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A PAIRING OF A CLASS OF EVOLUTION SYSTEMS WITH A CLASS OF GENERATORS

BY
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Abstract. Suppose that S is a Banach space and that A and M are functions such that if x and y are numbers, $x \geqq y$, and P is in S then each of $M(x, y)P$ and $A(y, P)$ is in S. This paper studies the relation

$$M(x, y)P = P + \int_x^y A(t, M(t, y)P)\, dt.$$

Classes OM and OA will be described and a correspondence will be established which pairs members of the two classes which are connected as M and A are by the relation indicated above.

Suppose that S is a Banach space, A is a function such that, if t is a number, then $A(t, \cdot)$ has domain all of S and values in S, and M is a function such that, if $x \geqq z$, then $M(x, z)$ is a function from S to S satisfying

$$\text{(E)} \qquad M(x, y)M(y, z)P = M(x, z)P$$

for all y between x and z and all P in S. This paper is a study of the relation

$$\text{(1)} \qquad M(x, y)P = P + \int_x^y A(t, M(t, y)P)\, dt$$

between A and M.

In [9] and [10], J. S. Mac Nerney defines classe OM and OA and a one-to-one correspondence $\mathscr{E}$ from OA onto OM. Members M of OM have the evolution property (E) and members V of OA have the property that

$$V(x, y)P + V(y, z)P = V(x, z)P$$

for all y between x and z and all P in S. The function $\mathscr{E}$ associates members M and V of OM and OA which are related by an equation similar to (1). Important in those papers, but not to be considered here, is the possibility of discontinuities of the solutions $M(\cdot, y)P$. The author's study in [4] and [5] of these discontinuities leads into the analysis in this paper.

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The development of Mac Nerney's (see also [13]) includes the case that, for $a>b$, the function $M(a, b)$ is "generated" by a Lipschitz function $A(t, \cdot)$ from S to S and $\lim_{a\to b} |[M(a, b)-1]P-[M(a, b)-1]Q|=0$. In case $A(t, \cdot)$ is (not necessarily Lipschitz) continuous and $\lim_{a\to b} |M(a, b)P-P|=0$ and, especially in case M arises from a one-parameter semigroup of nonlinear functions, many investigations have been made. Some of these are [1], [2], [3], [6], [8], [11], [12], [14], [15], [16], [17], [18], and [19]. However, in none of these more recent papers has the complete pairing of the solutions with their generators been made as provided by $\mathscr{E}$ in [9] and [10]. This paper will provide an extension of the function $\mathscr{E}$ to a nonlinear analogue of the linear, strong case (see [7, §11.5]).

1. **The main result.** The class OA will consist of all functions V having the property that if $a \geqq b$ then $V(a, b)$ is a function from S to S and

1A. there is a continuous function ρ which is of bounded variation on each finite interval such that if $a\geqq b$ and $\rho(a)-\rho(b)<1$ then $1-V(a, b)$ has range all of S and, if P and Q are in S, then

$$\{1-[\rho(a)-\rho(b)]\}|P-Q| \leqq |[1-V(a, b)]P-[1-V(a, b)]Q|,$$

2A. if $x\geqq y\geqq z$ and P is in S then $V(x, y)P+V(y, z)P=V(x, z)P$,

3A. if $a>b$ and B is a bounded subset of S then there is a nondecreasing, continuous function α such that if $a\geqq x\geqq y\geqq b$ and P is in B then $|V(x, y)P| \leqq \alpha(x)-\alpha(y)$, and

4A. if $a>b$ then there is a nondecreasing function β such that if $\varepsilon>0$ and P is in S then there is a positive number δ having the property that if Q is in S such that $|Q-P|<\delta$ and $a\geqq x\geqq y\geqq b$ then $|V(x, y)P-V(x, y)Q| \leqq [\beta(x)-\beta(y)]\varepsilon$.

The class OM will consist of all functions M having the property that if $a\geqq b$ then $M(a, b)$ is a function from S to S and

1M. there is a continuous function ρ which is of bounded variation on each finite interval such that if $a\geqq b$ and P and Q are in S then

$$|M(a, b)P-M(a, b)Q| \leqq \exp(\rho(a)-\rho(b))|P-Q|,$$

2M. if $x\geqq y\geqq z$ and P is in S then $M(x, y)M(y, z)P=M(x, z)P$,

3M. if $a>b$ and B is a bounded subset of S then there is a nondecreasing, continuous function α such that if $a\geqq x\geqq y\geqq b$ and P is in B then $|M(x, y)P-P| \leqq \alpha(x)-\alpha(y)$, and

