TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 157, June 1971

G-STRUCTURES ON SPHERES

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Abstract. G_n denotes one of the classical groups $SO(n)$, $SU(n)$ or $Sp(n)$ and H a closed connected subgroup of G_n. We ask whether the principal bundle $G_n \to G_{n+1} \to G_{n+1}/G_n$ admits a reduction of structure group to H. If n is even and G_n is $SO(n)$ or $SU(n)$ or if $n \not\equiv 11 \bmod 12$ and G_n is $Sp(n)$, we prove that there are no such reductions unless $n=6$, $G_6=SO(6)$ and $H=SU(3)$ or $U(3)$. In the remaining cases we consider the problem for H maximal. We divide the maximal subgroups into three main classes: reducible, nonsimple irreducible and simple irreducible. We find a necessary and sufficient condition for reduction to a reducible maximal subgroup and prove that there are no reductions to the nonsimple irreducible maximal subgroups. The remaining case is unanswered.

1. **Introduction.** In this paper we consider the problem of determining all G-structures on the standard n-sphere, S^n. More precisely, let G_n denote either the special orthogonal group, $SO(n)$, the special unitary group, $SU(n)$, or the symplectic group $Sp(n)$. Given a closed connected subgroup H of G_n we ask whether the principal G_n-bundle $G_n \to G_{n+1} \to G_{n+1}/G_n$ admits a reduction of structure group to H.

The problem has been solved in a number of significant cases. Adams [1] has obtained a complete solution for $G_n=SO(n)$ and H the standard subgroup $SO(n-k)$, $1 \leqq k<n$. The results of Atiyah and Todd [3] and Adams and Walker [2] completely solve the problem for $G_n=SU(n)$ and H the standard subgroup $SU(n-k)$, $1 \leqq k<n$. Finally, Borel and Serre [4] obtained the final solution for $G_{2n}=SO(2n)$ and $H=U(n)$.

For G_n equal to $SO(n)$ or $SU(n)$ and n even and for G_n equal to $S_p(n)$ and $n \not\equiv 11 \bmod 12$, we obtain a complete solution to the general problem. Namely, we prove

THEOREM 1. A. *For n even, the fibration*

$$SO(n) \to SO(n+1) \to SO(n+1)/SO(n) = S^n$$

cannot be reduced to a proper subgroup H of $SO(n)$ unless n is 6 and H is $SU(3)$ or $U(3)$.

Received by the editors April 6, 1970.

AMS 1969 *subject classifications*. Primary 5731, 5352.

Key words and phrases. n-sphere, G-structure, reduction of structure group, special orthogonal group, special unitary group, symplectic group, reducible subgroup, irreducible subgroup, maximal subgroup, homotopy exact sequence, fibration.

B. *For n even the fibration*

$$SU(n) \to SU(n+1) \to SU(n+1)/SU(n) = S^{2n+1}$$

cannot be reduced to a proper subgroup of $SU(n)$.

C. *For* $n \not\equiv 11 \bmod 12$, *the fibration*

$$Sp(n) \to Sp(n+1) \to Sp(n+1)/Sp(n) = S^{4n+3}$$

cannot be reduced to a proper subgroup of $Sp(n)$.

In the remaining cases we restrict ourselves to consideration of the maximal closed connected subgroups of G_n. Following Dynkin [6], we divide these subgroups into three main classes: the reducible maximal subgroups, the nonsimple irreducible maximal subgroups and the simple irreducible maximal subgroups. We obtain a necessary and sufficient condition for reduction to the reducible maximal subgroups. We prove

THEOREM 2. A. *Let H be a reducible maximal subgroup of* $SO(n)$, *n odd. Then H leaves invariant a subspace V of* $\mathbf{C}^n$ *such that* $V=\bar{V}$. *Let k be the larger of* dim V, codim V. *Then* $SO(n) \to SO(n+1) \to S^n$ *can be reduced to H if and only if there is a reduction to the standard subgroup* $SO(k)$.

B. *Let H be a reducible maximal subgroup of* $SU(n)$. *Then H leaves invariant a subspace V of* $\mathbf{C}^n$. *Let k be the larger of* dim V, codim V. *Then* $SU(n) \to SU(n+1) \to S^{2n+1}$ *can be reduced to H if and only if there is a reduction to the standard subgroup* $SU(k)$.

For the symplectic case we need the following definition. Let $J: \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ be defined by $J(x_1, \dots, x_{2n}) = (\bar{x}_2, -\bar{x}_1, \dots, \bar{x}_{2n}, -\bar{x}_{2n-1})$, and $(x, y)'$ the skew-symmetric bilinear form of $\mathbf{C}^{2n}$ defined by

$$(x, y)' = \sum_{k=1}^{n} (x_{2k-1}y_{2k} - x_{2k}y_{2k-1}).$$

Then we have

THEOREM 2. C. *Let H be a reducible maximal subgroup of* $Sp(n)$. *Then H leaves invariant a subspace V of* $\mathbf{C}^{2n}$ *such that either*

(a) $J(V) = V$, *or*

(b) $V \oplus J(V) = \mathbf{C}^{2n}$ *and* $(x, y)'$ *is either zero or nondegenerate on V.*

Let k be the larger of dim V, codim V. *Then* $Sp(n) \to Sp(n+1) \to S^{4n+3}$ *can be reduced to H if and only if* $J(V) = V$ *and there is a reduction to the standard subgroup* $Sp(k/2)$.

The nonsimple irreducible maximal subgroups are dealt with in

THEOREM 3. *If H is a nonsimple irreducible maximal subgroup of* G_n, *then* $G_{n+1} \to G_{n+1}/G_n$ *cannot be reduced to H.*

We are unable to solve the problem for the simple irreducible maximal subgroups. However, we prove the following

PROPOSITION 4. *The fibration* $SU(n+1) \to S^{2n+1}$ *can be reduced to the subgroup* $SO(n)$ *of* $SU(n)$ *if and only if* $n=3$.

The results of this paper are contained in the author's doctoral dissertation. The author wishes to thank Professor Bruno Harris of Brown University for suggesting the problem and for many helpful conversations during the preparation of this paper.

