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p-SOLVABLE LINEAR GROUPS OF FINITE ORDER

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Abstract. The purpose of this paper is to prove the following result.

THEOREM. *Let p be an odd prime and let G be a finite p-solvable group. Assume that G has a faithful representation of degree n over a field of characteristic zero or over a perfect field of characteristic p. Let P be a Sylow p-subgroup of G and let $O_p(G)$ be the maximal normal p-subgroup of G. Then $|P:O_p(G)|\leqq p^{\lambda_p(n)}$ where*

$$\lambda_p(n) = \sum_{i=0}^{\infty}\left[\frac{n}{(p-1)p^i}\right] \qquad \text{if } p \text{ is a Fermat prime,}$$
$$= \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right] \qquad \text{if } p \text{ is not a Fermat prime.}$$

1. **Introduction.** The above theorem is a generalization of some of the results of J. D. Dixon [1]. Dixon proved the theorem under the stronger hypothesis that G is solvable. Examples are given in [1] showing that the result is best possible for each n and each odd prime p. The author thanks Professor Dixon for some helpful suggestions.

2. **Preliminaries.** $C(S)$ and $C(x)$ will denote the centralizer in the group concerned of, respectively, the set S and the element x. $O_p(G)$ and $O_{p'}(G)$ denote, respectively, the maximal normal p-subgroup of G and the maximal normal subgroup of G whose order is relatively prime to p. $H^{\#}$ is the set of nonidentity elements of the group H while 1_H is the principal character of H. $\mathfrak{Q}$ denotes the field of rational numbers and $\mathfrak{C}$ the field of complex numbers.

In §4 frequent use is made of the following result of Schur [5].

(2.1) *Let p be a prime and let P be a finite p-group which has a faithful representation X of degree n over the complex number field. If the character of X is rational-valued, then $|P|\leqq p^{f_p(n)}$ where*

$$f_p(n) = [n/(p-1)]+[n/p(p-1)]+[n/p^2(p-1)]+\cdots.$$

If p is fixed in the discussion, we shall write $f(n)$ for $f_p(n)$.

(2.2) [2, Theorem 2]. *Let ζ be an irreducible complex character of a finite group N. Let A be a relatively prime operator group on N such that A fixes ζ. Then*:

(a) *There exists a unique irreducible character η of AN such that $\eta|A=\zeta$ and* $(\det\eta)(\alpha)=1$ *for all $\alpha\in A$.*

(b) *If η satisfies* (a), *then $\mathfrak{Q}(\eta)=\mathfrak{Q}(\zeta)$, and $\eta(\alpha)$ is a rational integer for every $\alpha\in A$.*

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Definition. The character η that satisfies part (a) of (2.2) is called the *canonical extension* of ζ to AN.

(2.3) *Let ζ be a faithful irreducible complex character of the group N and let A be a cyclic relatively prime operator group on N which fixes ζ. Let $T=C(A)\cap N$ and assume $T\neq N$. Also assume A has odd order and $T=C(\alpha)\cap N$ for all $\alpha\in A^{\#}$. Let χ be the canonical extension of ζ to AN. Then there exist characters λ, ψ of AT/A with λ irreducible such that one of the following occurs*:

(2.3)(a) $\chi|A\times T=\rho\psi+\lambda$.

(2.3)(b) $\chi|A\times T=\rho\psi-\lambda$.

Here ρ is the character of the regular representation of AT/T.

Proof. By [2, Corollary 5(a)], there exist a sign $\varepsilon=\pm 1$ and an irreducible character λ of T such that $\chi(\alpha t)=\varepsilon\lambda(t)$ for all $t\in T$ and all $\alpha\in A^{\#}$ (since A has odd order $\theta_0(\alpha)=1$ for all $\alpha\in A$ in the above reference). Let λ also denote the character of AT/A whose restriction to T is λ. Then by the theory of characters of a direct product, $(\chi|AT)-\varepsilon\lambda$, as a generalized character of $A\times T$, may be expressed as $\sum_{\theta,\mu}c_{\theta,\mu}\theta\mu$ where θ and μ range over the irreducible characters of AT/T and AT/A, respectively, and $c_{\theta,\mu}$ is an integer. But since this function vanishes outside T, it is easily seen from the inner product formula for $c_{\theta,\mu}$ that $c_{\theta,\mu}$ is independent of θ. Hence $(\chi|AT)-\varepsilon\lambda=\rho\psi$ for some generalized character ψ of AT/A. Since $\chi|AT=\rho\psi+\varepsilon\lambda$ is a character of AT, it follows that ψ is 0 or a character of AT/A. If $\psi=0$, then by [2, Corollary 5(b)] $T=N$. Hence $\psi\neq 0$ and (2.3) is proved.

