7) is valid for $|z| < 1$, $|a| < 1$. We formalize the above calculations as

THEOREM 3. *If $f(z) \in V_k$ then $f^{*\prime}(z) = f'(z) + \delta f'(z)$ is a variational formula for the class of derivative functions $V_k' = \{f'(z) \mid f(z) \in V_k\}$ where $\delta f'(z)$ is $f^{*\prime}(z) - f'(z)$ as given above in (4.7).*

Integrating both sides of (4.7) gives us a formula for functions f in V_k. This formula, however, has thus far proved to be unwieldy due to the integrals involved. We can, nevertheless, solve many extremal problems of interest with the use of formula (4.7).

5. **Applications of the variational formula.** For our first application of this section we shall need a result for the class of derivative functions f', $f \in V_k$, analogous to Corollary 1. That is, we are going to compute the extremal function for $F(f'(z))$ where $F(u)$ is a function of u such that $F(f'(z))$ is analytic for E.

Let $f_0(z)$ be the function for which

(5.1) .
$$\min_{f \in V_k} \min_{|z|=r} \operatorname{Re} \{F(f'(z))\} = \min_{|z|=r} \operatorname{Re} \{F(f_0'(z))\}.$$

Since V_k is a compact family, $f_0(z) \in V_k$ exists. By a suitable rotation of the disc E we may assume

$$(5.2) \qquad \min_{|z|=r} \text{Re}\,\{F(f_0'(z))\} = \text{Re}\,\{F(f_0'(r))\}.$$

Consider now the Taylor series for $F(a+b)$:

$$(5.3) \qquad F(a+h) = F(a) + F'(a)h + F''(a)(h^2/2) + \cdots$$

and evaluate (5.3) for $a = f'(r)$, $h = \rho\delta f'(r) + o(\rho)$. We have

$$(5.4) \qquad F(f^{*\prime}(r)) - F(f'(r)) = F'(f'(r))[\rho\delta f'(r)] + o(\rho).$$

By our assumption $\text{Re}\,\{F(f'(r))\}$ is minimal for the class V_k when $f(z) = f_0(z)$. Thus we have for $f(z) = f_0(z)$

$$(5.5) \qquad \text{Re}\,\{F'(f'(r))[\rho\delta f'(r)]\} + o(\rho) \geqq 0$$

for r fixed, $\rho > 0$, but sufficiently small. It follows that

$$(5.6) \qquad \text{Re}\,\{F'(f'(r))[\delta f'(r)]\} \geqq 0.$$

We let $F'(f'(r)) = \alpha$, $f'(r) = \beta$, $f''(r) = \gamma$ and the inequality (5.6), after we have substituted for $\delta f'(r)$ using formula (4.7), becomes

$$(5.7) \qquad \text{Re}\,\left\{\frac{\varepsilon}{a-r}\left[\gamma - \beta\frac{f''(a)}{f'(a)}\right]\alpha + \frac{\bar{\varepsilon}}{1-\bar{a}r}\left[2\beta + r\gamma + \bar{a}\beta\frac{(f''(a))^-}{(f'(a))^-}\right]\alpha\right\} \geqq 0.$$

We note that if $\alpha = 0$ no meaningful conclusions can be drawn from (5.7). However, a theorem of Kirwan [3] shows that $\alpha = 0$ is not possible. Therefore, in what follows, we may assume $\alpha \neq 0$ without loss of generality. We use the facts that $\text{Re}\,\{w\} = \text{Re}\,\{\bar{w}\}$ and that if $\text{Re}\,\{\varepsilon w\} \geqq 0$ for all ε, $|\varepsilon| = 1$, then $w = 0$. Upon substituting for a in (5.7) we have the differential equation

$$(5.8) \qquad -\frac{f''(z)}{f'(z)} = \frac{z[r((\alpha\gamma)^- - \alpha\gamma) + 2(\alpha\beta)^-] + [\alpha\gamma - 2(\alpha\beta)^-r - (\alpha\gamma)^-r^2]}{z^2((\alpha\beta)^-) + z(r(\alpha\beta - (\alpha\beta)^-)) - \alpha\beta}.$$

