LOCALLY FLAT IMBEDDINGS OF
TOPOLOGICAL MANIFOLDS IN CODIMENSION THREE

BY

GLENN P. WELLER[1]

Abstract. This paper presents an imbedding theorem for one topological manifold M^n in another topological manifold Q^q, provided that the codimension $(q-n)$ is at least three. The result holds even if the manifolds are of the recently discovered non-piecewise-linear type. Denote the boundaries of M and Q by $\dot{M}$ and $\dot{Q}$ respectively. Suppose that M is $2n-q$ connected and Q is $2n-q+1$ connected. It is then proved that any map $f: (M, \dot{M}) \to (Q, \dot{Q})$ such that $f|\dot{M}$ is a locally flat imbedding is homotopic relative to $\dot{M}$ to a proper locally flat imbedding $g: M \to Q$. It is also shown that if M is closed and $2n-q+1$ connected and Q is $2n-q+2$ connected, then any two homotopic locally flat imbeddings are locally flatly concordant.

Introduction. The category of topological manifolds has recently been shown to include manifolds possessing no piecewise linear (P.L.) structure [8]. These non-P.L. manifolds must be investigated in their own right. One approach to the study of topological manifolds is to obtain results analogous to those already known for differentiable and P.L. manifolds.

In the study of imbeddings of manifolds, one of the principal results in the P.L. category is the following:

IRWIN'S IMBEDDING THEOREM [7]. *If $n \leq q-3$, M^n is a compact, $2n-q$ connected P.L. manifold, Q^q is a $2n-q+1$ connected P.L. manifold and $f: (M, \dot{M}) \to (Q, \dot{Q})$ is a map such that $f|\dot{M}$ is a P.L. imbedding, then $f \sim g$ rel $\dot{M}$ where $g: M \to Q$ is a proper P.L. imbedding.*

This theorem is related to theorems of Haefliger [4] in the differentiable category, although he usually has the dimension requirement that $2q \geq 3(n+1)$. This paper presents a proof of the analog of Irwin's theorem for topological manifolds.

The map $g: M \to Q$ is proved to be a locally flat imbedding. A special case of this theorem was proved by Jack Lees [10] when $\dot{M} = \varnothing$ and Q is euclidean space.

The proof of Irwin's theorem uses the technique of engulfing. Two engulfing theorems for topological manifolds have recently been proved by Newman [13] and Lees [9] and are used extensively here. Furthermore, Miller's P.L. approximation theorem [11] is crucial to this paper.

Since topological manifolds have P.L. structures locally, P.L. techniques are applied in coordinate neighborhoods. The proof of Theorem 1 then follows by induction on the number of coordinate neighborhoods needed to cover the manifold M, with special consideration needed for the last such neighborhood.

1. **Some preliminary definitions and lemmas.** Let E^n denote n dimensional euclidean space and let H^n denote the half-space of all points whose last coordinate is nonnegative. An n dimensional topological manifold is a separable Hausdorff space M^n in which every point has an open neighborhood homeomorphic to E^n or H^n. If U is an open set in M and Ψ is a homeomorphism of E^n or H^n onto U, the pair (U, Ψ) is called a coordinate neighborhood. The symbol $|\Psi|$ will often be used to denote the image of Ψ when the set U is suppressed. In this case it is understood that Ψ is the relevant homeomorphism.

Let $|\Psi|$ be a coordinate neighborhood where the domain of Ψ is H^n. The euclidean space E^{n-1} is viewed as a subset of H^n in the obvious fashion and the points in ΨE^{n-1} are called boundary points of M. The set of all boundary points of M is an $n-1$ manifold called the boundary of M and denoted $\dot{M}$. Of course, $\dot{M}$ may be empty, but every manifold will be allowed to have a boundary unless the contrary is indicated. The interior of M is $M - \dot{M}$ and is denoted $\mathring{M}$.

Let $\alpha > 0$. The cube $\{x \mid |x_i| \leqq \alpha\}$ in E^n is denoted $C(\alpha)$. If $\Psi: E^n \to |\Psi|$ is a coordinate neighborhood, the symbol $|\Psi|_\alpha$ will denote $\Psi C(\alpha)$. Note that $|\Psi|$ is open while $|\Psi|_\alpha$ is closed. In the case that $\Psi: H^n \to |\Psi|$, the symbol $|\Psi|_\alpha$ will denote $\Psi[C(\alpha) \cap H^n]$.

Any topological manifold M can be imbedded as a closed subset of a sufficiently high dimensional euclidean space [12] and hence has a metric, which will usually be denoted d.

Various standard symbols will be used throughout this paper. For instance, $\overline{A}$ will denote the closure of the subspace A in a given space, and " $\sim$ " will mean "is homotopic to", and " $\cong$ " will mean "is isomorphic to".

If Q is a manifold with boundary, $\dot{Q}$ has a boundary collar [1]. That is, there is an open set U in Q, $\dot{Q} \subset U$, and there is a homeomorphism $\Psi: \overline{U} \to \dot{Q} \times [0, 1]$ such that if $x \in \dot{Q}$, $\Psi x = (x, 0)$. Of course, $[0, 1]$ may be replaced by any closed interval. The next complicated lemma provides rather nice boundary collars and will be useful in homotoping maps which are imbeddings on the boundary to maps which are imbeddings near the boundary. In the lemma M_+ denotes the manifold obtained from the disjoint union of $M \times 1$ and $\dot{M} \times I$ by gluing along the common subspace $\dot{M} \times 1$. If M is P.L., so is M_+.

LEMMA 1. *There is a homotopy $F: M \times I \to M_+$ so that*

(1) *$F_0: M \to M_+$ is a homeomorphism onto with $F_0 x = (x, 0)$ for $x \in \dot{M}$.*

(2) *$F_t: M \to M_+$ is a homeomorphism into:*

(3) *$F_t x = (x, t)$ for $x \in \dot{M}$.*

(4) $F_1(x) = (x, 1)$ *for all* $x \in M$.

(5) *If* $V = F_0^{-1}(\dot{M} \times [0, 1])$, *then* $F_0 | \overline{V} : \overline{V} \to \dot{M} \times [0, 1]$ *is a boundary collar.*

There is also a homotopy $G : M_+ \times I \to M$ *such that*

(1') $G_0 = F_1^{-1}$,

(2') $G_t(x, s) = x$ *if* $x \in \dot{M}$, $s \leq t$,

(3') $G_1(x, 1) = x$ *for all* $x \in M$,

(4') $G_1(x, t) = x$ *for all* $x \in \dot{M}$.

Furthermore, the boundary collar $\overline{V}$ *can be prescribed in advance. If* M *and* $\overline{V}$ *are P.L., then* F_0 *and* G_0 *can be chosen to be P.L.*

Proof. Take a boundary collar $\overline{U} \xrightarrow{\beta} \dot{M} \times [1, 3]$ where $\beta x = (x, 1)$ if $x \in \dot{M}$. If $\overline{V}$ is given in advance, find a larger collar $\overline{U}$ as above with $\overline{V} = \beta^{-1}(\dot{M} \times [1, 2])$. If $\overline{V}$ is a P.L. collar, $\overline{U}$ can be made a P.L. collar [6].

For each $0 \leq t \leq 1$ define $h_t : [1, 3] \to [t, 3]$ to be the P.L. homeomorphism defined by setting $h_t(3) = 3$, $h_t(2) = 1 + t$, $h_t(2 - t) = 1$, $h_t(1) = t$ and extending linearly. This defines

$$h : [1, 3] \times I \to [0, 3].$$

Similarly define $k_t : [0, 3] \to [1, 3]$ to be the P.L. map obtained by setting $k_t(3) = 3$, $k_t(1) = 2 - t$, $k_t(t) = 1$, $k_t(0) = 1$ and extending linearly. This defines $k : [0, 3] \times I \to [1, 3]$. Define

$$F : M \times I \to M_+, \qquad G : M_+ \times I \to M$$

by

$$
\begin{aligned}
F(x, t) &= (x, 1), & x &\notin U, \\
&= (\beta^{-1}(1 \times h_t)\beta x, 1), & x &\in \beta^{-1}(\dot{M} \times [2 - t, 3]), \\
&= (1 \times h_t)\beta x, & x &\in \beta^{-1}(\dot{M} \times [1, 2 - t]); \\
G(x, s, t) &= x, & x &\notin U, \quad s = 1, \\
&= \beta^{-1}(1 \times k_t)\beta x, & x &\in \overline{U}, \quad s = 1, \\
&= \beta^{-1}(x, k_t(s)), & x &\in \dot{M}, \quad s \geq t, \\
&= x, & x &\in \dot{M}, \quad s \leq t.
\end{aligned}
$$

It is routine that all the desired properties hold. Furthermore, if $H_t = G_0 F_t$, the proof yields the following:

LEMMA 2. *There is an isotopy* H_t *of* M *into itself such that*

(1) $H_0 = 1_M$.