4M. if $a>b$ then there is a nondecreasing function β such that if $\varepsilon>0$ and P is in S then there is a positive number δ and a positive number d having the property that if Q is in S such that $|Q-P|<\delta$ and $a\geqq x\geqq y\geqq b$ such that $x-y<d$ then

$$|[M(x, y)-1]P-[M(x, y)-1]Q| \leqq [\beta(x)-\beta(y)]\varepsilon.$$

The main result of this paper is the following:

THEOREM. *There is a reversible function $\mathscr{E}$ from OA onto OM such that if V is OA and M is in OM then these are equivalent*:

(a) $M=\mathscr{E}(V)$,

(b) *if* $a \geqq b$ *and* P *is in* S *then* $M(a, b)P = {}_a\prod^b [1-V]^{-1}P$,

(c) *if* $a \geqq b$ *and* P *is in* S *then* $V(a, b)P = {}_a\sum^b [M-1]P$, *and*

(d) *if* $a \geqq b$ *and* P *is in* S *then* $M(a, b)P = P + \int_a^b V[M(\cdot, b)P]$.

REMARK. If $a > b$ then a subdivision $\{s_p\}_0^n$ of $\{a, b\}$ is a decreasing sequence such that $s(0)=a$ and $s(n)=b$. Also, t is a refinement of the subdivision s provided that t is a subdivision of $\{a, b\}$ and s is a subsequence of t. The continuously continued product and sum in (b) and (c) above are defined in [9] and [10]; the integral in (d) is the Riemann-Stieltjes integral.

2. **From OA to OM.** In this section, suppose that V is in OA and ρ is as in condition 1A.

LEMMA 2.0. *Suppose that* $a > b$, P *is in* S, $\{s_p\}_0^n$ *is a subdivision of* $\{a, b\}$ *such that if* p *is an integer in* $[1, n]$ *then* $\int_{s(p)}^{s(p-1)} |d\rho| \leqq \frac{1}{2}$, *and* j *is an integer in* $[1, n]$. *Then*

$$\prod_{p=j}^{n} [1-V(s_{p-1}, s_p)]^{-1}P = P + \sum_{p=j}^{n} V(s_{p-1}, s_p) \prod_{i=p}^{n} [1-V(s_{i-1}, s_i)]^{-1}P$$

and

$$\left|\prod_{p=j}^{n} [1-V(s_{p-1}, s_p)]^{-1}P - P\right| \leqq \exp\left(2\int_b^a |d\rho|\right) \sum_{p=j}^{n} |V(s_{p-1}, s_p)P|.$$

Indication of proof. With the supposition of the lemma,

$$\begin{aligned}\prod_{p=j}^{n} [1-V(s_{p-1}, s_p)]^{-1}P - P &= \sum_{p=j}^{n} \{[1-V(s_{p-1}, s_p)]^{-1}-1\} \prod_{i=p+1}^{n} [1-V(s_{i-1}, s_i)]^{-1}P \\ &= \sum_{p=j}^{n} V(s_{p-1}, s_p) \prod_{i=p}^{n} [1-V(s_{i-1}, s_i)]^{-1}P.\end{aligned}$$

Also,

$$\begin{aligned}&\left|\prod_{p=j}^{n} [1-V(s_{p-1}, s_p)]^{-1}P - P\right| \\ &\quad = \left|\sum_{p=j}^{n} \left\{\prod_{i=j}^{p} [1-V(s_{i-1}, s_i)]^{-1}P - \prod_{i=j}^{p-1} [1-V(s_{i-1}, s_i)]^{-1}P\right\}\right| \\ &\quad \leqq \sum_{p=j}^{n} \prod_{i=j}^{p} \{1-[\rho(s_{i-1})-\rho(s_i)]\}^{-1} |V(s_{p-1}, s_p)P| \\ &\quad \leqq \exp\left(2\int_b^a |d\rho|\right) \sum_{p=j}^{n} |V(s_{p-1}, s_p)P|.\end{aligned}$$

This last inequality follows since if $0 \leqq z \leqq \frac{1}{2}$ then $[1-z]^{-1} \leqq 1+2z \leqq \exp(2z)$.

LEMMA 2.1. *Suppose that* $a > b$, β *is as in condition* 4A, $\{R_p\}_{p=1}^{\infty}$ *is a Cauchy sequence with values in* S, *and* $\varepsilon > 0$. *There is a positive number* δ *having the property* *that if n is a positive integer, P is in S such that* $|P-R_n|<\delta$, *and* $a \geqq x \geqq y \geqq b$ *then* $|V(x,y)P-V(x,y)R_n| \leqq [\beta(x)-\beta(y)]\varepsilon$.