3. **Initial reductions.** We shall describe some of Dixon's results which we may use directly because they do not require the solvability of G. By the proofs of Corollaries 1 and 2 of [1], it suffices to prove the theorem assuming the complete reducibility of the given representation. Further, we may assume that the underlying field $\mathfrak{F}$ is algebraically closed. This done, let G be a counterexample to the theorem of minimal order. Let P denote a fixed Sylow p-subgroup of G with, say, $|P|=p^a$. Let n be the smallest positive integer such that G has a faithful completely reducible representation of degree n over $\mathfrak{F}$ and $|P:O_p(G)|>p^{\lambda_p(n)}$. Let X denote such a representation and let χ be its character. As in [1, pp. 547–548] it may be shown that

(3.1) *X is irreducible and primitive.*

If $\mathfrak{F}$ has characteristic p, X may be lifted to a representation over $\mathfrak{C}$ [6, Theorem 6]. Hence we may assume that $\mathfrak{F}=\mathfrak{C}$ and shall do so from now on.

(3.2) *G has a series of normal subgroups $O_p(G)<N_1<G$ where $N_1/O_p(G)=O_{p'}(G/O_p(G))$ and G/N_1 is a p-group.*

Proof. Let $P_1/N_1=O_p(G/N_1)$ and suppose $P_1\neq G$. Then $PN_1\neq G$ and by induction $O_p(G)<O_p(PN_1)$. Since $P_1\leqq PN_1$, P_1 normalizes $O_p(PN_1)\cap P_1\geqq O_p(G)$. Since $P_1\lhd G$, $O_p(P_1)\lhd G$ and this implies $O_p(G)=O_p(PN_1)\cap P_1$. Hence $O_p(PN_1)P_1/N_1=(O_p(PN_1)N_1/N_1)\times(P_1/N_1)$ which is contradictory to Lemma 1.2.3 of Hall-Higman [3]. Hence $P_1=G$ as desired.

(3.3) $|P:O_p(G)|=p^{\lambda_p(n)+1}$.

Proof. G contains a normal subgroup H of index p containing N_1. Since $O_p(H)\lhd G$, $O_p(H)\leqq O_p(G)$. By minimality of $|G|$, $p^{\lambda_p(n)}\geqq|P\cap H:O_p(H)|\geqq|P:O_p(G)|/p>p^{\lambda_p(n)-1}$, which proves (3.3).

(3.4) *$N_1=O_p(G)\times N$ where N is a normal p-complement of G.*

Proof. By [4, Lemma 1] it suffices to show that $O_p(G)$ is contained in the Frattini subgroup of G. Let M be a maximal subgroup of G not containing $O_p(G)$. Then $G=O_p(G)M$. Now M and $O_p(G)$ normalize $O_p(G)O_p(M)$. Hence $O_p(M)=O_p(G)\cap M$. Let $|M|=p^s g'$ and $|G|=p^a g'$, $p\nmid g'$. By minimality of $|G|$, $|G|=|O_p(G)|\,|M|/|O_p(G)\cap M|\leqq|O_p(G)|p^{\lambda_p(n)}g'$. This contradicts (3.3) and (3.4) is proved.

(3.5) *$G=PN$, $P\cap N=\langle 1\rangle$, $\chi|N$ is irreducible, $O_p(G)=\langle 1\rangle$ and $\lambda_p(n)=a-1$.*