(2) $H_1 M \subset \mathring{M}$.

(3) *For some neighborhood* U *of* $\dot{M}$, $H_t | M - U = 1$.

At this point some preliminary steps toward proving the main result of this paper can be made. But first some definitions must be given.

DEFINITION 1. Let M^n and Q^q be topological manifolds and $f: M \to Q$ be a map. f is said to be proper if $f\dot{M} \subset \dot{Q}$ and $f\overset{\circ}{M} \subset \overset{\circ}{Q}$. f is said to be strictly proper if f is proper and there are boundary collars $\varphi: \dot{M} \times [0, 1] \to \overline{U}$ and $\Psi: \dot{Q} \times [0, 1] \to \overline{V}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
U & \xrightarrow{\quad f|U \quad} & V \\
\uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\Psi} \\
\dot{M} \times [0, 1) & \xrightarrow{(f|\dot{M}) \times 1} & \dot{Q} \times [0, 1)
\end{array}
$$

f is said to be locally flat if $f\overset{\circ}{M} \subset \overset{\circ}{Q}$ and for each $x \in M$ there exist neighborhoods (U, V) of (fx, x) in (Q, M) such that f is 1-1 on V and the pair (U, fV) is homeomorphic to (E^q, E^n) if $x \in \overset{\circ}{M}, fx \in \overset{\circ}{Q}$; to (H^q, H^n) if $x \in \dot{M}, fx \in \dot{Q}$; or to (E^q, H^n) if $x \in \dot{M}, fx \in \overset{\circ}{Q}$.

The concept of local flatness is related to certain concepts in the differentiable and P.L. categories. If M and Q are differentiable manifolds and f is a differentiable immersion, then f is locally flat by the Implicit Function Theorem [12]. If M and Q are P.L. manifolds and f is a proper P.L. imbedding, then f is called locally unknotted if for every point $x \in M$ there are closed neighborhoods (U, V) of (fx, x) in (Q, M) and a P.L. homeomorphism $h: (U, fV) \to (B^q, B^n)$, where (B^q, B^n) is the standard ball-pair. In [16], Zeeman proved that if $q - n \geq 3$, then f is locally unknotted. Hence, the following lemma is clear.

LEMMA 3. *If f is a P.L. imbedding of $C(1)$, the cube in E^n, into E^q where $q - n \geq 3$, then $f|\overset{\circ}{C}(1)$ is a locally flat imbedding.*

Another useful fact about locally flat maps is stated in the following lemma. In this paper a simplex is always a closed simplex.

LEMMA 4. *Let T be a triangulation of $C(1) \subset E^n$ with K a subcomplex of T. Let M^n and Q^q be topological manifolds and $f: M \to Q$ a proper map such that for some open subset O of $\overset{\circ}{M}, f|O$ is a locally flat imbedding. Then for any coordinate homeomorphism $\Psi: E^n \to |\Psi| \subset M$ such that $\Psi K \subset O$, $f\Psi|K$ is an imbedding which is locally flat on every simplex of K.*

Proof. Let Δ^i be a simplex in K and $x \in \Delta$. By hypothesis there are open neighborhoods (W, V') of $(f\Psi x, \Psi x)$ in $(\overset{\circ}{Q}, O)$ and a homeomorphism

$$
k: (W, fV') \to (E^q, E^n).
$$

Suppose first that $x \in \overset{\circ}{\Delta}$. Then there are open neighborhoods (V, U) of x in (E^n, Δ) with $U \subset V$, $\Psi V \subset V' \subset O$ and a homeomorphism $h: (V, U) \to (E^n, E^l)$. Then

$$(E^q, E^n, E^l) = (E^n \times E^{q-n}, E^n, E^l) \xrightarrow{h^{-1} \times 1} (V \times E^{q-n}, V, U)$$

$$\xrightarrow{(kf\Psi) \times 1} (kf\Psi V \times E^{q-n}, kf\Psi V, kf\Psi U).$$

Since V is open in E^n and $kf\Psi: E^n \to E^n$ is a homeomorphism, $kf\Psi V$ is open in E^n by the Theorem on Invariance of Domain. Hence $kf\Psi V \times E^{q-n}$ is open in E^q and $W' = k^{-1}(kf\Psi V \times E^{q-n})$ is open in $\overset{\circ}{Q}$. Thus, there are homeomorphisms $(E^q, E^l) \to (kf\Psi V \times E^{q-n}, kf\Psi U) \xrightarrow{k^{-1}} (W', f\Psi U)$.

If $x \in \dot{\Delta}$, then hU can be chosen to be H^l and (E^q, H^l) is homeomorphic to $(W', f\Psi U)$ as before. In either case, $f\Psi | \Delta$ is locally flat.

The following two lemmas establish the initial setup for the inductive argument which appears in §3.

LEMMA 5. *Let $f: (M, \dot{M}) \to (Q, \dot{Q})$ be a map such that $f | \dot{M}$ is a locally flat imbedding into $\dot{Q}$. Then there is a map $g: M \to Q$ such that*

(1) *with respect to open boundary collars U and V in M and Q respectively, g is strictly proper,*

(2) *$g | U$ is a locally flat imbedding,*

(3) *$g \sim$ rel $\dot{M}$.*

Furthermore, U may be prescribed in advance provided $\overline{U}$ is a closed boundary collar in M.

Proof. Use Lemma 1 twice to obtain U and F for M and V and G for Q with the properties listed in Lemma 1. Of course, U can be prescribed in advance. Set $g = G_0(f \times 1)F_0$. Schematically

$$M \xrightarrow{F_0} M_+ \xrightarrow{f \times 1} Q_+ \xrightarrow{G_0} Q.$$

Then $g \sim G_1(f \times 1)F_0 \sim G_1(f \times 1)F_1 = f$ and all homotopies are relative to $\dot{M}$.

The remainder of the proof will demonstrate that g is strictly proper and that $g | U$ is a locally flat imbedding. Clearly $g | U$ is an imbedding. Let $gx = y$ where $x \in U$. Let $F_0 x = (x', t)$, $x' \in \dot{M}$, $0 \le t \le 1$, $fx' = y'$. Then $y = gx = G_0(f \times 1)F_0 x = G_0(y', t)$. Since $f | \dot{M}$ is locally flat, there are neighborhoods U' and V' of x' and y' in $\dot{M}$ and $\dot{Q}$ respectively and homeomorphisms h, k such that the following diagram commutes, where i is the standard inclusion:

$$
\begin{array}{ccc}
U' & \xrightarrow{f|U'} & V' \\
\uparrow{\scriptstyle h} & & \uparrow{\scriptstyle k} \\
E^{n-1} & \xrightarrow{i} & E^{q-1}
\end{array}
$$

Then the following diagram commutes:

$$
\begin{array}{ccc}
U & \xrightarrow{\;g\,|\,U\;} & V \\[4pt]
\downarrow{\scriptstyle F_0} & & \uparrow{\scriptstyle G_0} \\[4pt]
\dot{M}\times[0,1] & \xrightarrow{\;f\times 1\;} & \dot{Q}\times[0,1] \\[4pt]
\uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} \\[4pt]
U'\times[0,1] & \xrightarrow{\;f\times 1\;} & V'\times[0,1] \\[4pt]
\uparrow{\scriptstyle h\times 1} & & \uparrow{\scriptstyle k\times 1} \\[4pt]
E^{n-1}\times[0,1] & \xrightarrow{\;i\times 1\;} & E^{q-1}\times[0,1]
\end{array}
$$

Since $i\times 1$ is obviously locally flat, so is $g\,|\,U$.

LEMMA 6. *Let $n\leqq q$ and let M^n and Q^q be topological manifolds with $\dot{M}=\varnothing$. Let $f\colon M\to Q$ be a map and $x_0\in M$. Then there is a map $g\colon M\to\dot{Q}$ and an open neighborhood U of x_0 such that $g\sim f$ and $g\,|\,U$ is a locally flat imbedding.*

Proof. By Lemma 2, after a preliminary homotopy of f, it can be assumed that $fM\subset\dot{Q}$. Let $\varepsilon>0$ be chosen so that any ε-approximation to f is homotopic to f. Let $|\varphi|$ be a coordinate neighborhood in $\dot{Q}$ at fx_0 of diameter $<\varepsilon$. Let $|\Psi|$ be a coordinate neighborhood at x_0 such that $f|\overline{\Psi}|\subset|\varphi|$.