By hypothesis $\text{Re}\,\{F(f'(r))\}$ is minimal for $\text{Re}\,\{F(f'(z))\}$ on $|z| = r$ and so it follows that

$$(5.9) \qquad \frac{\partial}{\partial\theta}\,\text{Re}\,\{F(f'(re^{i\theta}))\}_{\theta=0} = 0.$$

We therefore have

$$(5.10) \qquad \text{Re}\,\{F'(f'(re^{i\theta}))f''(re^{i\theta})re^{i\theta}i\}|_{\theta=0} = \text{Re}\,\{F'(f'(r))f''(r)\cdot ri\}.$$

Thus $\text{Im}\,\{\alpha\gamma\} = 0$ and $\alpha\gamma = (\alpha\gamma)^-$. We now have

$$(5.11) \qquad q(z) = \frac{-\alpha\beta z^2 + (r[\alpha\beta + (\alpha\beta)^-] + (r^2-1)\alpha\gamma)z - \alpha\beta}{(\alpha\beta)^- z^2 + r(\alpha\beta - (\alpha\beta)^-)z - \alpha\beta}.$$

Equation (5.11) is now of the form

$$(5.12) \qquad q(z) = \frac{G(z)}{g(z)} = \frac{G_2 z^2 + G_1 z + G_0}{g_2 z^2 + g_1 z + g_0}$$

where $G_n = (G_{2-n})^-$ and $g_n = -(g_{2-n})^-$. By a standard argument (Robertson [9]) it can be shown that

$$(5.13) \qquad q(z) = \lambda_1 \frac{1+z}{1-z} + \lambda_2 \frac{1-z}{1+z}, \qquad \lambda_1 + \lambda_2 = 1, \ |\lambda_1| + |\lambda_2| \leqq k/2.$$

We have the following result.

THEOREM 4. *Let $F(u)$ be an analytic function for which $F\{f'(z)\}$ is analytic for $|z| < 1$ whenever $f \in V_k$. Then $\min_{f \in V_k} \min_{|z|=r} \mathrm{Re}\,\{F(f'(z))\}$ occurs for*

$$(5.14) \qquad f(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{k/2} - 1\right] = z + \cdots.$$

COROLLARY 3. *Let $f \in V_k$, $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, then $\mathrm{Re}\,\{(f'(z))^{1/k}\} > 0$, for $|z| < 1$.*

Proof. By Theorem 4, $\min_{f \in V_k} \min_{|z|=r} \mathrm{Re}\,\{(f'(z))^{1/k}\}$ occurs for the extremal function (5.14). Differentiating in (5.14) we have

$$(5.15) \qquad f'(z) = \left(\frac{1+z}{1-z}\right)^{k/2-1} \frac{1}{(1-z)^2} = \frac{(1+z)^{k/2-1}}{(1-z)^{k/2+1}}.$$

It follows that

$$(5.16) \qquad \begin{aligned} |\arg f'(z)| &\leqq (k/2-1)|\arg(1+z)| + (k/2+1)|\arg(1-z)| \\ &\leqq k\sin^{-1} r \leqq k\pi/2. \end{aligned}$$

Hence $\mathrm{Re}\,\{(f'(z))^{1/k}\} \geqq 0$ for $|z| = r < 1$.

Notice that this gives us a method for forming univalent functions of arbitrarily high boundary rotation. We denote by S_k the subclass of V_k in which the function f of V_k are univalent.

COROLLARY 4. *Let $f \in V_k$, and let F be defined by*

$$(5.17) \qquad F(z) = \int_0^z [f'(z^k)]^{1/k}\, dz = z + a_{k+1} z^{k+1} + \cdots$$

then $F \in S_k$ whenever k is a positive integer.