2. **Notation.** $SU(n)$ denote the group of unitary $n \times n$ matrices of determinate 1, σ the automorphism of $SU(n)$ induced by complex conjugation, $SO(n)$ the subgroup of $SU(n)$ of fixed points. If $n=2m$, let τ be the automorphism of $SU(2m)$ defined by $\tau(A)=J^{-1}\sigma(A)J$, where J is the $2m \times 2m$ matrix with 2×2 blocks

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

down the main diagonal and zeros elsewhere. $Sp(m)$ denotes the subgroup of $SU(2m)$ of fixed points of τ.

By a subgroup of $SU(n)$ we will mean a closed connected subgroup. A subgroup is reducible if it leaves invariant a proper subspace of complex n-space C^n and irreducible otherwise.

If A and B are square matrices of orders m and n, respectively, then $A \times B$ denotes the square matrix of order $m+n$ of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

and $A \otimes B$ the square matrix of order mn with entries

$$c_{(j,l)}^{(i,k)} = a_j^i b_l^k,$$

where we use the ordered pairs (i, k), $1 \leqq i \leqq m$, $1 \leqq k \leqq n$, as indices. If M and N are two sets of square matrices of orders m and n, respectively, then $M \times N$ $(M \otimes N)$ is the set of all matrices $A \times B$ $(A \otimes B)$ where $A \in M$ and $B \in N$.

If G is a Lie group, X a CW-complex and η a principal fibre bundle with structure group G over the suspension SX of X, then η is classified by a map $c: X \to G$ or a map $c: SX \to B_G$, where B_G is a classifying space for G [9]. We will speak of either map as a classifying map for η.

Finally, if p is a prime integer, n an integer, $\nu_p(n)$ will denote the highest power of p dividing n.

3. **Proof of Theorem 1.**

LEMMA 3.1. *Let G be a Lie group and H, H_1, H_2 closed subgroups of G such that $H_i \subset H$, $i=1, 2$. The composition*

$$H_1 \subset H \to H/H_2$$

is essential if there is a principal G-bundle over the suspension of a CW-complex which can be reduced to H_1 but not H_2.

Proof. Let $\pi: E \to S(X)$ be such a bundle and $c: X \to G$ a classifying map. There is a homotopy commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{c} & G \\ & \searrow \qquad \nearrow{\scriptstyle i} & \\ & H_1 & \end{array}$$

where i is inclusion. If $H_1 \to H/H_2$ is inessential, then, by the homotopy lifting theorem, there is a homotopy commutative triangle

$$\begin{array}{ccc} H_1 & \xrightarrow{i} & H \\ & \searrow \qquad \nearrow{\scriptstyle i_2} & \\ & H_2 & \end{array}$$

where i, i_2 are inclusions, and $c: X \to G$ factors through H_2.

Corollary 3.2. *Let $\pi: E \to S(X)$ be a principal G_n-bundle which can be reduced to G_{n-k} but not G_{n-k-1}. If there is a reduction to a closed subgroup H of G_{n-k}, then H acts transitively on the sphere G_{n-k}/G_{n-k-1} through G_{n-k}.*

We proceed with the proof of Theorem 1.A. The fibration $SO(2n) \to SO(2n+1) \to S^{2n}$ cannot be reduced to $SO(2n-1)$ [1]. By Corollary 3.2, if there is a reduction to a subgroup H, then H must act transitively on $S^{2n-1} = SO(2n)/SO(2n-1)$ through $SO(2n)$ and must be one of the groups $SO(2n)$, $SU(n)$, $U(n)$, Spin (7) ($n=4$), Spin (9) ($n=8$), or if $n=2m$, $Sp(m)$ or $Sp(1) \times_{Z_2} Sp(m)$ [12], [14].

Reduction to $U(n)$ is possible if and only if $n=1$ or 3 and to $SU(n)$ if and only if $n=3$ [4].

Suppose that n is even. Reduction to $Sp(n/2)$ implies reduction to $SU(n)$ and this is impossible.

Suppose that n is even and that reduction to $Sp(1) \times_{Z_2} Sp(n/2)$ is possible. If $n=2$, $Sp(1) \times_{Z_2} Sp(1) = SO(4)$ and we need only consider the case $n \geqq 4$. The projection $Sp(1) \times Sp(n/2) \to Sp(1) \times_{Z_2} Sp(n/2)$ is a double covering and induces an isomorphism

$$\pi_{2n-1}(Sp(1) \times Sp(n/2)) \cong \pi_{2n-1}(Sp(1) \times_{Z_2} Sp(n/2)).$$

Thus we have a homotopy commutative diagram

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\quad c \quad} & SO(2n) \\ \downarrow & \overset{c^1}{\searrow} & \uparrow{\scriptstyle i} \\ Sp(1) \times Sp(n/2) & \longrightarrow & Sp(1) \times_{Z_2} Sp(n/2) \end{array}$$

where c is a classifying map, c^1 classifies the reduction to $Sp(1)\times_{Z_2} Sp(n/2)$ and i is inclusion. By Lemma 7.5, there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\;c\;} & SO(2n) \\ \downarrow & \nearrow & \uparrow j \\ Sp(1)\times Sp(n/2) & \longrightarrow & Sp(n/2) \end{array}$$

where j is inclusion. Thus, we are back to the previous case.

We now consider the fibration $SO(8)\to SO(9)\to S^8$ and the subgroup Spin (7). The exact sequence of homotopy groups

$$\pi_{11}(S^8)\xrightarrow{c_*}\pi_{11}(B_{SO(8)})\longrightarrow\pi_{11}(B_{SO(9)})\longrightarrow\pi_{10}(S^8)$$
$$\longrightarrow\pi_{10}(B_{SO(8)})\longrightarrow\pi_{10}(B_{SO(9)})$$

of the fibration $S^8\xrightarrow{c} B_{SO(8)}\to B_{SO(9)}$ is as follows [10]:

$$Z_{24}\xrightarrow{c_*}Z_{24}\oplus Z_8\longrightarrow Z_8\longrightarrow Z_2\longrightarrow Z_2\oplus Z_2\oplus Z_2\longrightarrow Z_2\oplus Z_2.$$

Thus,

$$\pi_{11}(S^8)\xrightarrow{c_*}\pi_{11}(B_{SO(8)})\longrightarrow\pi_{11}(B_{SO(9)})$$

is the exact sequence

$$0\longrightarrow Z_{24}\xrightarrow{c_*}Z_{24}+Z_8\longrightarrow Z_8\longrightarrow 0.$$

Since $\pi_{11}(B_{\mathrm{Spin}(7)})=Z_8$, c cannot factor through $B_{\mathrm{Spin}(7)}$.