Let $i\colon E^n\to E^q$ be the standard inclusion. Let μ,ν be a partition of unity on E^n subordinate to the open cover $\mathring{C}(2)$, $E^n-C(1)$. Let $U=|\Psi|_1$. Define $F\colon E^n\to E^q$ by $F(x)=\mu(x)i(x)+\nu(x)\varphi^{-1}f\Psi(x)$. Then $F|\mathring{C}(1)=i|\mathring{C}(1)$ and $F|E^n-C(2)=\varphi^{-1}f\Psi$. Define $g\colon M\to Q$ by

$$
\begin{aligned}
g(x) &= \varphi F\Psi^{-1}(x), & x &\in|\Psi|, \\
     &= f(x), & x &\notin|\Psi|.
\end{aligned}
$$

Then $g\,|\,U$ is a locally flat imbedding, $d(f,g)<\varepsilon$, and $f\sim g$.

GENERAL ASSUMPTION. Throughout the remainder of this paper all manifolds will be assumed to be connected unless otherwise stated.

2. **The main geometrical tools.** This section contains the main geometrical tools used to prove the principal result of this paper. Particular attention is given to engulfing theorems of Newman and Lees as well as to the approximation theorem of Miller. Specific applications of these important principles are then obtained. All of these tools are essential to the remaining sections of the paper.

DEFINITION 2. A subset X of a topological manifold M^n is called locally tame of dimension p if there is a covering of X by open neighborhoods $|\Psi_i|_1$ such that $\Psi_i^{-1}(X\cap|\Psi_i|_1)$ is a polyhedral set in E^n of dimension p.

NEWMAN'S ENGULFING THEOREM [13]. *Let V be an open set in a topological manifold M^n. Suppose that either of the following conditions holds:*

(1) *M is p-connected, V is $p-1$ connected.*

(2) *The pair (M, V) is p-connected.*

Then every locally tame closed set X of dimension $p \leq n-3$ such that $X-V$ is compact can be engulfed by V. That is, there is a homeomorphism $h: M \to M$ isotopic to the identity and fixed outside a compact set such that $X \subset hV$.

LEES' ENGULFING THEOREM [9]. *Let M^q be a $2n-q+2$ connected topological manifold without boundary, let L be an n-complex, $q-n \geq 3$ and let $f: L \to M$ be an imbedding, locally flat on every simplex of L.*

Then if fL is inessential in M, fL is contained in a coordinate neighborhood.

Outline of Lees' proof. Let K be a triangulation of the cone on L in which L is a subcomplex. Since fL is inessential in M, f extends to a map $f: K \to M$. Let $K = K_r \searrow K_{r-1} \searrow \cdots \searrow K_0 = \mathrm{pt.}$ represent the collapse of the cone where K is chosen so that if $K_i \subset K_{i-1} \cup a_i A_i$, $a_i \dot{A}_i \subset K_{i-1}$, then $fa_i A_i \subset |\Psi_i|$ for $1 \leq i \leq r$. It is important to note that the coordinate neighborhoods $|\Psi_i|$ are indexed as shown but otherwise arbitrarily chosen.

Maps $f_i: K \to M$ are then inductively constructed such that

(1) $f_i = f$ on L,

(2) $f_i a_j A_j \subset |\Psi_j|$, $1 \leq j \leq r$,

(3) $f_i K_i$ is contained in a coordinate neighborhood.

Then $g = f_r$ is defined. In particular, $gK \subset |\Psi|$ for some coordinate neighborhood. Hence, $gL = fL \subset |\Psi|$.

PROPOSITION 1. *If $n \leq q-2$, D^n is a P.L. n-disk, Q^q is a $2n-q$ connected topological manifold without boundary and $f: D \to Q$ is a map such that $f|\dot{D}$ is an imbedding which is locally flat on every simplex of some triangulation L of $\dot{D}$, then $f \sim g$ rel $\dot{D}$ where gD is contained in a coordinate neighborhood on Q.*

Proof. Note that $\dim L = n-1$, Q is $2(n-1)-q+2$ connected and $q-(n-1) \geq 3$. Extend the triangulation L to a triangulation K of D as the cone on L. Now there is an $\varepsilon > 0$ so that any ε-approximation to f which agrees with f on L is homotopic to f relative to L.

The proof of Lees' Engulfing Theorem is now repeated, following the outline above. Care is taken that K is triangulated finely enough and the coordinate neighborhoods $|\Psi_i|$ are chosen to have diameter $< \varepsilon$. Lees' proof produces $g: K \to M$ such that

(1) $g|L = f|L$,

(2) $ga_i A_i \subset |\Psi_i|$, $fa_i A_i \subset |\Psi_i|$, $1 \leq i \leq r$,

(3) $gD = gK \subset |\Psi|$ for some coordinate neighborhood $|\Psi|$ on Q.

From (2) $d(f, g) < \varepsilon$ follows. From (1) and the choice of ε, $f \sim g$ rel $\dot{D}$.

PROPOSITION 2. *If $n \leq q-2$, M^n is a topological manifold, Q^q is a $2n-q$ connected topological manifold and $f: (M, \dot{M}) \to (Q, \dot{Q})$ is a proper map such that, for some coordinate neighborhood $|\Psi|$ in $\dot{M}$ and some $0 < \rho' < 1$, $f|M - |\Psi|\rho'$ is a locally flat imbedding, then for any $1 > \rho > \rho'$ there is a proper map $g: M \to Q$ such that*

(1) *$g = f$ off $|\Psi|\rho$,*

(2) *$g|\Psi|\rho \subset |\varphi|$ for some coordinate neighborhood $|\varphi|$ on $\dot{Q}$,*

(3) *$f \sim g$ rel $\dot{M}$.*

Proof. Triangulate $C(1)$ with the subcomplex L triangulating $\dot{C}(\rho)$. Then $\Psi L \subset M - |\bar{\Psi}|\rho'$, an open set on which f is a locally flat imbedding. By Lemma 4, $f\Psi|L$ is locally flat on every simplex. By Proposition 1, $f\Psi|C(\rho) \sim \alpha$ rel $\dot{C}(\rho)$ via a homotopy $H: C(\rho) \times I \to Q$ where $\alpha C(\rho) \subset |\varphi|$ for some coordinate neighborhood $|\varphi|$ on $\dot{Q}$, which is also $2n-q$ connected. Define $G: M \times I \to Q$ by

$$G(x, t) = H(\Psi^{-1}x, t), \qquad x \in |\Psi|\rho,$$
$$= f(x), \qquad x \notin |\Psi|\rho.$$

G is well defined since if $x \in |\dot{\Psi}|\rho$, $H(\Psi^{-1}x, t) = f\Psi(\Psi^{-1}x) = fx$. So G is a homotopy from f to a map $G_1 = g$ which clearly satisfies (1) and (3). Finally $g|\Psi|\rho = \alpha\Psi^{-1}|\Psi|\rho = \alpha C(\rho) \subset |\varphi|$.

The next tools to be discussed here are the general position theorems.

FIRST GENERAL POSITION THEOREM [6]. *Let $X_0 \subset X^n$ and Y^k be polyhedra and Q^q a P.L. manifold with Y a subcomplex of Q. Let $f: X \to Q^q$ be a P.L. imbedding such that $f(X - X_0) \subset \dot{Q}$. Then there is an arbitrarily small ambient P.L. isotopy sending f into g and keeping $\dot{Q} \cup fX_0$ fixed, such that $g|X - X_0$ is in general position with respect to Y. That is,*

$$\dim g(X - X_0) \cap Y \leq n + k - q.$$

SECOND GENERAL POSITION THEOREM [6]. *Let $X_0 \subset X^n$ be polyhedra and Q^q a P.L. manifold. Let $f: X \to Q$ be a map such that $f|X_0$ is a P.L. imbedding. Suppose $2q > 3(n+1)$. Then $f \sim g$ via an arbitrarily small homotopy keeping X_0 fixed such that g is P.L. and in general position. That is, $\dim S_2(g) \leq 2n - q$ and $S_3(g) = \varnothing$ where $S_r(g)$ denotes the closure of $\{x \in X \mid g^{-1}gx$ contains at least r points$\}$.*

The final tools of this section are concerned with approximating maps by P.L. maps.

