TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 157, June 1971

ON THE COMPLEX BORDISM OF EILENBERG-MAC LANE SPACES AND CONNECTIVE COVERINGS OF BU

BY
PETER S. LANDWEBER[1]

Abstract. Explicit computations show that the universal coefficient spectral sequence from complex bordism to integral homology collapses for the spectra $\boldsymbol{K}(\boldsymbol{Z})$ and $\boldsymbol{bu}$, and also for their mod p reductions. Moreover the complex bordism modules of these spectra have infinite projective dimension.

1. **Introduction.** The aim of this paper is to continue the study of the complex bordism modules $\Omega_*^U(K(\boldsymbol{Z}, n))$ and $\Omega_*^U(BU(2n, \ldots, \infty))$ which was begun in the earlier note [5]. Since our interest is in the stable ranges, it is convenient to introduce the Eilenberg-Mac Lane spectrum $\boldsymbol{K}(\boldsymbol{Z}) = \{K(\boldsymbol{Z}, n)\}$ and the connective BU-spectrum (see [1] or [3, §10])

$$\boldsymbol{bu} = \{\ldots, BU(2n, \ldots, \infty), U(2n+1, \ldots, \infty), BU(2n+2, \ldots, \infty), \ldots\}.$$

Thus the objects of study are the bordism modules $\Omega_*^U(\boldsymbol{K}(\boldsymbol{Z}))$ and $\Omega_*^U(\boldsymbol{bu})$; in general if $\boldsymbol{M} = \{M_n\}$ is a spectrum, we define

$$\Omega_k^U(\boldsymbol{M}) = \lim_{n\to\infty} \tilde{\Omega}_{k+n}^U (M_n).$$

In [5] we determined the images of the Thom homomorphisms

$$\Omega_*^U(\boldsymbol{K}(\boldsymbol{Z})) \xrightarrow{\mu} H_*(\boldsymbol{K}(\boldsymbol{Z})), \qquad \Omega_*^U(\boldsymbol{bu}) \xrightarrow{\mu} H_*(\boldsymbol{bu}).$$

We are now able to obtain a more complete understanding of the relation between the complex bordism and homology of $\boldsymbol{K}(\boldsymbol{Z})$ and $\boldsymbol{bu}$. In [3, §4] P. Conner and L. Smith introduce a natural spectral sequence for finite (CW) complexes

$$E^r\langle X\rangle \Rightarrow H_*(X) \tag{1.1}$$

with

$$E^2_{p,q}\langle X\rangle = \operatorname{Tor}^{\Omega_*^U}_{p,q}(\Omega_*^U(X), \boldsymbol{Z}), \tag{1.2}$$

Received by the editors July 16, 1970.

AMS 1969 *subject classifications*. Primary 5530.

Key words and phrases. Complex bordism, Eilenberg-Mac Lane spectra, connective BU-spectrum, Koszul resolution, Hurewicz homomorphisms for MU.

[1] Research supported in part by NSF grant GP-7993.

where $\boldsymbol{Z}$ is made a module over Ω_*^U by the augmentation. By taking limits there are also spectral sequences (1.1) for the spectra $\boldsymbol{K}(\boldsymbol{Z})$ and $\boldsymbol{bu}$, as well as for their mod p reductions $\boldsymbol{K}(\boldsymbol{Z})\wedge \boldsymbol{Z}_p=\boldsymbol{K}(\boldsymbol{Z}_p)$ and $\boldsymbol{bu}\wedge \boldsymbol{Z}_p$. We shall compute the E^2-terms and prove the following:

THEOREM. *For each of the spectra* $\boldsymbol{K}(\boldsymbol{Z})$, $\boldsymbol{K}(\boldsymbol{Z}_p)$, $\boldsymbol{bu}$ *and* $\boldsymbol{bu}\wedge \boldsymbol{Z}_p$ (*p a prime*) *the spectral sequence* (1.1) *of* [3, §4] *collapses.*

Thus in each case $H_*(X)$ has a filtration by graded subgroups $0\subset F_0\subset F_1\subset\cdots$, $\bigcup F_n=H_*(X)$, such that F_p/F_{p-1} is isomorphic to $\mathrm{Tor}_{p,*}^{\Omega_*^U}(\Omega_*^U(X), \boldsymbol{Z})$. The edge homomorphism of (1.1) is the reduced Thom homomorphism

$$\tilde{\mu}\colon \Omega_*^U(X)\otimes_{\Omega_*^U}\boldsymbol{Z}\to H_*(X),$$

hence $\tilde{\mu}$ is an isomorphism onto $F_0\subset H_*(X)$ in each case of the theorem.