Indication of proof. A proof may be constructed similar to the usual proofs that continuous functions on closed and (sequentially) compact sets are uniformly continuous.

REMARK. The construction in the proof of the next lemma is similar to that of [6, Lemma 3].

LEMMA 2.2. *Suppose that* $a>b$, β *is as in condition* 4A, $\varepsilon>0$, *and P is in S. There is a subdivision* $\{s_p\}_0^m$ *of* $\{a,b\}$ *such that if k is an integer in* $[1,m]$, $\{t_p\}_0^n$ *is a subdivision of* $\{s_{k-1},s_k\}$, *j is an integer in* $[1,n]$, *and* $a \geqq x \geqq y \geqq b$ *then* $\int_{s(k)}^{s(k-1)} |d\rho| \leqq \frac{1}{2}$ *and*

$$\begin{aligned}\Bigg| V(x,y)\prod_{p=j}^{n}[1-V(t_{p-1},t_p)]^{-1}\prod_{q=k+1}^{m}[1-V(s_{q-1},s_q)]^{-1}P \\ - V(x,y)\prod_{q=k+1}^{m}[1-V(s_{q-1},s_q)]^{-1}P\Bigg| \\ \leqq [\beta(x)-\beta(y)]\varepsilon.\end{aligned}$$

Indication of proof. With the supposition of the lemma, let Δ be a function from S to the positive real numbers such that if Q is in S then $\Delta(Q)$ is the largest number δ not exceeding 1 and having the property that if R is in S, $|R-Q|<\delta$, and $a \geqq x \geqq y \geqq b$ then $|V(x,y)Q-V(x,y)R| \leqq [\beta(x)-\beta(y)]\varepsilon$. Let D be a function such that if $a>z\geqq b$ and Q is in S then $D(z,Q)$ is the largest number x not exceeding a and having the property that if $x \geqq y \geqq z$ and t is a subdivision of $\{y,z\}$ then $\sum_t |VQ| \leqq \Delta(Q)/\exp\left(2\int_b^a |d\rho|\right)$ and $\int_z^y |d\rho| \leqq \frac{1}{2}$. Let u be a sequence defined by $u(0)=b$ and, if n is a nonnegative integer, then

$$u(n+1) = D\left(u(n), \prod_{q=1}^{n}[1-V(u_{n-q+1},u_{n-q})]^{-1}P\right).$$

Suppose that u is infinite. The sequence u is increasing and the sequence R defined by $R(j)=\prod_{q=1}^{j}[1-V(u_{j-q+1},u_{j-q})]^{-1}P$, $j=1,2,3,\ldots$, converges. To see this latter: by Lemma 2.0, there is a bounded set B which contains P and the values of R. Let α be a nondecreasing, continuous function such that if $a \geqq x \geqq y \geqq b$ and Q is in B then $|V(x,y)Q| \leqq \alpha(x)-\alpha(y)$. Let m and n be positive integers.

$$\begin{aligned}|R_m-R_n| &= \left|\left\{\prod_{j=1}^{m-n}[1-V(u_{m-j+1},u_{m-j})]^{-1}-1\right\}\prod_{p=1}^{n}[1-V(u_{n-p+1},u_{n-p})]^{-1}P\right| \\ &\leqq \exp\left(2\int_b^a |d\rho|\right)\sum_{j=1}^{m-n}\left|V(u_{m-j+1},u_{m-j})\prod_{p=1}^{n}[1-V(u_{n-p+1},u_{n-p})]^{-1}P\right| \\ &\leqq \exp\left(2\int_b^a |d\rho|\right)[\alpha(u_m)-\alpha(u_n)].\end{aligned}$$

The convergence of R now follows from the continuity of α. By Lemma 2.1, there is a positive number δ such that if n is a positive integer then $\Delta(R_n) \geqq \delta$. By the uniform continuity of α on $[b, a]$, there is a number d such that if $a \geqq x \geqq y \geqq b$ and $x-y<d$ then $\alpha(x)-\alpha(y)<\delta/\exp(2\int_b^a |d\rho|)$ and so, if n is an integer, then $D(y, R_n)>x$. This contradicts the assumption that u is infinite. Let m be the least integer such that $u(m)=a$ and define $s(p)$ to be $u(m-p)$ for $p=0, 1, 2, \ldots, m$. If k is an integer in $[1, m]$, $\{t_p\}_0^n$ is a subdivision of $\{s_{k-1}, s_k\}$, and j is an integer in $[1, n]$ then $\prod_{q=k+1}^m [1-V(s_{q-1}, s_q)]^{-1}P=R_{m-k}$ and

$$\left|\prod_{p=j}^n [1-V(t_{p-1}, t_p)]^{-1}R_{m-k}-R_{m-k}\right| \leqq \exp\left(2\int_b^a |d\rho|\right)\sum_{p=j}^n |V(t_{p-1}, t_p)R_{m-k}| \leqq \Delta(R_{m-k}).$$

Hence the conclusion of the lemma.