THEOREM OF MILLER [11]. *Suppose $q \geq 5$, $q \geq n+3$, M^n is a compact P.L. manifold, $\hat{M}^n$ is a compact P.L. submanifold of M, and Q^q is a P.L. manifold. Let $g: M \to \dot{Q}$ be a locally flat imbedding such that $g|\hat{M}$ is P.L. Let N be a neighborhood of $g(M - \hat{M})$ in Q. Then for $\varepsilon > 0$, there is an ambient ε-isotopy H_t of Q that is fixed outside N such that $H_1 g$ is a P.L. imbedding.*

PROPOSITION 3. *Suppose $q \geq 5$, $q \geq n+3$, M^n and Q^q are P.L. manifolds, and C is a compact subset of M. Suppose $g: M \to \dot{Q}$ is a map such that $S_3(g) = \varnothing$, $S_2(g)$ is compact, g is P.L. near $S_2(g)$, and $g|M - S_2(g)$ is a locally flat imbedding. Then*

for any $\varepsilon > 0$ there is an ε-homeomorphism $h\colon Q \to Q$, fixed on $S_2(g)$ and outside a compact set, such that hg is P.L. near C.

Proof. Let N_1 be a regular neighborhood of $S_2(g)$ near which g is P.L. Let N_2 be a regular neighborhood of N_1 on which g is P.L. Let N_3 be a regular neighborhood of $N_2 \cup C$. Then apply Miller's Theorem to $M_1 = N_3 - \mathring{N}_1$ and $\hat{M}_1 = N_2 - \mathring{N}_1$. Let $h = H_1$.

3. The inductive argument. The first major step in the proof of this paper's main result, Theorem 1, proceeds by induction on the number of coordinate neighborhoods in a special cover of M. This section's first proposition isolates the inductive step.

Notation. Let $\alpha > 0$ and let $\{|\Psi_i|\}_i$ be a collection of coordinate neighborhoods in $\mathring{M}$ indexed by the integers. Denote

$$\Psi^q(\alpha) = \bigcup_{i \leq q} |\Psi_i|_\alpha, \qquad \mathring{\Psi}^q(\alpha) = \bigcup_{i \leq q} |\mathring{\Psi}_i|_\alpha.$$

PROPOSITION 4. *Suppose M^n and Q^q are topological manifolds, M^n is $2n - q$ connected and $q \geq n + 3$. Let $\varepsilon > 0$ and $0 < \alpha < 1$ be given.*

Suppose that for some integer $k \geq 1$, $f_{k-1}\colon M \to Q$ is a map such that

(1) *$f_{k-1}^{-1}C$ is compact for every compact subset C of Q.*

(2) *There is a locally finite collection $\{|\Psi_i|\}_{i \leq k}$ of coordinate neighborhoods in $\mathring{M}$ and a coordinate neighborhood $|\varphi|$ in $\mathring{Q}$ such that $f_{k-1}|\overline{\Psi}_k| \subset |\mathring{\varphi}|_1$.*

(3) *There are numbers δ and σ with $\alpha < \delta < 1$ and $\frac{1}{2} \leq \sigma < 1$ and a homeomorphism fixed outside a compact set $j_{k-1}\colon M \to M$ such that $f_{k-1}|\mathring{\Psi}^{k-1}(\delta) - j_{k-1}|\Psi_1|_\sigma$ is a locally flat imbedding.*

Then there is a map $f_k\colon M \to Q$, a homeomorphism $j_k\colon M \to M$ fixed outside a compact set and numbers β and ρ such that

(1) *$\alpha < \beta < \delta, \sigma < \rho < 1$,*

(2) *$j_k = j_{k-1}$ on $|\Psi_1|_\sigma$,*

(3) *$f_k = f_{k-1}$ on $\mathring{\Psi}^{k-1}(\beta) - j_k|\Psi_1|_\rho$,*

(4) *$f_k = f_{k-1}$ off $f_{k-1}^{-1}|\varphi|_2$,*

(5) *$f_k^{-1}C$ is compact for every compact subset C of Q,*

(6) *$f_k|\mathring{\Psi}^k(\beta) - j_k|\Psi_1|_\rho$ is a locally flat imbedding,*

(7) *$d(f_{k-1}, f_k) < \varepsilon$, where d is the metric on Q induced from a fixed imbedding of Q as a closed subset of E^s for some s.*

The proof of Proposition 4 is very similar to the proof of Proposition 1 in [10]. The idea of the proof is the same. Technical difficulties are encountered since euclidean space E^q is here replaced by an arbitrary manifold Q^q. The dimension requirement $q \geq n + 3$ is improved here because Miller's Theorem is now available. The details of the proof can be found in [16].

The next proposition actually accomplishes the promised inductive argument, wherein everything except one coordinate neighborhood is imbedded. The extension of the imbedding to the last neighborhood is then deferred until §5.

PROPOSITION 5. *If $q \geq n+3$, M^n and Q^q are topological manifolds such that M is compact and $2n-q$ connected, and $f: (M, \dot{M}) \to (Q, \dot{Q})$ is a map such that $f|\dot{M}$ is a locally flat imbedding, then $f \sim g$ rel $\dot{M}$ where $g: M \to Q$ is a strictly proper map and $g|M - |\Psi|_\rho$ is a locally flat imbedding, where $|\Psi|$ is a coordinate neighborhood in $\dot{M}$ and $0 < \rho < 1$.*

Proof. By Lemmas 5 and 6, after a preliminary homotopy of f, f may be assumed to be strictly proper. Also it may be assumed that there is an open neighborhood U of $\dot{M}$ or of some $x_0 \in M$ if $\dot{M} = \varnothing$ such that $f|U$ is locally flat. Let $0 < \alpha < 1$ and $\{|\Psi_i| \mid i \leq r\}$ be a locally finite family of coordinate neighborhoods in $\dot{M}$ such that

(1) $\{|\Psi_i|_\alpha \mid i \leq r\}$ covers $\dot{M}$,

(2) $\overset{\circ}{\Psi}{}^1(1) \subset U$,

(3) $\overset{\circ}{\Psi}{}^1(\alpha) \supset \overline{V} - \dot{M}$ where $\dot{M} \subset V \subset \overline{V} \subset U$ or $x_0 \in V \subset \overline{V} \subset U$,

(4) $f|\overline{\Psi}_i| \subset |\hat{\varphi}_i|_1$ for some coordinate neighborhood $|\varphi_i|$ on $\dot{Q}$.

By induction define maps $f_i: M \to Q$ for $0 \leq i \leq r$ such that

(A) $f_i \sim f$ rel $\dot{M}$.

(B) $f_i|\overline{\Psi}_j| \subset |\hat{\varphi}_j|_1$ for $j \leq r$.

(C) There are numbers δ_i and σ_i with $\alpha < \delta_i < 1$, $\frac{1}{2} \leq \sigma_i < 1$ and a homeomorphism $j_i: M \to M$, fixed outside a compact set, such that $f_i|\overset{\circ}{\Psi}{}^i(\delta_i) - j_i|\Psi_1|_{\sigma_i}$ is a locally flat imbedding.

Set $f_0 = f$, $\alpha < \delta_0 < 1$, $\sigma_0 = \frac{1}{2}$, $j_0 = 1_M$. Now suppose f_{k-1} is defined for some $k \geq 1$. Pick $\varepsilon > 0$ such that

(a) $\varepsilon < \operatorname{dist}(f_{k-1}|\overline{\Psi}_i|, Q - |\hat{\varphi}_i|_1)$ for those finitely many indices i for which $|\Psi_i|$ meets $f_{k-1}^{-1}|\varphi_k|_2$.

(b) Once f_k is defined such that f_k agrees with f_{k-1} on $\dot{M}$ and $d(f_{k-1}, f_k) < \varepsilon$, then $f_k \sim f_{k-1}$ rel $\dot{M}$.

By Proposition 4, there are f_k, j_k, δ_k, σ_k such that $\alpha < \delta_k < \delta_{k-1}$, $\sigma_{k-1} < \sigma_k < 1$, $d(f_{k-1}, f_k) < \varepsilon$, $f_k|\overset{\circ}{\Psi}{}^k(\delta_k) - j_k|\Psi_1|_{\sigma_k}$ is a locally flat imbedding. Hence (A) and (C) hold for k. Also $f_k = f_{k-1}$ on $\overset{\circ}{\Psi}{}^k{}^{-1}(\delta_k) - j_k|\Psi_1|_{\sigma_k}$ and $f_k = f_{k-1}$ off $f_{k-1}^{-1}|\varphi_k|_2$. So by choice (a), $f_k|\overline{\Psi}_i| \subset |\hat{\varphi}_i|_1$ for $i \leq r$.