The steps involved in the proof of the theorem are outlined in §2. The analysis for $\boldsymbol{K}(\boldsymbol{Z})$ and $\boldsymbol{K}(\boldsymbol{Z}_p)$ is made in §3, and in §4 we carry out the study of $\boldsymbol{bu}$ and $\boldsymbol{bu}\wedge \boldsymbol{Z}_p$.

From the computation of the E^2-terms (see (3.3) and (4.6)) we conclude the following:

COROLLARY. *The complex bordism of each spectrum* $\boldsymbol{K}(\boldsymbol{Z})$, $\boldsymbol{K}(\boldsymbol{Z}_p)$, $\boldsymbol{bu}$ *and* $\boldsymbol{bu}\wedge \boldsymbol{Z}_p$ (*p a prime*) *is an* Ω_*^U*-module of infinite projective dimension.*

In fact L. Smith has pointed out that from [3, §5] one can obtain the more precise result that $\Omega_*^U(K(\boldsymbol{Z}_p, n))$ has projective dimension $\geqq n$.

ACKNOWLEDGEMENT. It is a pleasure to thank Larry Smith for sending me preliminary versions of the paper [3], and for an illuminating correspondence that has been very helpful in the development of this paper.

2. **Sketch of the argument.** The first step is to note that, by switching the roles of the spectra involved, one obtains as in [5] the following isomorphisms:

$$\begin{aligned}
\Omega_*^U(\boldsymbol{K}(\boldsymbol{Z})) &\cong H_*(\boldsymbol{MU}),\\
\Omega_*^U(\boldsymbol{K}(\boldsymbol{Z}_p)) &\cong H_*(\boldsymbol{MU};\boldsymbol{Z}_p) = H_*(\boldsymbol{MU})\otimes \boldsymbol{Z}_p,\\
\Omega_*^U(\boldsymbol{bu}) &\cong k_*(\boldsymbol{MU})
\end{aligned}$$

and

$$\Omega_*^U(\boldsymbol{bu}\wedge \boldsymbol{Z}_p)\cong k_*(\boldsymbol{MU};\boldsymbol{Z}_p) = k_*(\boldsymbol{MU})\otimes \boldsymbol{Z}_p.$$

These are isomorphisms of Ω_*^U-modules if the graded rings on the right are made Ω_*^U-modules via the following diagram of Hurewicz homomorphisms and reductions mod p.

$$\begin{array}{ccccc}
 & & k_*(\boldsymbol{MU}) & \xrightarrow{\rho_p} & k_*(\boldsymbol{MU};\boldsymbol{Z}_p)\\
 & \overset{\mathscr{H}}{\nearrow} & \downarrow & & \downarrow\\
\Omega_*^U=\pi_*(\boldsymbol{MU}) & & & & \\
 & \underset{\mathscr{H}}{\searrow} & & & \\
 & & H_*(\boldsymbol{MU}) & \xrightarrow{\rho_p} & H_*(\boldsymbol{MU};\boldsymbol{Z}_p)
\end{array}$$

Next we describe convenient choices of polynomial generators for Ω_*^U, $H_*(\boldsymbol{MU})$ and $k_*(\boldsymbol{MU})$ following the detailed study of the complex bordism ring made by R. Stong in [8, Chapter 7] (see (3.1) and (4.3)). However it is rather difficult to deal directly with the Ω_*^U-modules $H_*(\boldsymbol{MU})$ and $k_*(\boldsymbol{MU})$ for the computations we have in mind. Thus we first compute the bigraded groups (see (3.3) and (4.6))

$$\operatorname{Tor}_{*,*}^{\Omega_*^U}(H_*(\boldsymbol{MU}; \boldsymbol{Z}_p), \boldsymbol{Z}), \qquad \operatorname{Tor}_{*,*}^{\Omega_*^U}(k_*(\boldsymbol{MU}; \boldsymbol{Z}_p), \boldsymbol{Z})$$

and by a comparison of their totalizations with $H_*(\boldsymbol{K}(\boldsymbol{Z}_p))$ and $H_*(\boldsymbol{bu} \wedge \boldsymbol{Z}_p)$ we conclude that the spectral sequences (1.1) for $\boldsymbol{K}(\boldsymbol{Z}_p)$ and $\boldsymbol{bu} \wedge \boldsymbol{Z}_p$ must collapse.