REMARK. The proof of the following theorem is similar to the proof of Theorem 1 of [19].

THEOREM 2.1. *If $a>b$, P is in S, β is as in condition* 4A *and $\varepsilon>0$ then there is a subdivision $\{s_p\}_0^m$ of $\{a, b\}$ such that if p is an integer in* $[1, m]$ *then $\int_{s(p)}^{s(p-1)} |d\rho| \leqq \frac{1}{2}$ and if t is a refinement of s then*

$$\left|\prod_s [1-V]^{-1}P-\prod_t [1-V]^{-1}P\right| \leqq \exp\left(2\int_b^a |d\rho|\right)[\beta(a)-\beta(b)]2\varepsilon.$$

Indication of proof. Suppose that $\{s_p\}_0^m$ is a subdivision of $\{a, b\}$ as indicated in Lemma 2.2. If k is an integer in $[1, m]$, $\{t_p\}_0^n$ is a subdivision of $\{s_{k-1}, s_k\}$, j is an integer in $[1, n]$, and $a \geqq x \geqq y \geqq b$, then

$$\Bigg|V(x, y)\prod_{p=j+1}^n [1-V(t_{p-1}, t_p)]^{-1}\prod_{q=k+1}^m [1-V(s_{q-1}, s_q)]^{-1}P$$
$$-V(x, y)\prod_{p=1}^n [1-V(t_{p-1}, t_p)]^{-1}\prod_{q=k+1}^m [1-V(s_{q-1}, s_q)]^{-1}P\Bigg| \leqq 2[\beta(x)-\beta(y)]\varepsilon.$$

Let $\{t_p\}_0^n$ be a refinement of s, u be an increasing sequence such that $u(0)=0$ and $t(u(p))=s(p)$, and define K to be the sequence given by

$$K_p = \prod_{1+u(p-1)}^{u(p)} [1-V(t_{q-1}, t_q)]^{-1}.$$

Then

$$\left|\prod_{q=1}^n [1-V(t_{q-1}, t_q)]^{-1}P-\prod_{p=1}^m [1-V(s_{p-1}, s_p)]^{-1}P\right|$$
$$= \left|\prod_{p=1}^m K_pP-\prod_{p=1}^m [1-V(s_{p-1}, s_p)]^{-1}P\right|$$
$$= \left|\sum_{p=1}^m\left\{\prod_{i=1}^p K_i\prod_{p+1}^m [1-V(s_{q-1}, s_q)]^{-1}P-\prod_{i=1}^{p-1} K_i\prod_{p}^m [1-V(s_{q-1}, s_q)]^{-1}P\right\}\right|$$
$$\leqq \sum_{p=1}^m\prod_{i=1}^{u(p-1)} \{1-[\rho(t_{i-1})-\rho(t_i)]\}^{-1}.$$

$$\cdot\left|K_p \prod_{p+1}^{m} [1-V(s_{q-1}, s_q)]^{-1}P - \prod_{p}^{m} [1-V(s_{q-1}, s_q)]^{-1}P\right|$$

$$\leqq \sum_{p=1}^{m} \prod_{i=1}^{u(p-1)} \{1-[\rho(t_{i-1})-\rho(t_i)]\}^{-1}\{1-[\rho(s_{p-1})-\rho(s_p)]\}^{-1}$$

$$\cdot\left|[1-V(s_{p-1}, s_p)]K_p \prod_{p+1}^{m} [1-V(s_{q-1}, s_q)]^{-1}P - \prod_{p+1}^{m} [1-V(s_{q-1}, s_q)]^{-1}P\right|$$

$$\leqq \exp\left(2\int_b^a |d\rho|\right) \sum_{p=1}^{m} \left|K_p \prod_{p+1}^{m} [1-V(s_{q-1}, s_q)]^{-1}P - \prod_{p+1}^{m} [1-V(s_{q-1}, s_q)]^{-1}P\right.$$

$$\left. - V(s_{p-1}, s_p)K_p \prod_{p+1}^{m} [1-V(s_{q-1}, s_q)]^{-1}P\right|$$

$$\leqq \exp\left(2\int_b^a |d\rho|\right) \sum_{p=1}^{m} \left| \sum_{q=1+u(p-1)}^{u(p)} V(t_{q-1}, t_q) \prod_{i=q}^{u(p)} [1-V(t_{i-1}, t_i)]^{-1}\right.$$