Proof. By Corollary 3, $|f'(z)|^{1/k}$ has a positive real part, $|z| < 1$. It follows that $[f'(z^k)]^{1/k}$ also has positive real part, $|z| < 1$. Therefore, $F(z)$ is univalent and is close-to-convex for $|z| < 1$. But

$$\int_0^{2\pi}\left|\,\mathrm{Re}\left\{1 + \frac{zF''(z)}{F'(z)}\right\}\right| d\theta = \int_0^{2\pi}\left|\,\mathrm{Re}\left\{1 + \frac{z^k f''(z^k)}{f'(z^k)}\right\}\right| d\theta$$

$$= \frac{1}{k}\int_0^{2k\pi}\left|\,\mathrm{Re}\left\{1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right\}\right|_{\arg\zeta = \phi} d\phi$$

$$= \int_0^{2\pi}\left|\,\mathrm{Re}\left\{1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right\}\right| d\phi \leqq k\pi.$$

Thus $F \in S_k$.

Corollary 3 also gives us a method of estimating the largest disc in which the derivative functions have positive real part. This provides a lower bound for the radius of univalency for $f \in V_k$. We formalize these results as follows.

COROLLARY 5. *If $f \in V_k$, then* $\operatorname{Re}\{f'(z)\} > 0$ *for* $|z| < r_k$ *where r_k is the smallest positive root of the transcendental equation*

$$(5.18) \qquad \frac{k-2}{2} \sin^{-1}\left(\frac{2r}{1+r^2}\right) + 2\sin^{-1} r = \frac{\pi}{2}.$$

Proof. By Theorem 4, $\min_{f \in V_k} \min_{|z|=r} \operatorname{Re}\{f'(z)\}$ occurs for

$$f'(z) = \left(\frac{1+z}{1-z}\right)^{k/2-1}(1-z)^{-2}.$$

Evidently $\operatorname{Re}\{f'(z)\} \geq 0$ if

$$(5.19) \qquad \left|\frac{k-2}{2}\arg\left(\frac{1+z}{1-z}\right) - 2\arg(1-z)\right| \leq \frac{\pi}{2}.$$

Notice that $\max_{|z|=r}|\arg((1+z)/(1-z))| = \sin^{-1}(2r/(1+r^2))$, $\max_{|z|=r}|\arg(1-z)|$ $= \sin^{-1} r$. It will be sufficient to require that

$$(5.20) \qquad \frac{k-2}{2}\sin^{-1}\frac{2r}{1+r^2} + 2\sin^{-1} r \leq \frac{\pi}{2}.$$

For clarity and for the sake of comparison with other results we shall give r_k for $k=2, 4$ and 6. For $k=2$, $r_k = 2^{-1/2} = .707\ldots$ which is known to be sharp. For $k=4$ we are led from (5.20) to the equation

$$(5.21) \qquad 4r^8 + 4r^6 + 8r^5 - 3r^4 + 4r^3 + 2r^2 - 4r + 1 = 0.$$

It can be shown that

$$r = \frac{1}{3} + \frac{2}{3}\left(\frac{7}{2}\right)^{1/2}\left(\sinh\frac{1}{3}\sinh^{-1}\left(\frac{2}{7}\right)^{3/2}\right) = .3966\ldots.$$

The method of Pinchuk [7] yields $\sin^{-1}(\pi/8) = .38268\ldots$. Furthermore, Robertson [11] has shown that for the function $f(z) = (z/(1-z)^2)$ $\operatorname{Re}\{f'(z)\}$ vanishes at a point on $|z| = .4035\ldots$. Thus, the sharp bound is between $.3966\ldots$ and $.4035\ldots$.

For $k=6$, Pinchuk's method yields $\sin^{-1}(\pi/12) = .2588\ldots$ whereas our method gives $r_6 = .2647\ldots$.