Recall from [5, p. 526] that for each prime p the homology groups $H_*(\boldsymbol{K}(\boldsymbol{Z}))$ and $H_*(\boldsymbol{bu})$ have no elements of order p^2, and that this property implies that a homology class vanishes if all its reduction mod p vanish. In order to show that the spectral sequences (1.1) for $\boldsymbol{K}(\boldsymbol{Z})$ and $\boldsymbol{bu}$ collapse, it suffices to show that also

$$\operatorname{Tor}_{*,*}^{\Omega_*^U}(H_*(\boldsymbol{MU}), \boldsymbol{Z}), \qquad \operatorname{Tor}_{*,*}^{\Omega_*^U}(k_*(\boldsymbol{MU}), \boldsymbol{Z})$$

have no elements of order p^2. This we do by applying the elementary result:

LEMMA 2.1. *Let* $\cdots \to C_n \xrightarrow{d} C_{n-1} \to \cdots$ *be a complex of free abelian groups, and let* ∂_p *denote the Bockstein homomorphism*

$$H(C; \boldsymbol{Z}_p) \xrightarrow{\partial} H(C) \xrightarrow{\rho} H(C; \boldsymbol{Z}_p)$$

where ∂ *is the boundary associated to the coefficient sequence* $0 \to \boldsymbol{Z} \xrightarrow{p} \boldsymbol{Z} \to \boldsymbol{Z}_p \to 0$ *and* ρ *is reduction* mod p. *If* ∂_p *has zero homology in dimension* $n+1$ *then* $H_n(C)$ *has no elements of order* p^2.

Proof. Let $x \in H_n(C)$ with $p^2x=0$ but $px \neq 0$; we shall reach a contradiction. Since $p(px)=0$ we have $px=\partial y$ for a class $y \in H_{n+1}(C; \boldsymbol{Z}_p)$. Then $\partial_p y = \rho\partial y = \rho px = 0$, hence $y \in \operatorname{Ker}(\partial_p)$. By assumption $y=\rho\partial(z)$ for some class $z \in H_{n+2}(C; \boldsymbol{Z}_p)$. But then $px=\partial y=\partial\rho\partial(z)=0$ since $\partial\rho=0$, violating the assumption that $px \neq 0$. Q.E.D.

We use this result as follows: Let

$$\cdots \longrightarrow F_n \xrightarrow{d} F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{\varepsilon} \boldsymbol{Z} \longrightarrow 0$$

be a free Ω_*^U-resolution of $\boldsymbol{Z}$, and let $\mathscr{M}_*$ denote either of the Ω_*^U-modules $H_*(\boldsymbol{MU})$, $k_*(\boldsymbol{MU})$ (both are free abelian). Then $\operatorname{Tor}_*^{\Omega_*^U}(\mathscr{M}_*, \boldsymbol{Z})$ is the homology of the complex of free abelian groups

$$(2.2) \quad \cdots \longrightarrow \mathscr{M}_* \otimes_{\Omega_*^U} F_n \xrightarrow{1 \otimes d} \mathscr{M}_* \otimes_{\Omega_*^U} F_{n-1} \longrightarrow \cdots \longrightarrow \mathscr{M}_* \otimes_{\Omega_*^U} F_0 \longrightarrow 0$$

and hence $\operatorname{Tor}_*^{\Omega_*^U}(\mathscr{M}_* \otimes \boldsymbol{Z}_p, \boldsymbol{Z})$ is the mod p homology of the complex (2.2) (for clarity we indicate only the homological degree). We shall compute the Bockstein homomorphism ∂_p in $\mathrm{Tor}_*^{\Omega_*^U}(\mathcal{M}_* \otimes Z_p, Z)$ and show that its homology is zero in positive dimensions (see (3.6) and (4.8)), hence by the lemma $\mathrm{Tor}_{*,*}^{\Omega_*^U}(\mathcal{M}_*, Z)$ has no elements of order p^2.

Finally we exploit the commutative diagrams of differentials

$$\begin{array}{ccc} E_r\langle X\rangle & \xrightarrow{d^r} & E_r\langle X\rangle \\ \downarrow{\rho_p} & & \downarrow{\rho_p} \\ E_r\langle X \wedge Z_p\rangle & \xrightarrow{d^r} & E_r\langle X \wedge Z_p\rangle \end{array}$$

to conclude, by an induction on $r \geqq 2$, that the spectral sequences (1.1) for $K(Z)$ and ***bu*** must collapse. That is, if we suppose that $d^2 = \cdots = d^{r-1} = 0$ for $X = K(Z)$ or ***bu*** then $E_r\langle X\rangle = E_2\langle X\rangle$ and we find a commutative diagram

$$\begin{array}{ccc} E_r\langle X\rangle & \xrightarrow{d^r} & E_r\langle X\rangle \\ \downarrow{\prod \rho_p} & & \downarrow{\prod \rho_p} \\ \prod_p E_r\langle X \wedge Z_p\rangle & \xrightarrow{\prod d^r} & \prod_p E_r\langle X \wedge Z_p\rangle \end{array}$$

in which $\prod \rho_p$ is a monomorphism and $\prod d^r = 0$, hence also $d^r = 0$ for the spectrum X.