$$\cdot \prod_{j=p+1}^{m} [1-V(s_{j-1}, s_j)]^{-1}P$$

$$- V(t_{q-1}, t_q) \prod_{i=1+u(p-1)}^{u(p)} [1-V(t_{i-1}, t_i)]^{-1}$$

$$\left. \cdot \prod_{j=p+1}^{m} [1-V(s_{j-1}, s_j)]^{-1}P\right|$$

$$\leqq \exp\left(2\int_b^a |d\rho|\right)[\beta(a)-\beta(b)]2\varepsilon.$$

THEOREM 2.2. *If* $M(x, y)P = {}_x\prod^y [1-V]^{-1}P$ *for all* $x \geqq y$ *and* P *in* S *then* M *is in* OM.

Indication of proof. For property 1M, suppose that $a > b$, P and Q are in S, and $\{s_p\}_0^m$ is a subdivision of $\{a, b\}$ such that p is an integer in $[1, m]$ and $\rho(s_{p-1}) - \rho(s_p) < 1$. Then

$$\left|\prod_s [1-V]^{-1}P - \prod_s [1-V]^{-1}Q\right| \leqq \prod_s [1+d\rho]^{-1}|P-Q|.$$

Finally, ${}_a\prod^b [1+d\rho]^{-1} = \exp(\rho(a)-\rho(b))$. For property 2M, suppose that $x \geqq y \geqq z$ and P is in S. Let u be a subdivision of $\{x, z\}$ for which there is an integer j such that $u(j) = y$ and such that if v refines u then $|M(x, z)P - \prod_v [1-V]^{-1}P| < \varepsilon$. Let $\{t_p\}_0^n$ be a subdivision of $\{y, z\}$ such that if $v(p) = u(p)$ for $0 \leqq p \leqq j$ and $v(p) = t(p-j)$ for $j \leqq p \leqq j+n$ then v is a refinement of u and $|M(y, z)P - \prod_t [1-V]^{-1}P| < \varepsilon/\exp(\rho(x)-\rho(y))$. Let $\{s_p\}_0^m$ be a subdivision of $\{x, y\}$ such that if $v(p) = s(p)$ for $0 \leqq p \leqq m$ and $v(p) = t(p-j)$ for $m \leqq p \leqq m+n$ then v is a refinement of u and

$$\left|M(x, y)\prod_t [1-V]^{-1}P - \prod_s [1-V]^{-1}\prod_t [1-V]^{-1}P\right| < \varepsilon.$$

Then

$$\begin{aligned}
&|M(x,y)M(y,z)P-M(x,z)P| \\
&\quad\le \left|M(x,y)M(y,z)P-M(x,y)\prod_t [1-V]^{-1}P\right| \\
&\qquad+\left|M(x,y)\prod_t [1-V]^{-1}P-\prod_s [1-V]^{-1}\prod_t [1-V]^{-1}P\right| \\
&\qquad+\left|\prod_s [1-V]^{-1}\prod_t [1-V]^{-1}P-M(x,z)P\right| < 3\varepsilon.
\end{aligned}$$

For property 3M, if $a>b$, B is a bounded subset of S, α is as indicated in condition 3A, and $\{s_p\}_0^m$ is a subdivision of $\{x,y\}$ such that $\int_{s(p)}^{s(p-1)} |d\rho| \le \frac{1}{2}$ then $|\prod_s [1-V]^{-1}P-P| \le \exp(2\int_b^a |d\rho|)[\alpha(x)-\alpha(y)]$.

For property 4M, suppose that $a>b$, β is as in condition 4A, $\varepsilon>0$, and P is in S. Corresponding to P and ε, let δ be as in 4A. Corresponding to the bounded set containing only the point P let α be as in condition 3A. Let d be a positive number such that if $a\ge x\ge y\ge b$ and $x-y<d$ then $\alpha(x)-\alpha(y)<\delta/2\exp(2\int_b^a |d\rho|)$. Let Q be in S such that $|P-Q|<\delta/2\exp(2\int_b^a |d\rho|)$ and $a\ge x\ge y\ge b$ such that $x-y<d$. It follows that if t is a subdivision of $\{x,y\}$ then $|\prod_t [1-V]^{-1}Q-\prod_t [1-V]^{-1}P|<\delta/2$, $|\prod_t [1-V]^{-1}P-P|<\delta/2$, and $|\prod_t [1-V]^{-1}Q-P|<\delta$. Thus

$$\begin{aligned}
&\left|\left[\prod_t [1-V]^{-1}P-P\right]-\left[\prod_t [1-V]^{-1}Q-Q\right]\right| \\
&= \left|\sum_{p=1}^{n}\left\{V(t_{p-1},t_p)\prod_{i=p}^{n}[1-V(t_{i-1},t_i)]^{-1}P-V(t_{p-1},t_p)\prod_{i=p}^{n}[1-V(t_{i-1},t_i)]^{-1}Q\right\}\right| \\
&\le [\beta(x)-\beta(y)]2\varepsilon.
\end{aligned}$$