Finally, consider the fibration $SO(16)\to SO(17)\to S^{16}$ and the subgroup Spin (9). The exact sequence of homotopy groups

$$\pi_{19}(S^{16})\xrightarrow{c_*}\pi_{19}(B_{SO(16)})\longrightarrow\pi_{19}(B_{SO(17)})\longrightarrow\pi_{18}(S^{16})$$
$$\longrightarrow\pi_{18}(B_{SO(16)})\longrightarrow\pi_{18}(B_{SO(17)})$$

is as follows [10]:

$$Z_{24}\xrightarrow{c_*}Z_{24}+Z_8\longrightarrow Z_8\longrightarrow Z_2\longrightarrow Z_2\oplus Z_2\oplus Z_2\longrightarrow Z_2\oplus Z_2.$$

Thus

$$\pi_{19}(S^{16})\xrightarrow{c_*}\pi_{19}(B_{SO(16)})\longrightarrow\pi_{19}(B_{SO(17)})$$

is the exact sequence

$$0\longrightarrow Z_{24}\xrightarrow{c_*}Z_{24}\oplus Z_8\longrightarrow Z_8\longrightarrow 0.$$

Mimura [11] has shown that

$$\pi_{19}(B_{\text{Spin}(9)}) = Z_{2835} \oplus Z_{16} \oplus Z_8 \oplus Z_2.$$

A simple argument shows that c_* cannot factor through $Z_{2835} \oplus Z_{16} \oplus Z_8 \oplus Z_2$. This completes the proof of Theorem 1.A.

We now prove Theorem 1.B. The fibration $SU(2n) \to SU(2n+1) \to S^{4n+1}$ cannot be reduced to $SU(2n-1)$ [3]. By Corollary 3.2, if there is a reduction to a subgroup H of $SU(2n)$, H must act transitively on $S^{4n-1} = SU(2n)/SU(2n-1)$ through $SU(2n)$ and must be one of the groups $SU(2n)$, $Sp(n)$ or Spin (9) ($n=4$).

There is a fibration

$$S^{4n+1} \xrightarrow{c} B_{SU(2n)} \longrightarrow B_{SU(2n+1)}$$

and c is a classifying map. From the exact sequence

$$\pi_{4n+1}(S^{4n+1}) \xrightarrow{c_*} \pi_{4n+1}(B_{SU(2n)}) \longrightarrow \pi_{4n+1}(B_{SU(2n+1)}),$$

we see that c_* is surjective since $\pi_{4n+1}(B_{SU(2n+1)}) = 0$ [5]. Since [10]

$$\begin{aligned} \pi_{4n+1}(B_{SU(2n)}) &= Z_{(2n)!}, \\ \pi_{4n+1}(B_{Sp(n)}) &= 0 \quad \text{if } n \text{ even}, \\ &= Z_2 \quad \text{if } n \text{ odd}, \end{aligned}$$

and [11]

$$\pi_{17}(B_{\text{Spin}(9)}) = Z_2 \oplus \cdots \oplus Z_2 \qquad \text{(6 copies)},$$

we see that there are no reductions to $Sp(n)$ or Spin (9) ($n=4$).

Finally, we prove Theorem 1.C. We first show that if $Sp(n) \to Sp(n+1) \to S^{4n+3}$ can be reduced to $Sp(n-1)$ then $n+1 \equiv 0 \bmod 12$. Since $Sp(n+1) \to S^{4n+3}$ is a reduction of $SU(2n+2) \to S^{4n+3}$, reduction of $Sp(n+1) \to S^{4n+3}$ to $Sp(n-1)$ implies reduction of $SU(2n+2) \to S^{4n+3}$ to $SU(2n-2)$. But this is possible if and only if $2n+2$ is divisible by the Atiyah-Todd number $M_4 = 24$ [2].

Therefore, if $n \not\equiv 11 \bmod 12$, and if $Sp(n+1) \to S^{4n+3}$ can be reduced to a subgroup H, H must act transitively on $S^{4n-1} = Sp(n)/Sp(n-1)$ through $Sp(n)$. But the only subgroup of $Sp(n)$ acting transitively on S^{4n-1} is $Sp(n)$.

4. **Proof of Theorem 2.** One may easily obtain the following description of the reducible maximal subgroups of $SO(n)$ (n odd), $SU(n)$ and $Sp(n)$.

PROPOSITION 4.1. *Let H be a reducible maximal subgroup of $SO(n)$ (n odd). Then H leaves invariant a proper subspace V of C^n such that $\overline{V} = V$ and is conjugate in $SO(n)$ to $SO(k) \times SO(n-k)$ if* $\dim V = k$.

PROPOSITION 4.2. *Let H be a reducible maximal subgroup of $SU(n)$. Then H leaves invariant a proper subspace V of C^n and is conjugate in $SU(n)$ to $S(U(k) \times U(n-k))$, where $k = \dim V$ and $S(U(k) \times U(n-k))$ is the subgroup of unimodular matrices in $U(k) \times U(n-k)$.*

PROPOSITION 4.3. *Let H be a reducible maximal subgroup of $Sp(n)$. Then H leaves invariant a proper subspace V of C^{2n} such that either*

(a) *$J(V)=V$, and H is conjugate in $Sp(n)$ to $Sp(k)\times Sp(n-k)$, where $2k=\dim V$,*

(b) *$V\oplus J(V)=C^{2n}$ and $(x, y)'$ is zero on V, and H is conjugate in $Sp(n)$ to $U(n)$, or*

(c) *$V\oplus J(V)=C^{2n}$ and $(x, y)'$ is nondegenerate on V.*

We will also need the following:

LEMMA 4.4. *Let G be a Lie group, H a closed subgroup and U_1, U_2 subgroups of H. Let α be an automorphism of G such that $\alpha(H)=H$ and $\alpha(U_1)=U_2$. Then $G\to G/H$ can be reduced to U_1 if and only if there is a reduction to U_2.*

Proof. α induces homeomorphisms $\alpha\colon G/H\to G/H$ and $\tilde{\alpha}\colon G/U_1\to G/U_2$ such that the diagram

$$\begin{array}{ccc} G/U_1 & \xrightarrow{\tilde{\alpha}} & G/U_2 \\ \downarrow & & \downarrow \\ G/H & \xrightarrow{\alpha} & G/H \end{array}$$

commutes. Since reduction of $G\to G/H$ to U_i, $i=1, 2$, is equivalent to the existence of a cross-section of $G/U_i\to G/H$, the lemma follows.