6. **The coefficient problem.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in V_k$. It has been conjectured that $|a_n| \leq b_n$ where b_n is the coefficient of z^n in the expansion of

$$(6.1) \qquad f(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{k/2} - 1\right] = z + \frac{k}{2}z^2 + \frac{k^2+2}{3!}z^3 + \frac{k^3+8k}{4!}z^4 + \cdots.$$

In the case $k \leq 4$, $f(z)$ is univalent (Paatero [5]) and the coefficients a_2, a_3 and a_4 have been shown to satisfy the conjectured bounds (Schiffer and Tammi [13]). The

methods of Schiffer and Tammi also solve the problem for a_2 and a_3 for general k for the subclasses S_k of univalent functions in V_k. They solve [14] the problem for a_4 only when $k \leq 4$.

With the theorems of this paper combined with Schiffer's and Tammi's methods we shall show that for any $k \geq 2$ that $|a_n| \leq b_n$ for all $f \in V_k$, $n = 2, 3, 4$.

We now characterize the extremal functions for V_k for the coefficient problem.

THEOREM 5. *Let* $f \in V_k$, $f(z) = z + a_2 z^2 + \cdots$. *Then* $|a_n|$ *is maximized when*

$$(6.2) \qquad f(z) = \sum_{i=1}^{2n-2} \lambda_i \frac{1 + \varepsilon_i z}{1 - \varepsilon_i z}$$

where $\sum \lambda_i = 1$, $\sum |\lambda_i| = k/2$, $|\varepsilon_i| = 1$.

Proof. We suppose a_n is real and maximal over the class V_k. Applying Theorem 3 we have the following identity between power series

$$(6.3) \qquad \sum_{n=1}^{\infty} n a_n^* z^{n-1} - \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=2}^{\infty} c_n z^{n-1} + o(\rho)$$

where $f^*(z) = \sum_{n=1}^{\infty} a_n^* z^n$, $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $\delta f'(z) = \rho \sum_{n=2}^{\infty} c_n z^{n-1} + o(\rho)$. Since a_n is assumed to be maximal, we have $\text{Re} \{a_n^* - a_n\} \leq 0$. Computing the coefficient ρc_n of z^{n-1} in $\delta f'(z)$ we find

$$(6.4) \qquad \begin{aligned} \rho c_n = \rho \Bigg\{ &\frac{\varepsilon}{f'(a)} \left[f'(a) \sum_{j=2}^{n} j(j-1) a_j \left(\frac{1}{a}\right)^{n-j} - f''(a) \sum_{j=1}^{n} j a_j \left(\frac{1}{a}\right)^{n-j-1} \right] \\ &+ \frac{\bar{\varepsilon}}{(f'(a))^-} \left[(f'(a))^- \sum_{j=1}^{n} j(j+1) a_j \bar{a}^{n-j-1} \right. \\ &\qquad\qquad\qquad\qquad \left. + (f''(a))^- \sum_{j=1}^{n} j a_j \bar{a}^{n-j-1} \right] \Bigg\} + o(\rho). \end{aligned}$$

Since $\rho c_n = a_n^* - a_n$ it follows that $\text{Re} \{\rho c_n\} \leq 0$. We use the fact that $\text{Re} \{w\} = \text{Re} \{\bar{w}\}$ and the fact that if $\rho \, \text{Re} \{w\} + o(\rho) \leq 0$ then $\text{Re} \{w\} \leq 0$ to conclude

$$(6.5) \qquad \begin{aligned} \text{Re} \Bigg\{ &\varepsilon \left[\sum_{j=2}^{n} j(j-1) a_j \left(\frac{1}{a}\right)^{n-j} + \sum_{j=1}^{n} j(j+1) \bar{a}_j a^{n-j-1} \right] \\ &+ \varepsilon \frac{f''(a)}{f'(a)} \left[\sum_{j=1}^{n} j \bar{a}_j a^{n-j-1} - \sum_{j=1}^{n} j a_j \left(\frac{1}{a}\right)^{n-j-1} \right] \Bigg\} \leq 0. \end{aligned}$$