Finally, $g = f_r$ is defined along with $\alpha < \delta_r < 1$, $\frac{1}{2} \leq \sigma_r < 1$ and j_r such that $g \sim f$ rel $\dot{M}$. Since $\overset{\circ}{\Psi}{}^r(\delta_r) \supset \dot{M}$, $g|\overset{\circ}{M} - j_r|\Psi_1|_{\sigma_r}$ is a locally flat imbedding. But $g = f$ on $\overset{\circ}{\Psi}{}^{r-1}(\delta_r) - j_r|\Psi_1|_{\sigma_r}$, which is a deleted neighborhood of $\dot{M}$. Hence, $g = f$ near $\dot{M}$ and g is locally flat on all of $M - j_r|\Psi_1|_{\sigma_r}$.

4. Connectivity considerations.

This short section considers various connectivity statements which are of use in this paper. The most important of these statements is Lemma 9, which is crucial to the completion of the proof of this paper's main result.

LEMMA 7. *Let Q^q be a topological manifold without boundary, $q \geq 5$. Let A be a closed subset of Q. Let C be a closed n-cube where $q \geq n+3$. Suppose $f: C \to Q$ is an imbedding which is locally flat on every simplex in some triangulation of C. Then any*

map $\alpha: S^1 \to Q-(A \cup fC)$ *which is null-homotopic in* $Q-A$ *is null-homotopic in* $Q-(A \cup fC)$.

Proof. Since $q > 2 \dim S^1 + 2$, after a preliminary homotopy inside $Q-(A \cup fC)$, α can be assumed to be a locally flat imbedding [2]. It then must be shown that α extends to a map $D^2 \to Q-(A \cup fC)$. At any rate, α extends to a map $\beta: D^2 \to Q-A$. By Proposition 2, $\beta \sim \beta'$ rel S^1 so that $\beta' D^2 \subset |\dot\varphi|_1$ for some coordinate neighborhood $|\varphi|$ in $Q-A$.

By Miller's Theorem, there is an ε-homeomorphism $\bar h: E^q \to E^q$ fixed outside the ε-neighborhood of $\varphi^{-1}\alpha S^1$ such that $\bar h \varphi^{-1}\alpha$ is P.L. Now $\bar h$ defines a homeomorphism $h: Q \to Q$. Then $\varphi^{-1}h\alpha$ is P.L., and if ε is small enough $\operatorname{Im} h\beta' \subset |\dot\varphi|_1$ and $\operatorname{Im} h\alpha \subset Q-(A \cup fC)$.

Triangulate C finely enough that hf is locally flat on every simplex and for no simplex σ does $hf\sigma$ meet both $|\dot\varphi|_2$ and $|\dot\varphi|_1$. Then $C = K \cup J$ where

$$K = \{\sigma \mid hf\sigma \subset |\dot\varphi|_2\}, \qquad J = \{\sigma \mid hf\sigma \cap |\varphi|_1 = \varnothing\}.$$

By Miller's Theorem again there is an ε-homeomorphism $\bar k: E^q \to E^q$ fixed outside the ε-neighborhood of $\varphi^{-1}hfK$ such that $\bar k \varphi^{-1}hf$ is a P.L. imbedding near K. $\bar k$ defines $k: Q \to Q$ so that $\varphi^{-1}khf|K$ is P.L. If ε is small enough $\operatorname{Im} kh\beta' \subset |\dot\varphi|_1$, $khfJ \cap |\varphi|_1 = \varnothing$, and k is fixed on $h\alpha S^1$, which misses hfC. Thus, $\varphi^{-1}kh\beta'|S^1 = \varphi^{-1}h\alpha$ is a P.L. imbedding and $\varphi^{-1}kh\beta' \sim \theta$ rel S^1 by an ε-homotopy where $\theta: D^2 \to E^q$ is P.L. in general position with respect to $\varphi^{-1}khfK$. Then

$$\dim (\operatorname{Im} \theta) \cap \varphi^{-1}khfK \leqq 2+n-q < 0,$$
$$\operatorname{Im} \theta \cap \varphi^{-1}khfK = \varnothing, \quad \text{and} \quad \operatorname{Im} \varphi\theta \cap khfK = \varnothing.$$

If ε is small enough, $\operatorname{Im} \varphi\theta \subset |\dot\varphi|_1$. Since $khfJ \cap |\varphi|_1 = \varnothing$, $\operatorname{Im} \varphi\theta \cap khfC = \varnothing$ and so $\operatorname{Im} h^{-1}k^{-1}\varphi\theta \cap fC = \varnothing$. Since h and k are fixed outside $|\varphi|$ and $|\varphi| \cap A = \varnothing$, it follows that $\operatorname{Im} \theta h^{-1}k^{-1}\varphi \cap (A \cup fC) = \varnothing$. Thus, $h^{-1}k^{-1}\varphi\theta: D^2 \to Q-(A \cup fC)$ and $h^{-1}k^{-1}\varphi\theta|S^1 = h^{-1}k^{-1}\varphi\varphi^{-1}kh\beta'|S^1 = \alpha$.

The proof of the important Lemma 9 uses homology and cohomology functors which are defined in sheaf theory. For the relevant definitions and notation see [15], especially Chapter XI.

LEMMA 8. *Let* $f: X \to Y$ *be a homeomorphism. Let* G *be a sheaf over* Y *and* Φ *a family of supports in* Y. *Let* $f^{-1}\Phi = \{f^{-1}A \mid A \in \Phi\}$ *and let* f^*G *be the sheaf over* X *induced by* f. *Then* f *induces an isomorphism of augmented* δ-*functors*

$$H^*_\Phi(Y, G) \to H^*_{f^{-1}\Phi}(X, f^*G).$$

Proof. f and f^{-1} induce unique maps of augmented δ-functors

$$H^*_\Phi(Y, G) \leftrightarrows H^*_{f^{-1}\Phi}(X, f^*G).$$

Since $G = (f^{-1})^* f^*G$ and both compositions are the unique maps induced by the identity, the isomorphism follows.

Lemma 9. *Let $q \geq n+3$, M^n a $2n-q$ connected topological manifold and Q^q a simply connected topological manifold. Suppose $f \colon M \to Q$ is a strictly proper map such that for some coordinate neighborhood $|\Psi|$ on $\mathring{M}$, $f|M - |\Psi|_1$ is a locally flat imbedding. Then $\pi_i(Q, Q - f(M - |\overset{\circ}{\Psi}|_2)) = 0$ for $i \leq n$.*

Proof. Since f is strictly proper, the pairs $(Q, Q - f(M - |\overset{\circ}{\Psi}|_2))$ and $(\mathring{Q}, \mathring{Q} - f(\mathring{M} - |\overset{\circ}{\Psi}|_2))$ are homotopy equivalent. It may then be assumed that $\mathring{M} = \mathring{Q} = \varnothing$.

Let S_Q denote the sheaf of singular chains on Q and $T_Q = H_q(S_Q)$. Similarly, $T_M = H_n(S_M)$. Since Q is simply connected, it is orientable and T_Q is the constant sheaf with stalk Z. Let Z_X denote the constant sheaf over any space X with stalk Z. Let $A = M - |\overset{\circ}{\Psi}|_2$. Then $(f|A)^*(T_Q|fA) = Z_A$. Let Φ_M and Φ_Q denote the paracompactifying families of compact subsets of M and Q respectively. Then

$\Phi_{fA} = \{B \in \Phi_Q \mid B \subset fA\} =$ set of compact sets in fA,

$\Phi_A = \{B \in \Phi_M \mid B \subset A\} =$ set of compact sets in A, and $f^{-1}\Phi_{fA} = \Phi_A$.