Proof. Suppose that $\chi|N$ is reducible. Since G/N is a p-group, there is a sequence of normal subgroups of G, $G=P_0>P_1>\cdots>P_k=N$, such that $|P_i:P_{i+1}|=p$. Choose i so that $\chi|P_i$ is irreducible and $\chi|P_{i+1}$ is reducible. Then it is well known that $\chi|P_{i+1}$ is a sum of p distinct conjugate characters. This contradicts the primitivity of X and so $\chi|N$ is irreducible. Therefore $p\nmid\chi(1)$ and since $O_p(G)\leqq C(N)$, $O_p(G)\leqq Z(G)$ by Schur's lemma. Hence $\chi|O_p(G)=\chi(1)\lambda$ where λ is a linear character of $O_p(G)$. Since $p\nmid\chi(1)$, $\chi|P$ contains a linear character μ of P and $\mu|O_p(G)=\lambda$. Regarding μ as a character of G/N, we see that $\bar{\mu}\chi$ is a faithful character of $G/O_p(G)$ of degree n. If $O_p(G)\neq\langle 1\rangle$, minimality of $|G|$ yields a contradiction. Hence $O_p(G)=\langle 1\rangle$ and the last statement of (3.5) follows from (3.3).

4. **Completion of the proof.** The canonical extension of $\chi|N$ to G must be faithful since its kernel is a p-group and $O_p(G)=\langle 1\rangle$. Hence we may assume that χ is the canonical extension of $\chi|N$ to G and shall do so from now on. By (2.2)(b) χ is rational-valued on P and so by (2.1), we may assume p is not a Fermat prime. In particular, $p\geqq 7$ and it is easily seen that $f(t)=\lambda_p((p/(p-1))t)$ for any positive rational number t.

We now let w be an element of $Z(P)$ of order p such that if v is any element of $Z(P)$ of order p $(\chi|\langle v\rangle, 1_{\langle v\rangle})_{\langle v\rangle}\leqq(\chi|\langle w\rangle, 1_{\langle w\rangle})_{\langle w\rangle}$. Let $A=\langle w\rangle$ and let $T=C(w)\cap N$. Then P normalizes T.

Let $1=\theta_1,\theta_2,\ldots,\theta_p$ be the distinct linear characters of $A\times T$ whose kernels contain T. Since $w\in Z(PT)$, we may write $\chi|PT=\alpha_1+\cdots+\alpha_p$ where α_i is a character of PT (or $\alpha_i=0$ is possible) such that $\alpha_i|A$ is a multiple of $\theta_i|A$ for $i=1,\ldots,p$. By (2.2)(b), χ is invariant under the Galois group $\mathfrak{G}=\mathrm{Gal}\,(\mathfrak{Q}(\varepsilon)/\mathfrak{Q}(\varepsilon^{p^a}))$ where ε is a primitive $|G|$th root of unity. It follows that $\alpha_2,\ldots,\alpha_p$ is a full set of conjugates of α_2 under this group and $\alpha_2+\cdots+\alpha_p$ as well as α_1 are rational valued on P. The remainder of the proof is split into the two possible cases $\chi|A\times T=\rho\psi+\lambda$ or $\chi|A\times T=\rho\psi-\lambda$ as described in (2.3).

Assuming that the first case holds, we have $\alpha_1|T=\psi+\lambda$ and $\alpha_i|T=\psi$ for $i=2,\ldots,p$. Let $\cup=\ker\,(\alpha_2+\cdots+\alpha_p)\cap P$. We require the

LEMMA. $(\psi, \lambda)_T \neq 0$. *In particular*, $\psi(1) \geqq \lambda(1)$.

Suppose $(\psi, \lambda)_T = 0$. Then $(\chi, \lambda)_T = 1$. $\lambda|T$ is fixed by P because P fixes $\chi|T$ and λ is the only irreducible constituent of $\chi|T$ whose multiplicity is not a multiple of p. It follows from Clifford's theorem that $\chi|PT$ contains a unique irreducible constituent β which is an extension of $\lambda|T$. Because $\chi|PT$ is invariant under $\mathfrak{G}$ so is β and β is therefore rational on P.

Let S be such that $\ker \beta \cap P \leqq S \leqq P$ and $S \ker \beta/\ker \beta = O_p(PT/\ker \beta)$. Then $ST/\ker \beta = S \ker \beta/\ker \beta \times T \ker \beta/\ker \beta$. Since $\beta|T$ is irreducible, $\beta|S = \beta(1)\mu$ for some linear character μ of S. Since β is rational on S, we must have $S \leqq \ker \beta$ and therefore $S = \ker \beta \cap P$. By our induction hypothesis, $|P:S| \leqq p^{\lambda_p(\beta(1))}$. Since $\lambda(1) = \beta(1)$, we have shown $|S| \geqq p^{a-\lambda_p(\lambda(1))}$.