3. $H_*(MU)$ **and** $H_*(MU; Z_p)$ **as** Ω_*^U**-modules.** We begin by recalling the relation of the complex bordism ring Ω_*^U to the ring $H_*(MU) \cong H_*(BU)$ (multiplication is provided by the Whitney sum). The Hurewicz homomorphism

$$\Omega_*^U = \pi_*(MU) \xrightarrow{\mathscr{H}} H_*(MU)$$

is a monomorphism (Ω_*^U has no torsion) which records the Chern numbers of stably complex manifolds. The following result of J. Cohen, taken from [8, p. 130], provides all the information we need.

PROPOSITION 3.1. *There exist polynomial generators* x_i $(i \geqq 1)$ *of* Ω_*^U *and* z_i $(i \geqq 1)$ *of* $H_*(MU)$, $\dim x_i = \dim z_i = 2i$ *such that* $\mathscr{H} x_i = m_i z_i$ *where* $m_i = p$ *if* $i+1 = p^s$ *for some prime p and* $m_i = 1$ *otherwise.*

Recall that we regard Z as a module over $\Omega_*^U = Z[x_1, x_2, \ldots]$ by means of the augmentation. The *Koszul resolution* (see [6, p. 204]) of the Ω_*^U-module Z consists of the bigraded exterior algebra

$$E_{*,*} = E_{\Omega_*^U}[y_1, y_2, \ldots]$$

where y_i has bidegree $(1, 2i)$ and elements of Ω_{2i}^U have bidegree $(0, 2i)$, and homomorphisms of Ω_*^U-modules

$$\cdots \longrightarrow E_{n,*} \xrightarrow{d} E_{n-1,*} \longrightarrow \cdots \longrightarrow E_{0,*} \xrightarrow{\varepsilon} Z \longrightarrow 0$$

such that d is a derivation satisfying $d(y_i)=x_i\cdot 1$ on the generators and ε is the augmentation $E_{0,*}=\Omega_*^U \to \boldsymbol{Z}$. This is a free resolution.

We now fix a prime p and continue to denote by z_i the images of the polynomial generators in $H_*(MU;\boldsymbol{Z}_p)$, hence $H_*(MU;\boldsymbol{Z}_p)=\boldsymbol{Z}_p[z_1,z_2,\ldots]$. Then the bigraded algebra

$$\mathrm{Tor}_{*,*}^{\Omega_*^U}(H_*(MU;\boldsymbol{Z}_p),\boldsymbol{Z}) \tag{3.2}$$

is the homology of

$$H_*(MU;\boldsymbol{Z}_p)\otimes E_{\boldsymbol{Z}}[y_1,y_2,\ldots]$$

under a derivation d' of $H_*(MU;\boldsymbol{Z}_p)$-modules which satisfies $d'(1\otimes y_i) =\rho\mathscr{H}x_i\otimes 1$ if $i+1\neq p^s$ (note that $\rho\mathscr{H}x_i$ is then a generator of $H_*(MU;\boldsymbol{Z}_p)$ in dimension $2i$) and $d'(1\otimes y_i)=0$ if $i+1=p^s$ for some $s>0$. Thus (3.2) is the homology of the tensor product of the complexes

$$\boldsymbol{Z}_p[\rho\mathscr{H}x_i;\, i+1\neq p^s]\otimes E_{\boldsymbol{Z}_p}[y_i;\, i+1\neq p^s],$$
$$\boldsymbol{Z}_p[z_{p^s-1};\, s>0]\otimes E_{\boldsymbol{Z}_p}[y_{p^s-1};\, s>0]$$

under a derivation d' which satisfies $d'(y_i)=\rho\mathscr{H}x_i$ if $i+1\neq p^s$ and which annihilates z_{p^s-1} and y_{p^s-1} for $s>0$. This is simply the tensor product of a Koszul resolution for $\boldsymbol{Z}_p$ and a complex with zero differential, hence the Künneth formula yields

PROPOSITION 3.3. *There is an algebra isomorphism*

$$\mathrm{Tor}_{*,*}^{\Omega_*^U}(H_*(MU;\boldsymbol{Z}_p),\boldsymbol{Z})\cong \boldsymbol{Z}_p[z_{p^s-1};\, s>0]\otimes E_{\boldsymbol{Z}_p}[y_{p^s-1};\, s>0]$$

where y_i has bidegree $(1,2i)$ and z_i has bidegree $(0,2i)$ for $i=p^s-1$.