Then by the duality theorems in [15], Lemma 8, and obvious isomorphisms

$$H_i(Q, Q - fA) \cong H_i^{\Phi_Q}(Q, Q - fA; Z_Q) \cong H_{\Phi_{fA}}^{q-i}(fA; (T_Q \otimes Z_Q)|fA)$$
$$\cong H_{f^{-1}\Phi_{fA}}^{q-i}(A; (f|A)^*(T_Q|fA)) \cong H_{\Phi_A}^{q-i}(A; Z_A).$$

Suppose $2n - q \leq 0$. Then for $i \leq n$, $q - i \geq q - n \geq n$, and

$$H_{\Phi_A}^{q-i}(A; Z_A) \cong H^{q-i}(\mathring{A}; Z) = 0.$$

Suppose $2n - q \geq 1$. Then M and A are orientable and $Z_A \cong T_M|A \cong (T_M \otimes Z_M)|A$. Then by more duality arguments:

$$H_{\Phi_A}^{q-i}(A; Z_A) \cong H_{\Phi_A}^{q-i}(A; (T_M \otimes Z_M)|A) \cong H_{n-(q-i)}^{\Phi_M}(M, M - A; Z_M)$$
$$\cong H_{n-(q-i)}(M, |\overset{\circ}{\Psi}|_2) \cong \tilde{H}_{n-(q-i)}(M) = 0$$

for $n - (q - i) \leq 2n - q$, i.e., for $i \leq n$.

So in either case $H_i(Q, Q - fA) = 0$ for $i \leq n$. The lemma then follows from the relative Hurewicz Theorem provided it is shown that $Q - fA$ is simply connected. It is certainly arcwise connected by dimension theory. Now there is a locally finite cover of A by coordinate neighborhoods $|\Psi_1|, |\Psi_2|, \ldots$ in $M - |\Psi|_1$ such that $A \subset |\Psi_1|_1 \cup |\Psi_2|_1 \cup \cdots$. Then $\{f|\Psi_i|_1 \mid i \geq 1\}$ is locally finite in Q. Let $M_k = \bigcup_{i \geq k} f|\Psi_i|_1$. Then M_k is closed in Q.

Let $\alpha \colon S^1 \to Q - fA$ be given. Since Q is simply connected, α is null-homotopic in Q via a homotopy H. Now Im H intersects $f|\Psi_i|_1$ for only finitely many values of i. Hence, Im $H \subset Q - M_k$ for some k and α is null-homotopic in $Q - M_k$. Lemma 7 can now be applied finitely many times to show that α is null-homotopic in $Q - M_1 \subset Q - fA$.

5. The imbedding theorem. The proof of the main result of this paper can now be accomplished. The opening part of this section consists of a moderately involved argument which reduces the proof to a special case. The special case is then

proved by means of an intricate argument combining most of the topological and P.L. techniques considered in the previous sections. Finally, some corollaries are included.

THEOREM 1. *If $q \geq n+3$, M^n is a compact, $2n-q$ connected topological manifold, Q^q is a $2n-q+1$ connected topological manifold and $f: (M, \dot{M}) \to (Q, \dot{Q})$ is a map such that $f|\dot{M}$ is a locally flat imbedding, then $f \sim g$ rel $\dot{M}$ where $g: M \to Q$ is a proper locally flat imbedding.*

Proof. By Proposition 5, after a preliminary homotopy, it can be assumed that f is strictly proper and $f|M - |\Psi|_{\rho'}$ is a locally flat imbedding for some coordinate neighborhood $|\Psi|$ in $\dot{M}$ and some $1 > \rho' > 0$. Pick $\rho' < \rho < 1$. By Proposition 2, $f \sim f_1$ rel $\dot{M}$ where $f_1 = f$ off $|\dot{\Psi}|_{\rho}$ and $f_1|\Psi|_{\rho} \subset |\varphi|$ for some coordinate neighborhood $|\varphi|$ on $\dot{Q}$. By choosing ρ slightly larger it may be assumed that f_1 is a locally flat imbedding near $M - |\dot{\Psi}|_{\rho}$.

Pick $\rho < \sigma < 1$ such that $f_1|\Psi|_{\sigma} \subset |\varphi|$. Let $C(\sigma)$ be triangulated so that $\dot{C}(\rho)$, $C(\rho)$, and $L = C(\sigma) - \mathring{C}(\rho)$ are triangulated as subcomplexes. Now

$$\varphi^{-1} f_1 \Psi | L : L \to E^q$$

is an imbedding which is locally flat on every simplex. By Miller's Theorem, there is an ε-homeomorphism $\bar{h}: E^q \to E^q$ fixed outside the ε-neighborhood of $\varphi^{-1} f_1 \Psi L$ such that $\bar{h}\varphi^{-1} f \Psi | L$ is a P.L. imbedding. Now if ε is small enough, $\bar{h}$ defines an ε-homeomorphism $h: Q \to Q$. Then $f_2 = h f_1$ has the same properties as f_1 and $f_1 \sim f_2$ rel $\dot{M}$ if ε is small enough.

Pick $\sigma > \tau_1 > \tau_2 > \tau > \rho$ and suppose $C(\sigma)$ is further subdivided so that $\dot{C}(\tau_1)$, $\dot{C}(\tau_2)$, and $\dot{C}(\tau)$ are triangulated as subcomplexes. Now $k = \varphi^{-1} f_2 \Psi | L = \varphi^{-1} h f_1 \Psi | L = \bar{h} \varphi^{-1} f_1 \Psi | L$ is a P.L. imbedding. Triangulate L and E^q such that k is simplicial and if $A = N(\dot{C}(\tau), L'')$ is the simplicial neighborhood in the second derived subdivision L'' of L, then $A \cap (C(\sigma) - \mathring{C}(\tau_1)) = \varnothing$. Let $D = N(k\dot{C}(\tau), (E^q)'')$ be the corresponding simplicial neighborhood in the second derived subdivision of E^q. Then k imbeds A properly into D and $k^{-1} D = A$. Now $(C(\sigma) - \mathring{C}(\tau_1)) \cup A$ and $[C(\sigma) - \mathring{C}(\tau_1)] \cup [C(\tau_2) - \mathring{C}(\rho)]$ are regular neighborhoods of $\dot{C}(\sigma) \cup \dot{C}(\tau)$ in $C(\sigma)$. By the uniqueness of regular neighborhoods there is a P.L. homeomorphism l of $C(\sigma)$ fixed on $\dot{C}(\sigma) \cup \dot{C}(\tau)$ such that

$$l([C(\sigma) - \mathring{C}(\tau_1)] \cup A) = [C(\sigma) - \mathring{C}(\tau_1)] \cup [C(\tau_2) - \mathring{C}(\rho)].$$

Then

$$k_1 = k(l^{-1}) : L, C(\tau) - C(\rho), \dot{C}(\tau_2) \cup \dot{C}(\rho) \to E^q, \mathring{D}, \dot{D}$$

is a P.L. imbedding and $k_1 = k$ on $\dot{C}(\sigma)$. Extend k_1 to $C(\sigma)$ by means of any extension of $k_1|\dot{C}(\rho)$ to $C(\rho)$. Define $f_2: M \to Q$ by

$$\begin{aligned} f_2 &= f_1 & &\text{on } M - |\dot{\Psi}|_{\sigma}, \\ &= \varphi k_1 \Psi^{-1} & &\text{on } |\Psi|_{\sigma}. \end{aligned}$$

Note that if $x \in |\dot{\Psi}|_\sigma$, $\varphi k_1 \Psi^{-1} x = \varphi k \Psi^{-1} x = f_1 x$. Hence f_2 is well defined. Since their images disagree only in the coordinate neighborhood $|\varphi|$, $f_2 \sim f_1$ rel $\dot{M}$. Note that $\varphi^{-1} f_2 \Psi | L$ is a P.L. imbedding.

The remainder of the proof is divided into two cases.

Case 1. Q is not simply connected.

Then $2n - q + 1 < 1$ and $2n < q$. Now $M_1 = f_2^{-1}|\varphi|$ is an open n-submanifold of M, and $M^* = \varphi^{-1} f_2(M_1 - |\dot{\Psi}|_\tau)$ is closed in E^q since $\varphi M^* = f_2(M - |\dot{\Psi}|_\tau) \cap |\varphi|$. Then $R_1 = E^q - (M^* \cup \mathring{D})$ is a P.L. q-manifold with boundary $\dot{D} - M^*$. By collapsing D onto $k_1 \dot{C}(\tau) = \varphi^{-1} f_2 |\dot{\Psi}|_\tau$, R_1 and $E^q - M^*$ are seen to be homotopy equivalent. Hence, $\pi_i(E^q - M^*, R_1) = 0$ for all i. By Lemma 9, $\pi_i(E^q, E^q - M^*) = 0$ for $i \leq n$. So from the exact homotopy sequence of the triple $(E^q, E^q - M^*, R_1)$ it follows that $\pi_i(E^q, R_1) = 0$ for $i \leq n$.