On the other hand, by Schur's theorem (2.1), $|P:\cup| \leqq p^{f((p-1)\psi(1))} = p^{\lambda_p(p\psi(1))} = p^{\lambda_p(n-\lambda(1))}$. Therefore,

$$p^a \geqq |S\cup| = \frac{|S|\,|\cup|}{|S\cap\cup|} \geqq \frac{p^{a-\lambda_p(\lambda(1))}p^{a-\lambda_p(n-\lambda(1))}}{|S \cap \cup|}$$

$$\geqq \frac{p^{2a-\lambda_p(\lambda(1)+n-\lambda(1))}}{|S \cap \cup|} = \frac{p^{a+1}}{|S \cap \cup|}$$

by (3.5). This shows that $|S \cap \cup| \neq 1$. Let u be an element of $S \cap \cup \cap Z(P)$ of order p. By our choice of w, $\lambda(1)+\psi(1) \geqq (\chi, 1_{\langle u\rangle})_{\langle u\rangle} \geqq \lambda(1)+(p-1)\psi(1)$. This contradiction proves the lemma.

Now suppose $\cup \neq \langle 1\rangle$. Let $u \in \cup \cap Z(P)$ have order p. Then $\psi(1)+\lambda(1) \geqq (\chi, 1_{\langle u\rangle})_{\langle u\rangle} \geqq (p-1)\psi(1) \geqq (p-2)\psi(1)+\lambda(1)$ by the lemma. This is a contradiction and so $\cup = \langle 1\rangle$. As in the proof of the lemma, $|P| = |P:\cup| \leqq p^{\lambda_p(n-\lambda(1))} \leqq p^{\lambda_p(n)} = p^{a-1}$. This is a contradiction and the proof in the first case is complete.

Assume now that (2.3)(b) holds for χ and A, i.e., $\chi|A \times T = \rho\psi - \lambda$. Write $\rho = 1+\theta$ where θ is the sum of the nonprincipal linear characters of AT/T. Because $\chi|AT$ is a character of AT, it must be a linear combination of irreducible characters of AT with positive integral coefficients and so λ must be a constituent of $\rho\psi$. Let $r = (\rho\psi, \lambda)_{AT} \geqq 1$. Then $\psi = r\lambda + \psi_1$ where $\psi_1 = 0$ or ψ_1 is a character of AT/A which does not contain λ. Therefore $\chi|AT = (1+\theta)(r\lambda+\psi_1) - \lambda = (r-1)\lambda + r\theta\lambda + \rho\psi_1$. It follows that $\alpha_1|AT = (r-1)\lambda + \psi_1$ and $\alpha_i|AT = r\theta_i\lambda + \theta_i\psi_1$ for $i = 2, \ldots, p$.

Assume first that $\psi_1 \neq 0$. Because $\lambda|T$ is the only irreducible constituent of $\chi|T$ of multiplicity congruent to $-1 \bmod p$, $\lambda|T$ is invariant under P. Therefore by Clifford's theorem, we may write, for $i > 1$, $\alpha_i = \mu_i + \nu_i$ where μ_i and ν_i are characters of PT such that $\mu_i|T = r\lambda$ and $\nu_i|T = \psi_1$. Since T is a p'-group, $\lambda|T$ is invariant under $\mathfrak{G}$. It follows that $\mu_2, \ldots, \mu_p$ is a full set of conjugates under $\mathfrak{G}$ and that $\mu = \mu_2 + \cdots + \mu_p$ is rational valued on P. Since χ, α_1 and μ are rational on P, $\chi - \alpha_1 - \mu = \nu_2 + \cdots + \nu_p = \nu$ is rational on P. Let $\cup = \ker \mu \cap P$, $V = \ker \nu \cap P$ and $S = \ker \alpha_1 \cap P$. By (2.1), $|\cup| \geqq p^{a-f((p-1)r\lambda(1))}$, $|V| \geqq p^{a-f((p-1)\psi_1(1))}$ and $|S| \geqq p^{a-f(\alpha_1(1))}$. By our choice of w, we must have $S \cap \cup = \langle 1\rangle$ and $S \cap V = \langle 1\rangle$. This leads to two inequalities.