3. **From** OM **to** OA. In this section, suppose that M is in OM and ρ is as in condition 1M.

LEMMA 3.1. *Suppose that $a>b$, β is as in condition* 4M, *$\{R_p\}_{p=1}^\infty$ is a Cauchy sequence with values in S, and $\varepsilon>0$. There is a positive number δ and a positive number d having the property that if n is a positive integer, P is in S such that $|P-R_n|<\delta$, and $a\ge x\ge y\ge b$ such that $x-y<d$ then*

$$|[M(x,y)-1]P-[M(x,y)-1]R_n| \le [\beta(x)-\beta(y)]\varepsilon.$$

Indication of proof. Techniques applicable in the proof of Lemma 2.1 are also applicable here.

LEMMA 3.2. *Suppose that $a>b$, β is as in condition* 4M, *$\varepsilon>0$, and P is in S. There is a positive number e such that if $a\ge u\ge v\ge z\ge b$ and $u-z\le e$, then*

$$|[M(u,v)-1]M(v,z)P-[M(u,v)-1]P| \le [\beta(u)-\beta(v)]\varepsilon.$$

Indication of proof. With the supposition of the lemma, let δ and d be as indicated in condition 4M. Corresponding to the bounded set containing only the point P, let α be as indicated in 3M. Let c be a positive number such that if $a \geqq x \geqq y \geqq b$ and $x-y \leqq c$ then $\alpha(x)-\alpha(y)<\delta$ and e be the minimum of c and d. If z is in $[b, a]$ and $z+e \geqq u \geqq v \geqq z$ then $|[M(u, v)-1]M(v, z)P-[M(u, v)-1]P| \leqq [\beta(u)-\beta(v)]\varepsilon$.

THEOREM 3.1. *Suppose that $a>b$, P is in S, and $\varepsilon>0$. There is a subdivision s of $\{a, b\}$ such that if t refines s then* $|\sum_s [M-1]P-\sum_t [M-1]P| \leqq [\beta(a)-\beta(b)]\varepsilon$.

Indication of proof. With the supposition of the theorem, let e be as in the previous lemma and $\{s_p\}_0^m$ be a subdivision of $\{a, b\}$ such that $s(0)=a$ and, if p is a positive integer and $s(p-1)>b$, then $s(p)$ is the maximum of b and $s(p-1)-e$. Let t be a refinement of s and u be an increasing sequence such that $s(p)=t(u(p))$ for each integer p in $[0, m]$. Then

$$\begin{aligned} &\left|\sum_s [M-1]P-\sum_t [M-1]P\right| \\ &\quad = \left|\sum_{i=1}^{m}\left\{\sum_{p=1+u(i-1)}^{u(i)} [M(t_{p-1}, t_p)-1]M(t_p, s_i)P-[M(t_{p-1}, t_p)-1]P\right\}\right| \\ &\quad \leqq [\beta(a)-\beta(b)]\varepsilon. \end{aligned}$$

THEOREM 3.2. *If $V(x, y)P={}_x\sum^y [M-1]P$ for all $x \geqq y$ and P in S then V is in OA.*

Indication of proof. That V has properties 2A, 3A, and 4A is established with nearly the same techniques as used in the corresponding parts of Theorem 2.2. To prove that V has property 1A, consider the following propositions:

PROPOSITION 1. *If $a>b$, $\{t_p\}_0^n$ is a subdivision of $\{a, b\}$, $c>0$, and P and Q are in S then*

$$\left|\left\{1-c\sum_t [M-1]\right\}P-\left\{1-c\sum_t [M-1]\right\}Q\right| \geqq \left\{1-c\sum_t [\exp(-d\rho)-1]\right\}|P-Q|.$$

To see this, let A be $\{1-c\sum_t [M-1]\}P$ and B be $\{1-c\sum_t [M-1]\}Q$. Then

$$(1+cn)P = A+c\sum_{p=1}^{n} M(t_{p-1}, t_p)P \quad \text{and} \quad (1+cn)Q = B+c\sum_{p=1}^{n} M(t_{p-1}, t_p)Q.$$