Now $k_1 | C(\rho) : (C(\rho), \dot{C}(\rho)) \to (E^q, R_1)$. So $k_1 | C(\rho) \sim k_2$ rel $\dot{C}(\rho)$ where

$$k_2 : C(\rho) \to R_1.$$

Now $k_2 | \dot{C}(\rho) = k_1 | \dot{C}(\rho)$ is a P.L. imbedding into $\dot{R}_1$. Since $2n < q$, the pairs $(C(\rho), \dot{C}(\rho))$ and $(R_1, \dot{R}_1)$ are certainly $2n - q$ and $2n - q + 1$ connected, respectively. By Irwin's Imbedding Theorem, $k_2 | C(\rho) \sim k_3$ rel $\dot{C}(\rho)$ where $k_3 : C(\rho) \to R_1$ is a proper P.L. imbedding. Extend k_3 to $C(\sigma)$ by setting $k_3 = k_2$ off $C(\rho)$. Then $k_3 : C(\sigma) \to E^q$ is a P.L. imbedding and is locally flat by Lemma 3. Define $g : M \to Q$ by

$$g = \varphi k_3 \Psi^{-1} \qquad \text{on } |\Psi|_\sigma,$$
$$= f_2 \qquad\qquad \text{off } |\Psi|_\sigma.$$

g is well defined for if $x \in |\dot{\Psi}|_\sigma$, $\varphi k_3 \Psi^{-1} x = \varphi k_1 \Psi^{-1} x = f_2 x$. g is locally flat since it is locally flat on $|\dot{\Psi}|_\sigma$ and off $|\dot{\Psi}|_\tau$. Thus g is a locally flat imbedding, and $g \sim f_2$ rel $\dot{M}$ since their images disagree only inside $|\varphi|$.

Case 2. Q is simply connected.

Let $R = \mathring{Q} - (M' \cup \varphi \mathring{D})$ where $M' = f_2(M - |\dot{\Psi}|_\tau)$.

Claim. $\pi_i(\mathring{Q}, R) = 0$ for $i \leq n$.

As in Case 1, the collapse of D onto $k_1 \dot{C}(\tau)$ implies that $\pi_i(\mathring{Q} - M', R) = 0$ for all i. By Lemma 9, $\pi_i(\mathring{Q}, \mathring{Q} - M') = \pi_i(Q, Q - M') = 0$ for $i \leq n$. The claim follows from the exact homotopy sequence of the triple $(\mathring{Q}, \mathring{Q} - M', R)$.

Since $n + 3 \leq q$, it follows that $2n - q + 2 \leq n - 1$. Since $\mathring{Q}$ is $2n - q + 1$ connected, it follows from the homotopy sequence of $(\mathring{Q}, R)$ that R is $2n - q + 1$ connected.

Since $f_2 | |\Psi|_\rho : (|\Psi|_\rho, |\dot{\Psi}|_\rho) \to (\mathring{Q}, R)$, $f_2 | |\Psi|_\rho \sim f_3 : |\Psi|_\rho \to R$ rel $|\dot{\Psi}|_\rho$. f_3 extends to M by setting $f_3 = f_2$ off $|\Psi|_\rho$. Then $f_3 \sim f_2$ rel $\dot{M}$ and

$$f_3 | |\Psi|_\rho : (|\Psi|_\rho, |\dot{\Psi}|_\rho) \to (R, \dot{R}).$$

Now $\dot{R} = \varphi(\dot{D} - k_1 \dot{C}(\tau_1))$ is a P.L. manifold via the P.L. structure of $\dot{D} - k_1 \dot{C}(\tau_1)$. Now $|\Psi|_\sigma$ has a P.L. structure via Ψ, and $\dot{R}$ has neighborhoods in Q and R which have P.L. structures via φ. With respect to these P.L. structures, $f | |\Psi|_\sigma - |\dot{\Psi}|_\rho$ is a P.L. imbedding.

Suppose for the moment that $f_3 \mid |\Psi|_\rho \sim g$ rel $|\dot{\Psi}|_\rho$ where $g: |\Psi|_\rho \to R$ is a proper locally flat imbedding which is P.L. near $|\dot{\Psi}|_\rho$ in $|\Psi|_\rho$. Then extend g to M by setting $g=f_3$ off $|\Psi|_\rho$. Then clearly g is a proper imbedding and $g \sim f_3 \sim f$ rel $\dot{M}$. g is clearly locally flat on $M-|\Psi|_\rho$ and on $|\Psi|_\rho$. Since g is a P.L. imbedding near $|\dot{\Psi}|_\rho$, it is locally flat there by Lemma 3. So g is locally flat everywhere.

So the proof of Theorem 1 has been reduced to the proof of the following special case.

PROPOSITION 6. *Suppose that* $q \geqq n+3$, D^n *is a P.L. n-disk,* R^q *is a* $2n-q+1$ *connected topological manifold, and* $\dot{R}$ *is a P.L. manifold. Then if* $f: (D, \dot{D}) \to (R, \dot{R})$ *is a map such that* $f|\dot{D}$ *is a P.L. imbedding,* $f \sim g$ *rel* $\dot{D}$ *where* $g: D \to R$ *is a proper locally flat imbedding and* g *is P.L. near* $\dot{D}$ *with respect to some P.L. structure near* $\dot{R}$ *which is compatible with the given P.L. structure on* $\dot{R}$.

Proof. Pick $0<t<1$ and let $D_0=\{x \in D \mid |x| \leqq t\}$ and $W=\{x \in D \mid |x|>t\}$.

By Lemma 5, after a preliminary homotopy, f may be assumed to be strictly proper with respect to collar neighborhoods W and U in D and R respectively, and $f|W$ is a locally flat imbedding. $\overline{W}$ can be taken to be a P.L. subspace of D.

The map f will be changed several additional times during this proof. In each case the change will be effected by a proper homotopy, but mention of this fact will usually be omitted. For example, by Proposition 2 it may be assumed that f is changed so that $fD_0 \subset |\varphi|$ for some coordinate neighborhood $|\varphi|$ in $\dot{R}$.

Let $|\varphi|$ have the P.L. structure induced from E^q by φ. Let N be a compact neighborhood of fD_0 in $|\varphi|$. Let X be a finite polyhedron such that $f^{-1}N \subset X \subset f^{-1}|\varphi|$. Let P_1 be a regular neighborhood of $P_0=X \cap \overline{W}$ in $W \cap f^{-1}|\varphi|$. Then $f|P_1$ is locally flat. By Miller's Theorem, there is a small enough homeomorphism h of $|\varphi|$ which is fixed outside a neighborhood of fP_1, and is such that $hf|P_1$ is a P.L. imbedding into $|\varphi|$. h extends by the identity to a homeomorphism of R. Rename hf to be f.

Let $0<\varepsilon<\text{dist}(fD_0, |\varphi|-N)$. Let $P=P_0 \cup D_0$. By the Second General Position Theorem, $f|P \sim \eta$ rel P_0 via an ε-homotopy, where η is a P.L. map in general position. That is $S_3(\eta)=\varnothing$ and dim $S_2(\eta) \leqq 2n-q$. Define $f': D \to R$ by

$$f' = \eta \quad \text{on } P,$$
$$= f \quad \text{on } \overline{W}.$$

f' is well defined since $P \cap \overline{W} \subset P_0$ and $f=\eta$ on P_0. Then f' is P.L. on $P_2=P \cup P_1$.

Claim 1. $S_2(f')=S_2(f'|P)$.

Let $x, y \in D$, $x \neq y$, $f'x=f'y$ and suppose $x \notin P$. Then $x \in W$ and since $f'|W$ is an imbedding, $y \notin W$. Then $y \in D_0$. Now $f'y=f'x=fx$. So $d(fx, fy)=d(f'y, fy)=d(\eta y, fy)<\varepsilon$. By choice of ε, $fx \in N$ and $x \in X \cap \overline{W}=P_0 \subset P$. This is a contradiction.

If f' is renamed f, then $S_2(f)$ is a P.L. subspace of $\dot{D}$ of dimension $\leqq 2n-q$. By Theorem 7.7 in [6], $S_2(f) \subset C$ where C is a collapsible P.L. subspace in $\dot{D}$ of dimension $\leqq 2n-q+1$.

Claim 2. It may be assumed $fC \subset |\varphi|$. (This is not a priori true.)