We now prove Theorem 2.A. Since the reducible maximal subgroups of $SO(n)$ (n odd) are conjugate in $SO(n)$ to one of the subgroups $SO(k)\times SO(n-k)$, by Lemma 4.4, it suffices to prove the theorem for these subgroups. Let $c\colon S^{n-1}\to SO(n)$ be a classifying map for $SO(n+1)\to S^n$. There is a reduction to $SO(k)\times SO(n-k)$ if and only if there is a homotopy commutative triangle

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{c} & SO(n) \\ & \searrow \quad \nearrow{\scriptstyle j} & \\ & SO(k)\times SO(n-k) & \end{array}$$

where j is inclusion. By Corollary 7.2, such a triangle can be completed to a homotopy commutative diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\quad c \quad} & SO(n) \\ \downarrow & \overset{j}{\nearrow} & \uparrow{\scriptstyle i} \\ SO(k)\times SO(n-k) & \longrightarrow & SO(q) \end{array}$$

where $q=\max\{k, n-k\}$ and i is inclusion. Thus, $SO(n+1)\to S^n$ can be reduced to $SO(k)\times SO(n-k)$ if and only if there is a reduction to $SO(q)$, $q=\max\{k, n-k\}$.

To prove Theorem 2.B, it suffices to prove the theorem for the subgroups $S(U(k)\times U(n-k))$ of $SU(n)$. We show that $SU(n+1)\to S^{2n+1}$ can be reduced to

$S(U(k)\times U(n-k))$ if and only if it can be reduced to $SU(k)\times SU(n-k)$. The theorem will then follow, as above, from Corollary 7.2. The inclusion of $SU(k)\times SU(n-k)$ into $S(U(k)\times U(n-k))$ induces an isomorphism

$$\pi_{2n}(SU(k)\times SU(n-k)) \simeq \pi_{2n}(S(U(k)\times U(n-k))),$$

since $S(U(k)\times U(n-k))/SU(k)\times SU(n-k)=S^1$. Thus, a homotopy commutative triangle

$$\begin{array}{ccc} S^{2n} & \longrightarrow & SU(n) \\ & \searrow \quad \nearrow & \\ & S(U(k)\times U(n-k)) & \end{array}$$

can be completed to a homotopy commutative diagram

$$\begin{array}{ccc} S^{2n} & \longrightarrow & SU(n) \\ \downarrow & \searrow & \uparrow \\ SU(k)\times SU(n-k) & \longrightarrow & S(U(k)\times U(n-k)) \end{array}$$

We now prove Theorem 2.C. Let H be a reducible maximal subgroup of $Sp(n)$. We must consider three cases.

(a) H is conjugate in $Sp(n)$ to one of the subgroups $Sp(k)\times Sp(n-k)$. $Sp(n+1)\to S^{4n+3}$ can be reduced to H if and only if there is a reduction to $Sp(k)\times Sp(n-k)$ which is equivalent to reduction to $Sp(q)$, $q=\max\{k, n-k\}$, by Corollary 7.2.

(b) H is conjugate to $U(n)$. We show that reduction to $U(n)$ is impossible. Since $Sp(n+1)\to S^{4n+3}$ is a reduction of $SU(2n+2)\to S^{4n+3}$ to $Sp(n)$, reduction of $Sp(n+1)\to S^{4n+3}$ to $U(n)$ implies reduction of $SU(2n+2)\to S^{4n+3}$ to $U(n)$ under the inclusion

$$j\colon U(n)\to Sp(n)\to SU(2n+1)$$

which is given by $j(A)=B\times I_1$, where B is the $2n\times 2n$ matrix consisting of 2×2 blocks of the form

$$\begin{bmatrix} a_{ij} & 0 \\ 0 & \bar{a}_{ij} \end{bmatrix}$$

where $A=[a_{ij}]$. We can write $j(A)=A_1A_2$ where $A_1=B_1\times[\det A]$, $A_2=B_2\times[\det \bar{A}]$ and B_1, B_2 are $2n\times 2n$ matrices consisting of 2×2 blocks of the form

$$\begin{bmatrix} a_{ij} & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & \bar{a}_{ij} \end{bmatrix},$$

respectively. Let C_1 be the $2n\times 2n$ matrix with blocks

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

down the main diagonal and zeros elsewhere, and $C=C_1\times(-1)^n I_1$. Then $C\in SO(2n+1)$ and $CA_2C^{-1}=\bar{B}_1\times(\det A)I_1$. Let D be the matrix of the linear transformation of C^{2n+1} defined by

$$\begin{aligned} D(e_{2k-1}) &= e_k, & 1 \leqq k \leqq n,\\ D(e_{2k}) &= e_{n+k}, & 1 \leqq k \leqq n,\\ D(e_{2n+1}) &= \pm e_{2n+1}, \end{aligned}$$

where the sign is chosen so that D is unimodular. Then

$$D(A_1CA_2C^{-1})D^{-1} = A\bar{A}\times I_{n+1}.$$

Thus, there is a homotopy commutative triangle

$$\begin{array}{ccc} U(n) & \xrightarrow{\ j\ } & SU(2n+1) \\ & \searrow \quad \nearrow i & \\ & SU(n) & \end{array}$$

where $i(A)=A\times I_{n+1}$, and reduction to $U(n)$ under j implies reduction to the standard subgroup $SU(n)$. But then $2n+2$ must be divisible by the Atiyah-Todd number M_{n+2} [3], [2]. Since $\nu_2(M_{n+2})\geqq n+1$, 2^{n+1} must divide $2n+2$, which is impossible, if $n>1$. Since $\nu_2(M_3)=3$, M_3 does not divide 4.