First,

$$1 = |S \cap U| = |S|\,|U|/|SU| \geqq p^{2a-\{f(\alpha_1(1))+f((p-1)r\lambda(1))\}}/p^a.$$

Hence,

$$\begin{aligned} 0 &\geqq a-\{f(\alpha_1(1))+f((p-1)r\lambda(1))\} \geqq a-f(\alpha_1(1)+(p-1)r\lambda(1)) \\ &= a-\lambda_p((p/(p-1))\alpha_1(1)+pr\lambda(1)). \end{aligned}$$

Since $\lambda_p(n)=a-1$ by (3.5), this implies $(p/(p-1))\alpha_1(1)+pr\lambda(1)>n$. Because $\alpha_1(1)=(r-1)\lambda(1)+\psi_1(1)$ and $n=(pr-1)\lambda(1)+p\psi_1(1)$, this inequality reduces to

$$(pr-1)\lambda(1) > p(p-2)\psi_1(1). \tag{1}$$

Second,

$$1 = |S \cap V| = |S|\,|V|/|SV| \geqq p^{a-f(\alpha_1(1))}p^{a-f((p-1)\psi_1(1))}/p^a.$$

Hence

$$\begin{aligned} 0 &\geqq a-\{f(\alpha_1(1))+f((p-1)\psi_1(1))\} \geqq a-f(\alpha_1(1)+(p-1)\psi_1(1)) \\ &= a-\lambda_p((p/(p-1))\alpha_1(1)+p\psi_1(1)). \end{aligned}$$

This implies $(p/(p-1))\alpha_1(1)+p\psi_1(1)>n$ which reduces to

$$p\psi_1(1) > (p^2r-2pr+1)\lambda(1). \tag{2}$$

Combining (1) and (2), we have $(p^2r-2pr+1)\lambda(1)(p-2)<p(p-2)\psi_1(1)<(pr-1)\lambda(1)$ and hence $(p^2r-2pr+1)(p-2)<pr-1$. This is equivalent to $(p^3r-4p^2r)+(p+3pr-1)<0$ which is impossible both terms on the left being positive.

Finally, take $\psi_1=0$. Then $\alpha_1|A\times T=(r-1)\lambda$, $\alpha_i|A\times T=r\theta_i\lambda$ for $i>1$ and $\chi|T=(pr-1)\lambda$. In particular, λ is a faithful character of T. Therefore $\ker\alpha_1$ is a p-group and setting $R=O_p(PT)$, we have $|P:R|\leqq p^{\lambda_p(\alpha_1(1))}$ or $|R|\geqq p^{a-\lambda_p(\alpha_1(1))}$.

We have $(\chi|A, 1_A)_A=\alpha_1(1)<\alpha_2(1)$. By our choice of w, $\ker\alpha_2\cap P=\langle 1\rangle$. It follows that $\alpha_2+\cdots+\alpha_p$ is the character of a faithful matrix representation Y of PT over $\mathfrak{C}$ with $\alpha_2+\cdots+\alpha_p$ rational on P. By a suitable choice of basis of the underlying vector space we may assume that $Y(t)=A(t)\otimes I_{(p-1)r}$. Here $A(t)$ is an irreducible matrix representation of T with character λ and I_s denotes the $s\times s$ identity matrix. Since R centralizes T, it is easily verified using Schur's lemma that $Y(r)=I_{\lambda(1)}\otimes B(r)$, $r\in R$, where $B(r)$ is a faithful representation of R with character $(\alpha_2+\cdots+\alpha_p)/\lambda(1)$. By (2.1), $|R|\leqq p^{f((p-1)r)}=p^{\lambda_p(pr)}$. Combining this with our previous inequality, we get $a\leqq\lambda_p(\alpha_1(1))+\lambda_p(pr)\leqq\lambda_p((r-1)\lambda(1)+pr)$. By (3.5), $(r-1)\lambda(1)+pr>n=(r-1)\lambda(1)+(p-1)r\lambda(1)$ or $pr>(p-1)r\lambda(1)$. This implies $\lambda(1)<p/(p-1)<2$ and so $\lambda(1)=1$. But then $\chi|T=\chi(1)\lambda$ and $T=Z(G)$. Since w acts fixed-point-free on N/T, a well-known result of Thompson yields that N/T is nilpotent. G is therefore solvable and the result of [1] completes the proof.

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