But if $\text{Re} \{\varepsilon w\} \leq 0$ for all ε, $|\varepsilon| = 1$ then $w = 0$. Now substituting z for a in the above expression we have the following equation for the extremal function $f(z)$:

$$(6.6) \qquad 1 + \frac{z f''(z)}{f'(z)} = \frac{G_n(z)}{g_n(z)} = \frac{\displaystyle\sum_{j=1}^{n-1} j^2 a_j / z^{n-j} + n(n-1) a_n + \sum_{j=1}^{n-1} j^2 \bar{a}_j z^{n-j}}{\displaystyle\sum_{j=1}^{n-1} j a_j / z^{n-j} - \sum_{j=1}^{n-1} j \bar{a}_j z^{n-j}}.$$

Notice that if the numerator and denominator are multiplied by z^{n-1} we have

$$(6.7) \qquad 1 + z f''(z)/f'(z) = H_n(z)/h_n(z)$$

where both $H_n(z)$ and $h_n(z)$ are polynomials of degree $N \leq 2n-2$. Furthermore, the coefficients of these polynomials have a certain symmetry. For $H_n(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_{2n-2} z^{2n-2}$ we have that $\alpha_i = (\alpha_{2n-2-i})^-$, $i = 0, 1, \ldots, n-2$, and α_{n-1} is $n(n-1)a_n$ which is real. For $h_n(z) = \beta_0 + \beta_1 z + \cdots + \beta_{n-2} z^{2n-2}$ we have that $\beta_i = (-\beta_{2n-2-i})^-$, $i = 0, 1, \ldots, n-2$, and that $\beta_{n-1} = 0$. For either of these functions it follows that if ζ is a zero then $1/\bar{\zeta}$ is a zero. Since the left-hand side of the equation is regular in $E = \{z \mid |z| < 1\}$ it follows that any zero of h_n in E must also be a zero of H_n. By the reflection principle observed above any zero of h_n outside $|z| \leq 1$ must again be a zero of H_n. Therefore H_n/h_n is a function with all its poles on the unit circle $|z| = 1$. It is easily seen that $G_n(e^{i\theta})$ is real and $g_n(e^{i\theta})$ is purely imaginary. Thus $q(e^{i\theta}) = 1 + e^{i\theta} f''(e^{i\theta})/f'(e^{i\theta})$ is purely imaginary. By a partial fraction decomposition and because it is known that $q(e^{i\theta})$ cannot have any poles of order higher than one we may write

$$(6.8) \qquad q(z) = \frac{H(z)}{h(z)} = \sum_{i=1}^{N} \lambda_i \left(\frac{1 + \varepsilon_i z}{1 - \varepsilon_i z} \right)$$

where N is the number of poles on $|z| = 1$ and the ε_i are the conjugates of the poles. Because the degree of h_n is at most $2n-2$ it is clear that $N \leq 2n-2$. We now determine the character of the λ_i, $i = 1, 2, \ldots, N$. We observe by letting z approach $\bar{\varepsilon}_i$, for a fixed value of i, along the circle $|z| = 1$ that $(1 + \varepsilon_i z)/(1 - \varepsilon_i z)$ becomes unbounded. But since $q(e^{i\theta})$ is purely imaginary, it follows that, for $|z| = 1$, $\text{Re}\{\lambda_i((1 + \varepsilon_i z)/(1 - \varepsilon_i z))\} = 0$, thus $\text{Im}\,\lambda_i = 0$ and all the λ_i are real. From $q(0) = 1$ we have $\sum_{i=1}^{N} \lambda_i = 1$. Therefore, the right-hand side of (6.8) is only a special case of the general Stieltjes integral representation (2.3) and it follows that $\sum_{i=1}^{N} |\lambda_i| \leq k/2$.

Thus the same extremal functions obtained by Schiffer and Tammi for S_k are now proven to be extremal for the larger classes V_k. We can therefore state:

THEOREM 6. *Let* $f \in V_k$, $f(z) = z + a_2 z^2 + \cdots$. *Then* $|a_2| \leq k/2$, $|a_3| \leq (k^2+2)/3!$.