(c) H leaves invariant a subspace V such that $V\oplus J(V)=C^{2n}$ and $(x, y)'$ is nondegenerate on V. The dimension of V is even, and, thus, n is even. In particular $n\not\equiv 11 \bmod 12$ and, by Theorem 1.C, reduction to H is impossible.

5. **Proof of Theorem 3.** We first obtain the following description of the nonsimple irreducible maximal subgroups of $SO(n)$, $SU(n)$ and $Sp(n)$.

PROPOSITION 5.1. *Every nonsimple irreducible maximal subgroup of $SO(n)$ is conjugate in $O(n)$ to one of the groups $Sp(s)\otimes Sp(t)$ $(4st=n,\ 1\leqq t\leqq s)$ or $SO(s)\otimes SO(t)$ $(st=n,\ 3\leqq t\leqq s,\ s,\ t\neq 4)$.*

PROPOSITION 5.2. *Every nonsimple irreducible maximal subgroup of $SU(n)$ is conjugate in $SU(n)$ to one of the groups $SU(s)\otimes SU(t)$ $(st=n,\ 2\leqq t\leqq s)$.*

PROPOSITION 5.3. *Every nonsimple irreducible maximal subgroup of $Sp(n)$ is conjugate in $Sp(n)$ to one of the groups $SO(s)\otimes Sp(t)$ $(st=n,\ t\geqq 1,\ s\geqq 3,\ s\neq 4$ or $t=1,\ s=4)$.*

Proof of Propositions 4.1, 4.2, 4.3. Let $Sl(n)$ denote the group of all complex, unimodular matrices of order n, $SO(n, C)$ the subgroup of $Sl(n)$ leaving invariant the standard symmetric bilinear form on C^n and $Sp(n, C)$ the subgroup of $Sl(2n)$ leaving invariant the standard skew-symmetric bilinear form on C^{2n}. Then $SU(n)$, $SO(n)$ and $Sp(n)$ are compact real forms of $Sl(n)$, $SO(n, C)$ and $Sp(n, C)$, respectively.

Let $\tilde{G}$ be a simple complex Lie group and G a compact real form of $\tilde{G}$. Since G is maximal among the real subgroups of $\tilde{G}$ [6, p. 256], G is a maximal compact subgroup of $\tilde{G}$, and every compact subgroup of $\tilde{G}$ is contained in a conjugate of G. Let $(\tilde{G}, G)$ denote one of the pairs $(SO(n, C), SO(n))$, $(Sl(n), SU(n))$ or $(Sp(n, C), Sp(n))$.

The propositions follow immediately from Dynkin's Theorems 1.3 and 1.4 [6, p. 253] and the following 2 lemmas.

LEMMA 5.4. *Let H be a closed subgroup of G with Lie algebra $\mathscr{L}$. If $\tilde{H}$ is the subgroup of $\tilde{G}$ with Lie algebra $\mathscr{L} \otimes_R C$, then H is an irreducible maximal subgroup of G if and only if $\tilde{H}$ is an irreducible maximal subgroup of $\tilde{G}$.*

Proof. Let H be an irreducible maximal subgroup of G. Suppose $\tilde{H}$ is contained in a subgroup $\tilde{U}$ of $\tilde{G}$. Then $\tilde{H}$, $\tilde{U}$ are irreducible groups of unimodular linear transformations and, therefore, semisimple. Let U be a maximal compact subgroup of $\tilde{U}$ containing H. Since U is compact, there exists $b \in \tilde{G}$ such that

$$bHb^{-1} \subseteq bUb^{-1} \subseteq G.$$

If $b \in G$, then either $U=H$ or $U=G$ and, thus, $\tilde{U}=\tilde{H}$ or $\tilde{U}=\tilde{G}$. We now show that b does belong to G. Let $h \in H$. Since $bhb^{-1} \in G$, $bhb^{-1}=(bh^{-1}b^{-1})^{-1}=(\bar{b}^t)^{-1}h\bar{b}^t$ and $\bar{b}^t bh=h\bar{b}^t b$, for all $h \in H$. By Schur's Lemma, $\bar{b}^t b$ is a scalar matrix, say $\bar{b}^t b=\lambda I_n$. But $\lambda>0$ and $\lambda^n=1$. Thus $\bar{b}^t b=I_n$ and $b \in G$.

If $\tilde{H}$ is an irreducible maximal subgroup of $\tilde{G}$, then $\mathscr{L} \otimes_R C$ is an irreducible Lie algebra of linear transformations. Thus, $\mathscr{L}$ is an irreducible Lie algebra of linear transformations, and H is an irreducible group of linear transformations. If H is not maximal, then H is contained in a subgroup U of G, and $\tilde{H}$ is contained in $\tilde{U}$. Thus, H is an irreducible maximal subgroup of G.

LEMMA 5.5. *Let H_1, H_2 be irreducible subgroups of G. H_1 and H_2 are conjugate in G if and only if $\tilde{H}_1$ and $\tilde{H}_2$ are conjugate in $\tilde{G}$.*

Proof. If $aH_1a^{-1}=H_2$, $a \in G$, then $a\tilde{H}_1a^{-1}=\tilde{H}_2$.

Suppose $a\tilde{H}_1a^{-1}=\tilde{H}_2$, $\tilde{a} \in \tilde{G}$. Then aH_1a^{-1} is a compact subgroup of $\tilde{H}_2$, and there is a $b \in \tilde{H}_2$ such that $baH_1(ba)^{-1} \subset H_2$. Let $c=ba$ and $h \in H_1$. Then $chc^{-1} =(ch^{-1}c^{-1})^{-1}=(\bar{c}^t)^{-1}h\bar{c}^t$ and $\bar{c}^t ch=h\bar{c}^t c$, for all $h \in H_1$. As above, $c \in G$.