Pick $0 < s < s'$ such that $fP_2 \subset |\dot\varphi|_s$. Let $C_0 = f^{-1}(fC - |\dot\varphi|_s)$. Now the collar neighborhood U inherits a P.L. structure from $\mathring{R} \times [0, 1]$. The map $f|W$ is locally flat. By Proposition 3, there is a homeomorphism $u: U \to U$ such that $uf|W$ is P.L. There are triangulations such that $uf|f^{-1}(fC - |\varphi|_s)$ is a simplicial map into U. By considering stars in U of vertices in $u(fC - |\varphi|_s)$, it is seen that $u(fC - |\varphi|_s)$ and $fC - |\varphi|_s$ are locally tame of dimension $\leq 2n - q + 1$.

Now $\mathring{R} - |\varphi|_s$ and $|\dot\varphi|_{s'} - |\varphi|_s$ are $2n - q + 1$ connected and $fC - |\dot\varphi|_{s'}$ is compact. By Newman's Engulfing Theorem, there is a homeomorphism k of $\mathring{R} - |\varphi|_s$ onto itself such that k is fixed outside a compact set and $k(|\dot\varphi|_{s'} - |\varphi|_s) \supset fC - |\varphi|_s$. Extend k by the identity to a homeomorphism of R. Then $k\varphi$ defines a new coordinate neighborhood which can be renamed $|\varphi|$. Then $fC \subset |\varphi|_{s'}$. Since k is fixed on $|\varphi|_s$, $f|P_2$ is P.L. as a map into the new $|\varphi|$.

By Lemma 3, f is locally flat on $P_2 - S_2(f)$. Also $f|W$ is locally flat. So if $M_0 = f^{-1}|\varphi|$, then f is locally flat on $M_0 - S_2(f)$.

Claim 3. It may be assumed that f is P.L. near C.

$f|M$ is P.L. near P. By Proposition 3 there is a homeomorphism h of $|\varphi|$ fixed outside a compact set such that $hf|M_0$ is P.L. near C. h extends to a homeomorphism of R. Let hf be renamed f.

By Theorem 7.7 in [6], $fC \subset K$ where K is a collapsible P.L. subspace of $|\varphi|$ of dimension $\leq 2n - q + 2$.

Claim 4. It may be assumed that there is a compact P.L. neighborhood T of $f^{-1}K$ on which f is P.L.

Let F be a compact neighborhood of $K \cap fD$ in $|\varphi|$. Let $0 < \alpha < \text{dist}\,(K, fD - F)$. By Proposition 3 there is an α-homeomorphism h of $|\varphi|$ fixed outside a compact set such that $hf|M_0$ is P.L. near $f^{-1}F$. Extend h to a homeomorphism of R. Let T be a compact P.L. neighborhood of $f^{-1}F$ on which hf is P.L. Let $x \in (hf)^{-1}K$. Then $hfx \in K$, $d(hfx, fx) < \alpha$, $fx \in F$, and $x \in f^{-1}F \subset \mathring{T}$. So $(hf)^{-1}K \subset \mathring{T}$. If hf is renamed f, then Claim 4 is established.

Let $0 < \beta < \text{dist}\,(K, fD - f\mathring{T})$. By the First General Position Theorem, there is a P.L. β-homeomorphism h of $|\varphi|$ fixed on fC such that $h(K - fC)$ is in general position with respect to fT. Then $\dim h(K - fC) \cap fT \leq 2n - q + 2 + n - q \leq 2n - q - 1$.

Claim 5. $h(K - fC) \cap fD \subset f\mathring{T}$.

Suppose $y \in K - fC$, $x \in D$ and $hy = fx$. Then $d(y, fx) = d(y, hy) < \beta$. Thus $hy \in f\mathring{T}$, which establishes Claim 5.

Let $K_1 = hK$, $C_1 = C$. Then it follows from Claim 5 and $h|fC = 1$ that $(K_1 - fC) \cap fD \subset f\mathring{T}$ and $\dim (K_1 - fC_1) \cap fD \leq 2n - q - 1$. Then $f^{-1}K_1 = C_1 \cup X_1$ where $\dim X_1 \leq 2n - q - 1$. Since $fX_1 \subset f\mathring{T}$ and $S_2(f) \subset \mathring{T}$, it follows that $X_1 \subset \mathring{T}$. So f is P.L. near $C_1 \cup X_1$.

Now suppose by induction there are collapsible P.L. subspaces $C_i \subset \mathring{D}$ and $K_i \subset |\varphi|$ and also a P.L. subspace $X_i \subset \mathring{D}$ such that $S_2(f) \subset C_i$, $f^{-1}K_i = C_i \cup X_i$, f is P.L. near $C_i \cup X_i$ as a map into $|\varphi|$, and $\dim X_i \leq 2n - q - i \leq n - 3$.

By Theorem 7.7 in [6] there is a collapsible P.L. subspace $C_{i+1} \subset \mathring{D}$ with $C_i \cup X_i \subset C_{i+1}$ and dim $(C_{i+1} - C_i) \leq$ dim $X_i + 1$. As in Claims 2 and 3, it may be assumed that $fC_{i+1} \subset |\varphi|$ and f is P.L. near C_{i+1}. Again by Theorem 7.7 in [6], there is a collapsible P.L. subspace K of $|\varphi|$ such that $K_i \cup fC_{i+1} \subset K$ and dim $(K - K_i) \leq$ dim $X_i + 2 \leq 2n - q - i + 2$. As in Claim 4, it may be assumed that there is a compact P.L. neighborhood T of $f^{-1}K$ on which f is P.L.

Let $0 < \beta < $ dist $(K, fD - f\mathring{T})$. By the First General Position Theorem there is a P.L. β-homeomorphism h of $|\varphi|$ fixed on fC_{i+1} such that

$$\dim h(K - fC_{i+1}) \cap fT \leq 2n - q - i + 2 + n - q \leq 2n - q - (i+1).$$

Let $K_{i+1} = hK$. Then as before $f^{-1}K_{i+1} = C_{i+1} \cup X_{i+1}$ where dim $X_{i+1} \leq 2n - q - (i+1)$ and f is P.L. near $X_{i+1} \cup C_{i+1}$. So for some integer k, $X_k = \varnothing$ and $f^{-1}K_k = C_k$.

Let V be a regular neighborhood of C_k in $\mathring{D}$ on which f is P.L. as a map into $|\varphi|$. Let V and $|\varphi|$ be given second derived subdivisions of some triangulations in which $f|V$ is simplicial. Let $B_1 = N(C_k, V)$ and $B_2 = N(K_k, |\varphi|)$ be the corresponding simplicial neighborhoods. Since C_k and K_k are collapsible, B_1 and B_2 are P.L. balls of dimension n and q respectively in $\mathring{D}$ and $B_1 = f^{-1}B_2$ and $S_2(f) \subset \mathring{B}_1$.

Since $f|\mathring{B}_1$ is a P.L. imbedding into $\mathring{B}_2$, it extends to a proper P.L. imbedding $g: B_1 \to B_2$. Finally, g extends to D by setting $g = f$ off B_1. Then g is an imbedding since it has no double points. g is locally flat near B_1 by Lemma 3. Hence, g is a locally flat imbedding and $g \sim f$ rel $\mathring{D}$.

The next result follows directly from Theorem 1.

THEOREM 2. *If $q \geq n + 3$, M^n is a closed $2n - q$ connected topological manifold, Q^q is a $2n - q + 1$ connected topological manifold, and $f: M \to Q$ is any map, then $f \sim g$ where $g: M \to \mathring{Q}$ is a locally flat imbedding.*

COROLLARY. *If Q^q is a k-connected topological manifold, every element of $\pi_r(Q)$ is representable by a locally flatly imbedded sphere provided*

$$r \leq \min (q - 3, (k + q - 1)/2).$$

Proof. Theorem 2 and the inequalities $q \geq r + 3$ and $2r - q + 1 \leq k$.

THEOREM 3. *If $q \geq n + 3$, M^n is a closed $2n - q + 1$ connected topological manifold, Q^q is a $2n - q + 2$ connected topological manifold, and $f, g: M \to Q$ are homotopic locally flat imbeddings of M in Q, then f and g are locally flatly concordant. That is, there is a locally flat imbedding $H: M \times I \to Q \times I$ such that $H(x, 0) = (fx, 0)$ and $H(x, 1) = (gx, 1)$.*

Proof. Let $F: M \times I \to Q \times I$ be given by the known homotopy from f to g. Now $q + 1 \geq (n+1) + 3$, $M \times I$ is $2(n+1) - (q+1)$ connected and $Q \times I$ is $2(n+1) - (q+1) + 1$ connected. By Theorem 1, $F \sim H$ rel $M \times \mathring{I}$ where H is a proper locally flat imbedding.

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UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE,
CHICAGO, ILLINOIS 60680