Thus we have computed the E^2-term of the spectral sequence

$$E^r\langle K(\boldsymbol{Z}_p)\rangle \Rightarrow H_*(K(\boldsymbol{Z}_p)). \tag{3.4}$$

By interchanging the roles of the Eilenberg-Mac Lane spectra $K(\boldsymbol{Z})$ and $K(\boldsymbol{Z}_p)$ as in [5] we find that $H_*(K(\boldsymbol{Z}_p))\cong H_*(K(\boldsymbol{Z});\boldsymbol{Z}_p)$. It is well known that

$$H_*(K(\boldsymbol{Z});\boldsymbol{Z}_p)\to H_*(K(\boldsymbol{Z}_p);\boldsymbol{Z}_p)$$

is a monomorphism whose image we now describe. Recall from [7] that $H_*(K(\boldsymbol{Z}_p);\boldsymbol{Z}_p)$ is a Hopf algebra dual to the mod p Steenrod algebra and that there is an algebra isomorphism

$$H_*(K(\boldsymbol{Z}_p);\boldsymbol{Z}_p) = E_{\boldsymbol{Z}_p}[\eta_i;\, i\geqq 0]\otimes \boldsymbol{Z}_p[\zeta_i;\, i>0] \tag{3.5}$$

where $\deg \eta_i=2p^i-1$ and $\deg \zeta_i=2(p^i-1)$. Then the image of $H_*(K(\boldsymbol{Z});\boldsymbol{Z}_p)$ is the subalgebra generated by the η_i and ζ_i for $i>0$, and so we conclude by a comparison of (3.3) and (3.5) that $H_*(K(\boldsymbol{Z}_p))$ is (algebra) isomorphic to the totalization of $E^2_{*,*}\langle K(\boldsymbol{Z}_p)\rangle$. Since both are $\boldsymbol{Z}_p$-vector spaces of finite type, the spectral sequence (3.4), i.e. the spectral sequence (1.1) for $K(\boldsymbol{Z}_p)$, must collapse.

It remains to compute the Bockstein ∂_p in $\mathrm{Tor}^{\Omega^U_*}_{*,*}(H_*(MU; Z_p), Z)$, as in §2, in order to show that also the spectral sequence (1.1) for $K(Z)$ collapses. We shall prove

PROPOSITION 3.6. *The Bockstein ∂_p is a derivation of the algebra*

$$\mathrm{Tor}^{\Omega^U_*}_{*,*}(H_*(MU; Z_p), Z)$$

which satisfies $\partial_p(y_{p^s-1}) = z_{p^s-1}$ and $\partial_p(z_{p^s-1}) = 0$ on the generators.

It then follows that the homology of ∂_p is isomorphic to Z_p (in bidegree (0, 0)) since we have simply the Koszul resolution for Z_p over the polynomial algebra $Z_p[z_{p^s-1}; s>0]$. We then argue as in §2 that the spectral sequence (1.1) for $K(Z)$ must also collapse.

Proof of (3.6). We omit the standard argument that ∂_p is a derivation and concentrate on identifying $\partial_p(y_i)$, where the y_i are the exterior algebra generators of (3.3). Thus we consider the projection

$$H_*(MU) \otimes E_Z[y_1, y_2, \ldots] \to H_*(MU; Z_p) \otimes E_Z[y_1, y_2, \ldots]$$

of complexes, lift $1 \otimes y_i$ $(i = p^s - 1)$ back to $1 \otimes y_i$ in $H_*(MU) \otimes E_Z[y_1, y_2, \ldots]$, apply the differential to obtain $\mathscr{H}x_i \otimes 1 = pz_i \otimes 1$ (see (3.1)), divide by p to obtain $z_i \otimes 1$ and finally apply reduction mod p which yields z_i as desired. Q.E.D.