We can now prove Theorem 3. For the fibration $SO(n+1) \to S^n$ we need only consider the case n odd, and the subgroups $SO(s) \otimes SO(t)$, where $st=n$, $2<t \leqq s$. The natural projection j: $SO(s) \times SO(t) \to SO(s) \otimes SO(t)$ is a covering map and, if $n>2$, induces isomorphisms

$$\pi_{n-1}(SO(s) \times SO(t)) \approx \pi_{n-1}(SO(s) \otimes SO(t)).$$

Let c: $S^{n-1} \to SO(n)$ be a classifying map. A homotopy commutative triangle

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{c} & SO(n) \\ & \searrow \quad \nearrow & \\ & SO(s) \otimes SO(t) & \end{array}$$

can be completed to a homotopy commutative diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\quad c \quad} & SO(n) \\ \downarrow & \searrow & \uparrow \\ SO(s)\times SO(t) & \xrightarrow{j} & SO(s)\otimes SO(t) \end{array}$$

which, by Lemma 7.3, yields a homotopy commutative triangle

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{c} & SO(n) \\ & \searrow \quad \nearrow_{i} & \\ & SO(s) & \end{array}$$

where i is the standard inclusion. Thus reduction to $SO(s)\otimes SO(t)$ implies reduction to $SO(s)$.

We show that there is no reduction to $SO(s)$, where $n=st$, $2\leqq t\leqq s$ and n odd. The result of Adams [1] may be stated as follows: Define $\zeta(n)$ by

$$\begin{aligned} \zeta(n) &= 2\nu_2(n)+1 \quad \text{if } \nu_2(n) \equiv 0 \bmod 4, \\ &= 2\nu_2(n) \qquad\ \ \text{if } \nu_2(n) \equiv 1, 2 \bmod 4, \\ &= 2\nu_2(n)+2 \quad \text{if } \nu_2(n) \equiv 3 \bmod 4. \end{aligned}$$

Then $SO(n+1)\to S^n$ can be reduced to $SO(s)$ if and only if $n-s\leqq\zeta(n+1)-1$. Let $n+1=2^\beta\alpha$, where $\alpha\geqq 1$ is odd. Then $2(s+1)<ts=n$ and $s+1\leqq(n+1)/2-1$. Therefore, $n-s\geqq(n+1)/2+1>2\beta+1\geqq\zeta(n+1)-1$, if $\beta\geqq 5$ or $\alpha>1$. If $n+1=2^k$, $1\leqq k\leqq 4$, then n is prime for $k=1, 2, 3$ and the only remaining case is $n=15$, $t=3$, $s=5$. But $\zeta(16)-1=8<n-s=10$. Thus, $\zeta(n+1)-1<n-s$.

For the fibration $SU(n+1)\to S^{2n+1}$ we need only consider the case n odd and the subgroups $SU(s)\otimes SU(t), n=st, 2\leqq t\leqq s$. As above, reduction to $SU(s)\otimes SU(t)$, implies reduction to $SU(s)$. But Adams and Walker [2] have shown that this is possible if and only if $n+1$ is divisible by the Atiyah-Todd number M_{n-s+1}. In particular, $n+1$ must be divisible by 2^α, where

$$\alpha = \nu_2(M_{n-s+1}) = \max\{r+\nu_2(r) \mid 1 \leqq r \leqq n-s\} \geqq n-s.$$

Thus 2^{n-s} must divide $n+1$. But

$$n+1 \leqq s^2+1 < 2^{2s} \leqq 2^{s(t-1)} = 2^{n-s},$$

since $t\geqq 3$.

For the fibration $Sp(n+1)\to S^{4n+3}$ we need only consider the subgroups $Sp(s)\otimes SO(t)$, $n=st$, $t>1$. Using Lemma 7.4, we see that reduction to $Sp(s)\otimes SO(t)$ implies reduction to $Sp(q)$ where $q=\max\{s, t\}$. Since $Sp(n+1)\to S^{4n+3}$ is a reduction of $SU(2n+2)\to S^{4n+3}$, this implies a reduction of $SU(2n+2)\to S^{4n+3}$ to $Sp(q)$ and, hence, to $SU(2q)$. Thus, $2n+2$ would have to be divisible by the Atiyah-Todd number $M_{2n+2-2q}$ [2]. In particular, $2n+2$ would be divisible by 2^{β} where

$$\beta = \nu_2(M_{2n+2-2q}) = \max\{r+\nu_2(r) \mid 1 \leqq r \leqq 2n-2q+1\} \geqq 2n-2q+1.$$

Thus, $2^{2n-2q+1}$ must divide $2n+2$. If $r=\min\{s, t\}$, then $n=rq\leqq q^2$, and if $s>1$, $n-q\geqq q$. So, if $s>1$,

$$2^{2n-2q} \geqq 2^{2q} > 2q^2+1 \geqq n+1,$$

and $2^{2n-2q+1}$ cannot divide $2n+2$. If $s=1$, then $t=n$ and $Sp(1) \otimes SO(n) = SU(2) \otimes SO(n)$. Reduction to $SU(2) \otimes SO(n)$ implies reduction of $SU(2n+2) \to S^{4n+3}$ to $SU(2) \otimes SU(n)$. But reduction to $SU(2) \otimes SU(n)$ implies reduction to $SU(n)$. Thus $2n+2$ must be divisible by M_{n+1}. But $\nu_2(M_{n+1})\geqq n+1$ and 2^n does not divide $n+1$ if $n>1$.

6. **Proof of Proposition 4.** Since $SU(n+1) \to S^{2n+1}$ is a reduction of $SO(2n+2) \to S^{2n+1}$, there is a homotopy commutative triangle

$$\begin{array}{ccc} S^{2n} & \xrightarrow{c} & SO(2n+1) \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle i} & \\ & SU(n) & \end{array}$$

where c and f are classifying maps and i is inclusion. Reduction of $SU(n+1) \to S^{2n+1}$ to $SO(n)$ would result in a homotopy commutative triangle

$$\begin{array}{ccc} S^{2n} & \xrightarrow{c} & SO(2n+1) \\ & \searrow \quad \nearrow{\scriptstyle j} & \\ & SO(n) & \end{array}$$

where j is the composition

$$SO(n) \to SU(n) \to SO(2n+1),$$

i.e. $j(A)=A\times A\times I_1$. Corollary 7.2 readily implies the existence of a homotopy commutative triangle

$$\begin{array}{ccc} SO(n) & \xrightarrow{j} & SO(2n+1) \\ & \searrow \quad \nearrow{\scriptstyle i} & \\ & SO(n) & \end{array}$$

where $i(A)=A\times I_{n+1}$. Thus, $SU(n+1) \to S^{2n+1}$ can be reduced to $SO(n)$ only if $n+1<\zeta(2n+2)$, where

$$\begin{aligned} \zeta(2n+2) &= 2\nu_2(n+1)+2 \quad \text{if } n+1 \equiv 0 \bmod 2, \\ &= 2\nu_2(n+1)+1 \quad \text{if } n+1 \equiv 1 \bmod 2. \end{aligned}$$