Proof. The proof of Schiffer and Tammi [13] applies in view of the observation that the function f given by (6.4) is now known to be extremal for V_k and by use of the method of proof used in Theorem 7.

THEOREM 7. *Let* $f \in V_k$, $f(z) = z + a_2 z^2 + \cdots$. *Then* $|a_4| \leq (k^3 + 8k)/4!$.

Proof. In this case Schiffer and Tammi [14] have solved the problem only for $k \leq 4$. We may, however, modify their proof to obtain the result for $k > 4$. We start with a modification of their observation that the number of summands N in (6.2), $N \leq 2n-2 = 6$, cannot be 5 or 6 since the maximum value of $|a_n|$ for $N = 5$ or 6 is less than $(k+6)/12$ for $k > 2$ but the maximum value of $|a_n|$ for V_k is known to be larger than $(k+6)/12$. Furthermore we have [14, Formula 15]

$$(6.9) \qquad \begin{aligned} & \sum_{i=1}^{2n-2} \frac{1 + 2\lambda_i}{(\bar{\varepsilon}_i^r)} = 0 \qquad (r = 1, 2, \ldots, n-2), \\ & \sum_{i=1}^{2n-2} \frac{1 + 2\lambda_i}{(\bar{\varepsilon}_i^{n-1})} = n(n-1)a_n. \end{aligned}$$

Letting $\delta_i = 1 + 2\lambda_i$, $i = 1, 2, \ldots, 6$, and utilizing the fact that $N \leq 4$ we may write from (6.9)

$$(6.10) \qquad \sum_{i=1}^{4} \frac{\delta_i}{\bar{\varepsilon}_i} = -\left(r + \frac{1}{r}\right)e^{-i\phi},$$

$$(6.11) \qquad \sum_{i=1}^{4} \frac{\delta_i}{\bar{\varepsilon}_i^2} = -\left(r^2 + \frac{1}{r^2}\right)e^{-2i\phi},$$

$$(6.12) \qquad \sum_{i=1}^{4} \frac{\delta_i}{\bar{\varepsilon}_i^3} + \left(r^3 + \frac{1}{r^3}\right)e^{-3i\phi} = 12a_4.$$

(To see the validity of (6.10) notice that, if $N \leq 4$, $\lambda_5 = \lambda_6 = 0$ and $\varepsilon_5 z$, $\varepsilon_6 z$ may be replaced in (6.2) by $re^{i\phi}$, $(1/r)e^{i\phi}z$, $r > 0$.)

In view of (6.2) we can now make the estimate

$$(6.13) \qquad r^2 + \frac{1}{r^2} \leq \sum_{i=1}^{4} |\delta_i| \leq 4 + k$$

whence

$$(6.14) \qquad (r + 1/r)^2 = r^2 + 1/r^2 + 2 \leq 6 + k$$

which leads to the estimate

$$(6.15) \qquad \left(r^3 + \frac{1}{r^3}\right) = \left(r + \frac{1}{r}\right)\left(r^2 + \frac{1}{r^2} - 1\right) \leq (6+k)^{1/2}(3+k).$$

Formulas (6.12) and (6.15) now imply

$$(6.16) \qquad 12|a_4| \leq \sum_{i=1}^{4} |\delta_i| + (6+k)^{1/2}(3+k) \leq 4 + k + (3+k)(6+k)^{1/2}.$$

Formula (6.16) gives us a bound on a_4 in the case $N = 4$. On the other hand the extremal function for $N = 2$ is easily seen to be (6.1) and we have $|a_4| = (k^3 + 8k)/24$ in this case. We now notice that

$$(6.17) \qquad (4 + k + (3+k)(6+k)^{1/2})/12 \leq (k^3 + 8k)/24$$

for all $k \geq k_0$ where k_0 is clearly less than 4. Since the problem for $k \leq 4$ has been solved the desired result is now shown for all $k \geq 2$. This result has also recently been obtained by Brannan [1].

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