4. **$k_*(MU)$ and $k_*(MU; Z_p)$ as Ω^U_*-modules.** First recall that $k_*(\ )$ is the multiplicative homology theory represented by the connective BU-spectrum ***bu***, and that the coefficient ring $k_* = Z[\beta]$, $\dim \beta = 2$ (for example see [3, §10]). For a spectrum $M = \{M_n\}$ we define

$$k_m(M) = \lim_{n\to\infty} \tilde{k}_{m+n}(M_n).$$

Maps which result from the Whitney sum make $k_*(MU)$ an algebra over k_*. Since $H_*(MU)$ is a polynomial algebra over Z, hence has no torsion, it follows easily that

$$k_*(MU) = k_*[t_1, t_2, \ldots], \qquad \dim t_i = 2i. \tag{4.1}$$

It is possible to choose the generators so that the Hurewicz homomorphism

$$\Omega^U_* = \pi_*(MU) \xrightarrow{\mathscr{H}} k_*(MU), \tag{4.2}$$

induced by the map of the sphere spectrum $S = \{S^n\}$ into ***bu*** which consists of maps $S^{2n} \to BU(2n, \ldots, \infty)$ and $S^{2n+1} \to U(2n+1, \ldots, \infty)$ that lift generators of $\pi_{2n}(BU)$ and $\pi_{2n+1}(U)$, takes a convenient form. Namely, the generators can be chosen so that we have

$$\mathscr{H}[M^{2n}] = \sum [M^{2n}]_{i_1,\ldots,i_r} t_{i_1} \cdots t_{i_r} \beta^{n-i_1-\cdots-i_r}$$

where the coefficient is the tangential K-theory characteristic number (see [2] or [8])

$$[M^{2n}]_{i_1,\dots,i_r} = s_{(i_1,\dots,i_r)}(\gamma(\tau))[M^{2n}]$$

of the U-manifold M^{2n} determined by the partition $(i_1,\dots,i_r)$. (We choose tangential rather than normal K-theory numbers in order to agree with [8].)

It follows directly from the celebrated theorem of A. Hattori [4] and R. Stong [8, p. 129] that the Hurewicz homomorphism (4.2) is (additively) a split monomorphism. We shall deduce the following further information from [8, Chapter 7].

PROPOSITION 4.3. *There exist polynomial generators* x_i $(i\geqq 1)$ *of* Ω_*^U *and* z_i $(i\geqq 1)$ *of* $k_*(MU)$ *over* $k_*=Z[\beta]$, $\dim x_i=\dim z_i=2i$, *such that* $\mathscr{H}x_{p-1}=pz_{p-1}+\beta^{p-1}$, $\mathscr{H}x_i=pz_i \bmod \beta^{p-1}$ *if* $i+1=p^s$ $(s>1)$ *and* $\mathscr{H}x_i=z_i$ *if* $i+1$ *is not a prime power.*

We remark that (3.1) is an immediate consequence of this result, in view of the natural transformation $\lambda\colon k_*(\ )\to H_*(\ )$ of [5, §2]. For the generators z_i of $k_*(MU)$ over k_* are carried under λ to generators of $H_*(MU)$, and on the coefficient rings λ is the augmentation $k_*=Z[\beta]\to Z$ (so $\lambda(\beta)=0$).

When $i+1=p^s$ and $s>1$ we have an equality $\mathscr{H}x_i=pz_i+\beta^{p-1}w^{(s)}$ where $w^{(s)}\in k_{2(p^s-p)}(MU)$. Although this knowledge is sufficient for our purposes, it would be interesting to have a better grasp on the elements $w^{(s)}$. We conjecture that it is possible to choose the generators so that

$$\mathscr{H}x_{p^s-1} = pz_{p^s-1}+(\mathscr{H}x_{p-1})(z_{p^{s-1}-1})^p$$

for $s>1$. The best supporting evidence is the lemma on p. 121 of [8].

Proof of (4.3). We assume familiarity with the relevant portion of [8, Chapter 7]. If $i+1$ is not a prime power let $x_i=[M^{2i}]$ be any generator of Ω_*^U in dimension $2i$; then by [8, p. 128] we have $s_{(i)}(c(\tau))[M^{2i}]=\pm 1$ and therefore $\mathscr{H}x_i$ is a generator of $k_*(MU)$ over k_* in dimension $2i$. We put $z_i=\mathscr{H}x_i$.

If $i+1=p$ then put $x_i=[CP(p-1)]$. Since $s_{(p-1)}(c(\tau))[CP(p-1)]=p$ it follows from [8, p. 128] that $[CP(p-1)]$ is a generator of Ω_*^U in dimension $2(p-1)$. From [2, (14.1)] we see that $s_\omega(c(\tau))[CP(p-1)]$ is zero $\bmod\, p$ unless ω is the empty partition, and then as is well known we obtain the Todd genus of $CP(p-1)$ which is 1. The equation $\mathscr{H}x_{p-1}=pz_{p-1}+\beta^{p-1}$ now defines the generator z_{p-1}.