Let $j=\nu_2(n+1)$ and $n+1=2^j\alpha$. Then $n+1\geqq 2^j>2(1+j)$ if $j\geqq 4$, and $n+1\geqq 3 2^j >2(1+j)$ if $\alpha>1$. Thus $n+1>\zeta(2n+2)$ if $n+1\neq 2^j$, $1\leqq j\leqq 3$ and the proposition is proved for $n\neq 3, 7$.

Suppose there is a homotopy commutative triangle

$$\begin{array}{ccc} S^{14} & \xrightarrow{f} & SU(7) \\ & \searrow \qquad \nearrow_{i} & \\ & SO(7) & \end{array}$$

where f classifies $SU(8)\to S^{15}$. Then there is a commutative triangle

$$\begin{array}{ccc} \pi_{14}(S^{14}) & \xrightarrow{f_*} & \pi_{14}(SU(7)) \\ & \searrow \qquad \nearrow_{i_*} & \\ & \pi_{14}(SO(7)) & \end{array}$$

Since $\pi_{14}(SU(8))=0$, f_* is surjective [9, p. 90] and i_* must also be surjective. But $\pi_{14}(SU(7))=Z_{7!}$, and [12] $\pi_{14}(SO(7))=Z_{2520}+Z_8+Z_2$, and i_* cannot be surjective.

Thus, we are left with the case $n=3$. Let f: $S^6\to SU(3)$ be a classifying map for $SU(4)\to S^7$. We show that there is a map g: $S^6\to SO(3)$ such that the triangle

$$\begin{array}{ccc} S^{6} & \xrightarrow{f} & SU(3) \\ & {}_{g}\searrow \qquad \nearrow_{i} & \\ & SO(3) & \end{array}$$

is homotopy commutative. Since the homotopy class of f generates $\pi_6(SU(3))$ [9], it is sufficient to show that i_*: $\pi_6(SO(3))\to\pi_6(SU(3))$ is surjective.

The composition

$$SO(3)\xrightarrow{i} SU(3)\longrightarrow SU(3)/SU(2)$$

is inessential since it factors through $SO(3)/SO(2)$, and there is a homotopy commutative triangle

$$\begin{array}{ccc} SO(3) & \xrightarrow{i} & SU(3) \\ & {}_{\varphi}\searrow \qquad \nearrow_{j} & \\ & SU(2) & \end{array}$$

where j is inclusion. Let π: $S^3\to SO(3)$ be the universal covering of $SO(3)$. We show that $\varphi\pi$: $S^3\to S^3$ has degree ± 1. Since π_*: $\pi_3(S^3)\to\pi_3(SO(3))$ and j_*: $\pi_3(SU(2))\to\pi_3(SU(3))$ are isomorphisms, it suffices to show that

$$i_*\colon \pi_3(SO(3))\to\pi_3(SU(3))$$

is an isomorphism. If $\mathscr{C}$ is the class of 2-primary abelian groups, there is a $\mathscr{C}$-isomorphism [7]

$$\pi_3(SU(3)) \approx \pi_3(SO(3)) \oplus \pi_3(SU(3)/SO(3)).$$

Since $\pi_3(SU(3)) \approx Z$, $\pi_3(SO(3)) \approx Z$, $\pi_3(SU(3)/SO(3)) \in \mathscr{C}$ and

$$i_*: \pi_3(SO(3)) \to \pi_3(SU(3))$$

is a $\mathscr{C}$-isomorphism. Thus, i_* is an isomorphism.

Now consider the commutative diagram

$$\begin{array}{ccccc} \pi_6(S^3) & \xrightarrow{\pi_*} & \pi_6(SO(3)) & \xrightarrow{i_*} & \pi_6(SU(3)) \\ & & {\scriptstyle\varphi_*}\searrow & & \nearrow{\scriptstyle j_*} \\ & & & \pi_6(SU(2)) & \end{array}$$

Since $\varphi_*\pi_*$ is an isomorphism and j_* is surjective, i_* is surjective.

7. We now establish some results which were needed for the proof of the main propositions.

LEMMA 7.1. *Let $k<n$ and $j_l: G_k \to G_n$ be defined by $j_l(A)=I_l \times A \times I_{n-k-l}$ if $G_n=SO(n)$ or $SU(n)$ and by $j_l(A)=I_{2l} \times A \times I_{2(n-k-l)}$ if $G_n=Sp(n)$. Then j_l is homotopic to j_0 for $0 \leqq l \leqq n-k$.*

Proof. Let

$$C_l = \begin{bmatrix} 0 & I_k \\ EI_l & 0 \end{bmatrix} \times I_{n-k-l}, \qquad E = (-1)^{kl}$$

in the real or complex case. In the symplectic case replace I_j by I_{2j} and let $E=1$. Then $C_l \in G_n$ and

$$C_l j_l(A) C_l^{-1} = j_0(A).$$

Since G_n is path connected, we are finished.

COROLLARY 7.2. *Let $K_1, \dots, K_m$ be positive integers, $n=K_1+\cdots+K_m$ and $q=\max\{K_j\}$. There is a homotopy commutative triangle*

$$\begin{array}{ccc} G_{K_1}\times\cdots\times G_{K_m} & \xrightarrow{j} & G_n \\ \searrow & & \nearrow{\scriptstyle i} \\ & G_q & \end{array}$$

where $j(A_1,\dots,A_m)=A_1\times\cdots\times A_m$ and $i(A)=A\times I_{n-q}$ ($i(A)=A\times I_{2(n-q)}$ in the symplectic case).