Hence

$$(1+cn)|P-Q| \leqq |A-B|+c\sum_{p=1}^{n} \exp(\rho(t_{p-1})-\rho(t_p))|P-Q|$$

or

$$\left\{1-c\sum_{p=1}^{n} [\exp(\rho(t_{p-1})-\rho(t_p))-1]\right\}|P-Q| \leqq |A-B|.$$

PROPOSITION 2. *If $a>b$, $c>0$, and P and Q are in S then*

$$|[1-cV(a, b)]P-[1-cV(a, b)]Q| \geqq \{1-c[\rho(a)-\rho(b)]\}|P-Q|.$$

PROPOSITION 3. *If $a>b$ and $\rho(a)-\rho(b)<1$ then $1-V(a,b)$ has range all of S.*

To see this, let R be in S and A and B be functions from S to S defined as follows: $A(P)=V(a,b)P+R-P$ and $B(P)=V(a,b)P-P$ for each P in S. Then

$$\begin{aligned}\lim_{h\to 0-}&\frac{|[P-Q]+h[AP-AQ]|-|P-Q|}{h}\\ &=\lim_{h\to 0-}\frac{|[P-Q]+h[BP-BQ]|-|P-Q|}{h}\\ &=\lim_{h\to 0-}\frac{|[P-Q]+h[V(a,b)P-V(a,b)Q]|-|P-Q|}{h}-|P-Q|\\ &\leqq \{[\rho(a)-\rho(b)]-1\}|P-Q|.\end{aligned}$$

As in [12] each of these limits exists and by [12, Theorem 1], for each P in E, there is a function $U(\cdot)P$ from $[0,\infty)$ into S such that if P and Q are in S and x and y are nonnegative numbers, then

$$U(0)P=P,\qquad U(x)P=P-\int_x^0 A(U(\cdot)P)\,dI,\qquad U(x)U(y)=U(x+y),$$

and

$$|U(x)P-U(x)Q|\leqq \exp(\{[\rho(a)-\rho(b)]-1\}x)|P-Q|.$$

Hence, if $x>0$ then $U(x)$ is a contraction mapping and there is only one point Z_x such that $U(x)Z_x=Z_x$. However, if x and y are positive then $Z_x=Z_y$ for $U(y)Z_x=U(y)U(x)Z_x=U(x)U(y)Z_x$; so that $Z_x=U(y)Z_x$ or $Z_y=Z_x$. Hence, there is only one member Z of S such that $U(x)Z=Z$ for all nonnegative numbers x. Thus $Z=Z+xA(Z)$ for all $x\geqq 0$, or $Z=Z+V(a,b)Z+R-Z$, or $[1-V(a,b)]Z=R$.

4. **The one-to-one correspondence.** In §2, a mapping is defined from OA to OM and, in §3, a mapping is defined from OM to OA. This section will show that the composite of these mappings is the identity mapping.

LEMMA 4.1. *Suppose that V is in OA, M is in OM, V and M are related as in Theorem 2.2, $a>b$, β is as in condition 4A, P is in S, and $\varepsilon>0$. There is a positive number d such that if $a\geqq x\geqq y\geqq b$ and $x-y<d$ then*

$$|V(x,y)P-M(x,y)P+P|\leqq[\beta(x)-\beta(y)]\varepsilon.$$

Indication of proof. With the supposition of the lemma, let δ be as indicated in condition 4A. Corresponding to the bounded set containing only P, let α be as in condition 3A. Let d be a positive number such that if $a\geqq x\geqq y\geqq b$ and $x-y<d$ then $\alpha(x)-\alpha(y)<\delta/\exp(2\int_b^a|d\rho|)$. Let x and y be such that $x-y<d$ and $\{t_p\}_0^n$ be a subdivision of $\{x,y\}$. If p is an integer in $[1,n]$ then

$$\left|\prod_{i=p}^{n}[1-V(t_{i-1},t_i)]^{-1}P-P\right|\leqq\exp\left(2\int_b^a|d\rho|\right)\sum_t|VP|<\delta.$$

Thus

$$\left|\prod_t [1-V]^{-1}P-P-V(x,y)P\right|$$
$$= \left|\sum_{p=1}^{n}\left\{V(t_{p-1},t_p)\prod_{i=p}^{n}[1-V(t_{i-1},t_i)]^{-1}P-V(t_{p-1},t_p)P\right\}\right|$$
$$\leqq [\beta(x)-\beta(y)]\varepsilon.$$

THEOREM 4.1. *If V is in OA, $M(x,y)P={}_x\prod^y[1-V]^{-1}P$, and $U(x,y)P={}_x\sum^y[M-1]P$ for all $x\geqq y$ and P in S, then $U=V$.*

Indication of proof. Suppose that $a>b$, P is in S, $\varepsilon>0$, and d is as in the previous lemma. Let $\{t_p\}_0^n$ be a subdivision of $\{a,b\}$ such that if p is an integer in $[1,n]$ then $t_{p-1}-t_p<d$. Then $|\sum_t[M-1]P-V(a,b)P|\leqq[\beta(a)-\beta(b)]\varepsilon$.