Finally suppose $i+1=p^{s+1}$ and $s>0$. Let $x_i=[M^{2i}]$ be a generator of Ω_*^U in dimension $2i=2(p^{s+1}-1)$ which is congruent to a multiple of $[H_{p^s,\dots,p^s}] \bmod p$ (see [8, p. 121]). Then the key lemma on p. 121 of [8] implies that $\mathscr{H}x_i$ is divisible by $\beta^{p-1} \bmod p$; for the mod p K-theory characteristic numbers of M^{2i} are a multiple of those of $H_{p^s,\dots,p^s}$, and therefore $s_{(i_1,\dots,i_r)}(\gamma(\tau))[M^{2i}]=0 \bmod p$ if $i_1+\cdots+i_r>p^{s+1}-p=i-(p-1)$. From [8, p. 128] we have $s_{(i)}(c(\tau))[M^{2i}]=\pm p$, hence any solution of the congruence $\mathscr{H}x_i=pz_i \bmod \beta^{p-1}$ is a generator of $k_*(MU)$ in dimension $2i$. Q.E.D.

We are now ready to compute the bigraded algebra

$$\mathrm{Tor}^{\Omega_*^U}_{*,*}(k_*(MU;Z_p),Z) \tag{4.4}$$

for a fixed prime p. We continue to denote by z_i the reductions of the generators for $k_*(MU)$, so that $k_*(MU; Z_p)=Z_p[\beta; z_1, z_2, \ldots]$. Then, as in §3, (4.4) is the homology of

$$k_*(MU; Z_p) \otimes_Z E_Z[y_1, y_2, \ldots] \tag{4.5}$$

under a derivation d' of $k_*(MU; Z_p)$-modules which satisfies $d'(1 \otimes y_i)=\rho\mathscr{H}x_i \otimes 1$ if $i+1\neq p^s$ (note that $\rho\mathscr{H}x_i$ is then a generator of $k_*(MU; Z_p)$ in dimension $2i$) and $d'(1 \otimes y_{p^s-1})=\rho\mathscr{H}x_{p^s-1} \otimes 1=\beta^{p-1}w^{(s)} \otimes 1$ for some $w^{(s)} \in k_{2(p^s-p)}(MU; Z_p)$. In particular we have $d'(1 \otimes y_{p-1})=\beta^{p-1} \otimes 1$.

For $s>1$ we shall replace the exterior algebra generator $1 \otimes y_{p^s-1}$ by the cycle $y'_{p^s-1}=1 \otimes y_{p^s-1}-w^{(s)} \otimes y_{p-1}$. One checks easily that these cycles generate an exterior algebra (each y_i has odd total degree). Hence the complex (4.5) is the tensor product of the complexes

$$Z_p[\rho\mathscr{H}x_i; i+1 \neq p^s] \otimes E_{Z_p}[y_i; i+1 \neq p^s], \qquad Z_p[\beta] \otimes E_{Z_p}[y_{p-1}]$$

and

$$Z_p[z_{p^s-1}; s > 0] \otimes E_{Z_p}[y'_{p^s-1}; s > 1]$$

under a derivation d' which satisfies $d'(y_i)=\rho\mathscr{H}x_i$ if $i+1\neq p^s$, $d'(\beta)=0$ and $d'(y_{p-1})=\beta^{p-1}$, and which annihilates z_{p^s-1} and y'_{p^s-1}. In view of the Koszul resolution and the Künneth formula we now find

PROPOSITION 4.6. *There is an isomorphism*

$$\mathrm{Tor}^{\Omega^U_*}_{*,*}(k_*(MU; Z_p), Z) = B_* \otimes P_* \otimes E_*$$

with the tensor product of the truncated polynomial ring $B_*=Z_p[\beta]/(\beta^{p-1})$, *the polynomial ring* $P_*=Z_p[z_{p^s-1}; s>0]$ *and the exterior algebra* $E_*=E_{Z_p}[y'_{p^s-1}; s>1]$.

Notice that this result immediately implies that the spectral sequence

$$E^r\langle bu \wedge Z_p\rangle \Rightarrow H_*(bu \wedge Z_p) \tag{4.7}$$

collapses. For J. F. Adams showed in [1] that $H^*(bu; Z_p)$ is isomorphic to a direct sum of cyclic modules $A_p/A_pQ_0+A_pQ_1$ over the mod p Steenrod algebra on generators in $H^{2i}(bu; Z_p)$ for $i=0, 1, \ldots, p-2$ (recall that $Q_0 \in A^1_p$ and $Q_1 \in A^{2p-1}_p$). Hence $H_*(bu; Z_p)$ and the totalization of $E^2\langle bu \wedge Z_p\rangle \cong \mathrm{Tor}^{\Omega^U_*}_{*,*}(k_*(MU; Z_p), Z)$ are graded Z_p-modules of finite type which have the same dimension in each degree, so they are isomorphic. Therefore the spectral sequence (4.7), i.e. the spectral sequence (1.1) for $bu \wedge Z_p$, must collapse.