LEMMA 7.3. *Let $G_n = SO(n)$ or $SU(n)$. If $t \leqq s$, then there is a homotopy commutative triangle*

$$\begin{array}{ccc} G_s \times G_t & \xrightarrow{K} & G_{st} \\ & \searrow \quad \nearrow_{i} & \\ & G_s & \end{array}$$

where $K(A, B) = A \otimes B$ and $i(A) = A \times I_{s(t-1)}$.

Proof. Let $K_1 \colon G_s \to G_{st}$ and $K_2 \colon G_t \to G_{st}$ be defined by $K_1(A) = A \otimes I_t$, $K_2(B) = I_s \otimes B$. Then $K(A, B) = K_1(A)K_2(B)$, and it suffices to show that K_1, K_2 can be factored through G_s, G_t respectively.

Since $K_2(B) = B \times \cdots \times B$ (s copies), a simple application of Corollary 7.2 shows that K_2 has the desired factorization.

There is a real orthogonal matrix C_1 such the

$$C_1(A \otimes I_t)C_1^{-1} = I_t \otimes A.$$

Let $\varepsilon = \det C_1$ and $C = ([\varepsilon] \times I_{st-1})C_1$. Then

$$CK_1(A)C^{-1} = \tilde{A} \times (A \times \cdots \times A) \qquad (t-1 \text{ copies})$$

where $\tilde{A} = ([\varepsilon] \times I_{s-1})A([\varepsilon] \times I_{s-1})$. An application of Corollary 7.2 shows that K_1 has the desired factorization.

LEMMA 7.4. *Let $q = \max\{s, t\}$. There is a homotopy commutative triangle*

$$\begin{array}{ccc} SO(s) \times Sp(t) & \xrightarrow{K} & Sp(st) \\ & \searrow \quad \nearrow_{i} & \\ & Sp(q) & \end{array}$$

where $K(A, B) = A \otimes B$ and $i(A) = A \times I_{2(st-q)}$.

Proof. Let $K_1 \colon SO(s) \to Sp(st)$ and $K_2 \colon Sp(t) \to Sp(st)$ be defined by $K_1(A) = A \otimes I_{2t}$, $K_2(B) = I_s \otimes B$. Then $K(A, B) = K_1(A)K_2(B)$ and it suffices to show that K_1, K_2 factor through $Sp(q)$.

Since $K_2(B) = B \times \cdots \times B$ (s copies), Corollary 7.2 can be applied to obtain the desired factorization.

There is a symplectic matrix C such that

$$C(A \otimes I_{2t})C^{-1} = I_t \otimes (A \otimes I_2).$$

Thus, K_1 is homotopic to the composition

$$SO(s) \xrightarrow{l} Sp(s) \xrightarrow{j} Sp(st),$$

where $l(A) = A \otimes I_2$ and $j(B) = B \times \cdots \times B$ (t copies). Hence, Corollary 7.2 can be applied to obtain the desired factorization.

The concluding lemma will require the following description of $Sp(n)$. Let H denote the quaternions and H^n quaternionic n-space as a right vector space over H with quaternionic inner product $\langle x, y\rangle_H = \sum \bar{y}_i x_i$. H^n can be considered as a $4n$-dimensional vector space over the reals with inner product $\langle x, y\rangle_R$ defined by taking the real part $\langle x, y\rangle_H$. The group of quaternionic linear transformations of H^n leaving $\langle x, y\rangle_H$ invariant is $Sp(n)$. The group of real linear transformations of H^n leaving $\langle x, y\rangle_R$ invariant is $O(4n)$, and we have a natural inclusion $j\colon Sp(n) \to SO(4n)$.

If $\lambda \in Sp(1)$, the unit quaternions, define $L(\lambda)$, $R(\lambda)\colon H^n \to H^n$ by $L(\lambda)v = \lambda v$, $R(\lambda)v = v\bar{\lambda}$. Then $L(\lambda) \in Sp(n)$, $R(\lambda) \in SO(4n)$. If

$$g\colon Sp(1) \times Sp(n) \to SO(4n)$$

is defined by $g(\lambda, A) = R(\lambda)j(A)$, then g is a homomorphism with kernel generated by $(-1, -Id)$. We write

$$Sp(1) \times_{Z_2} Sp(n) = Sp(1) \times Sp(n)/\ker g;$$

g induces an inclusion $Sp(1) \times_{Z_2} Sp(n) \to SO(4n)$, and $Sp(1) \times_{Z_2} Sp(n)$ is a proper subgroup if and only if $n > 1$.

LEMMA 7.5. *If $n > 1$, there is a homotopy commutative triangle*

$$\begin{array}{ccc} Sp(1) \times Sp(n) & \xrightarrow{\;g\;} & SO(4n) \\ & {}_{K}\searrow \quad \nearrow_{j} & \\ & Sp(n) & \end{array}$$

where g, j are as above and $K(\lambda, A) = L(\lambda)A$.

Proof. Define $C_k\colon H^n \to H^n$ by $C_k(y) = (y_1, \ldots, \bar{y}_k, \ldots, y_n)$. Then $C_k \in O(4n)$ and has degree -1. If $\lambda \in Sp(1)$, define $L_k(\lambda)$, $R_k(\lambda)\colon H^n \to H^n$ by

$$L_k(\lambda)(y) = (y_1, \ldots, \lambda y_k, \ldots, y_n), \quad \text{and} \quad R_k(\lambda)(y) = (y_1, \ldots, y_k\bar{\lambda}, \ldots, y_n).$$

Then $L_k(\lambda) \in Sp(n)$, $R_k(\lambda) \in SO(4n)$ and $C_jC_kR_k(\lambda)(C_jC_k)^{-1} = L_k(\lambda)$ if $j \neq k$.

Let $D_k = C_nC_k$ for $1 \leqq k < n$ and $D_n = C_1C_n$. There is a path $D_k(t)$ in $SO(4n)$ such that $D_k(0) = Id$ and $D_k(1) = D_k$. Define a homotopy

$$h_t\colon Sp(1) \times Sp(n) \to SO(4n)$$

by $h_t(\lambda, A) = D_1(t)R_1(\lambda)D_1(t)^{-1} \cdots D_n(t)R_n(\lambda)D_n(t)^{-1}A$. Then $h_0 = g$ and $h_1 = j \circ K$.

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