It only remains to compute the Bockstein ∂_p in $\mathrm{Tor}^{\Omega^U_*}_{*,*}(k_*(MU; Z_p), Z)$, as in §2, in order to show that also the spectral sequence (1.1) for bu collapses. We shall prove, in the notation of (4.6), PROPOSITION 4.8. *The Bockstein* ∂_p *is a derivation of the algebra*

$$\mathrm{Tor}^{\Omega^U_*}_{*,*}(k_*(\boldsymbol{MU}; \boldsymbol{Z}_p), \boldsymbol{Z})$$

which satisfies $\partial_p(y'_{p^s-1}) = z_{p^s-1}$ *for* $s>1$, $\partial_p(\beta)=0$ *and* $\partial_p(z_{p^s-1})=0$ *for* $s>0$.

Proof. By a standard argument ∂_p is a derivation sending $\mathrm{Tor}_{r,s}$ to $\mathrm{Tor}_{r-1,s}$, and so we concentrate on identifying $\partial_p(y'_{p^s-1})$. Thus we consider the projection

$$k_*(\boldsymbol{MU}) \otimes E_Z[y_1, y_2, \dots] \to k_*(\boldsymbol{MU}; \boldsymbol{Z}_p) \otimes E_Z[y_1, y_2, \dots]$$

of complexes, lift the cycle y'_{p^s-1} back to $1 \otimes y_{p^s-1} - w^{(s)} \otimes y_{p-1}$ (where $w^{(s)} \in k_{2(p^s-p)}(\boldsymbol{MU})$ satisfies $\mathscr{H}x_{p^s-1} = pz_{p^s-1} + \beta^{p-1}w^{(s)}$), apply the differential to obtain $(\mathscr{H}x_{p^s-1} - \beta^{p-1}w^{(s)}) \otimes 1 = pz_{p^s-1} \otimes 1$, divide by p to obtain $z_{p^s-1} \otimes 1$, and finally apply reduction mod p which yields z_{p^s-1} as desired. Q.E.D.

We now obtain immediately

COROLLARY 4.9. *The homology of the Bockstein* ∂_p *in* $\mathrm{Tor}^{\Omega^U_*}_{*,*}(k_*(\boldsymbol{MU}; \boldsymbol{Z}_p), \boldsymbol{Z})$ *is algebra isomorphic to* $\boldsymbol{Z}_p[\beta, z_{p-1}]/(\beta^{p-1})$, *where* β *has bidegree* $(0, 2)$ *and* z_{p-1} *has bidegree* $(0, 2p-2)$.

Thus the homology of ∂_p is concentrated in bidegrees $(0, *)$, and then (2.1) implies that $\mathrm{Tor}^{\Omega^U_*}_{*,*}(k_*(\boldsymbol{MU}), \boldsymbol{Z})$ has no elements of order p^2. We now may conclude as in §2 that the spectral sequence (1.1) for ***bu*** must also collapse.

REFERENCES

1. J. F. Adams, *On Chern characters and the structure of the unitary group*, Proc. Cambridge Philos. Soc. **57** (1961), 189–199. MR **22** #12525.

2. P. E. Conner and E. E. Floyd, *The relation of cobordism to K-theories*, Lecture Notes in Math., no. 28, Springer-Verlag, Berlin and New York, 1966. MR **35** #7344.

3. P. E. Conner and L. Smith, *On the complex bordism of finite complexes*, Inst. Hautes Études Sci. Publ. Math. **37** (1970).

4. A. Hattori, *Integral characteristic numbers for weakly almost complex manifolds*, Topology **5** (1966), 259–280. MR **33** #742.

5. P. S. Landweber, *Steenrod representability of stable homology*, Proc. Amer. Math. Soc. **18** (1967), 523–529. MR **37** #4815.

6. S. MacLane, *Homology*, Die Grundlehren der math. Wissenschaften, Band 114, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR **28** #122.

7. J. W. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171. MR **20** #6092.

8. R. E. Stong, *Notes on cobordism theory*, Mathematical Notes, Princeton Univ. Press, Princeton, N. J.; Univ. of Tokyo Press, Tokyo, 1968. MR **40** #2108.

YALE UNIVERSITY,
NEW HAVEN, CONNECTICUT 06520
RUTGERS UNIVERSITY,
NEW BRUNSWICK, NEW JERSEY 08903