LEMMA 4.2. *Suppose that M is in OM, V is in OA, M and V are related as in Theorem 3.2, $a>b$, P is in S, β is as in condition 4M, and $\varepsilon>0$. There is a subdivision $\{s_p\}_0^m$ of $\{a,b\}$ such that if k is an integer in $[1,m]$ then*

$$|[M(s_{k-1},s_k)-1]M(s_k,b)P-V(s_{k-1},s_k)M(s_k,b)P|\leqq[\beta(s_{k-1})-\beta(s_k)]\varepsilon.$$

Indication of proof. With the supposition of the lemma, let x be an increasing sequence defined inductively as follows: $x(0)=b$ and, if n is a positive integer such that $a>x(n-1)\geqq b$ then $x(n)$ is the largest number c not exceeding a such that if $c\geqq u\geqq v\geqq x(n-1)$ then

$$|[M(u,v)-1]M(v,x_{n-1})M(x_{n-1},b)P-[M(u,v)-1]M(x_{n-1},b)P|\leqq[\beta(u)-\beta(v)]\varepsilon.$$

The existence of such a number c follows from Lemma 3.2. Lemma 3.1 can be used to show that the supposition that x is an infinite sequence leads to a contradiction. Hence, there is an integer m such that $x(m)=a$. Let $s(p)$ be $x(m-p)$ for p an integer in $[0,m]$. Let k be an integer in $[1,m]$. If t is a subdivision of $\{s_{k-1},s_k\}$ then

$$\left|[M(s_{k-1},s_k)-1]M(s_k,b)P-\sum_t[M-1]M(s_k,b)P\right|$$
$$=\left|\sum_{p=1}^{n}[M(t_{p-1},t_p)-1]M(t_p,b)P-[M(t_{p-1},t_p)-1]M(s_k,b)P\right|$$
$$\leqq[\beta(s_{k-1})-\beta(s_k)]\varepsilon.$$

LEMMA 4.3. *Suppose that V is in OA, M is in OM, $a>b$, P is in S, β is as in condition 4A, and $\varepsilon>0$. There is a positive number d such that if $a\geqq x\geqq y\geqq b$, $x-y<d$, and $a\geqq u\geqq v\geqq b$ then $|V(u,v)M(x,b)P-V(u,v)M(y,b)P|\leqq[\beta(u)-\beta(v)]\varepsilon$.*

Indication of proof. A proof may be constructed using the fact $M(\cdot,b)P$ is continuous on $[b,a]$ and so $M([b,a],b)P$ is closed and compact.

THEOREM 4.2. *If M is in OM, $V(x,y)P={}_x\sum^y [M-1]P$, and $W(x,y)P={}_x\prod^y [1-V]^{-1}P$ for all $x \geqq y$ and P in S, then $W=M$.*

Indication of proof. Suppose that $a>b$, P is in S, $\varepsilon>0$, d is as in the previous lemma, and $\{s_p\}_0^m$ is a subdivision of $\{a,b\}$ having the property indicated in Lemma 4.2 and the property that if k is an integer in $[1,m]$ then $s_{k-1}-s_k<d$. Then

$$\left|\prod_{p=1}^{n}[1-V(s_{p-1},s_p)]^{-1}P-M(a,b)P\right|$$

$$=\left|\sum_{p=1}^{n}\prod_{i=1}^{p-1}[1-V(s_{i-1},s_i)]^{-1}M(s_{p-1},b)P-\prod_{i=1}^{p}[1-V(s_{i-1},s_i)]^{-1}M(s_p,b)P\right|$$

$$\leqq \exp\left(2\int_b^a |d\rho|\right)\sum_{p=1}^{n}|M(s_{p-1},b)P-M(s_p,b)P-V(s_{p-1},s_p)M(s_{p-1},b)P|$$

$$\leqq \exp\left(2\int_b^a |d\rho|\right)[\beta(a)-\beta(b)]2\varepsilon.$$

5. **The integral equation.** With the usual arguments, it can be shown that if $a>b$, f is a continuous function from $[b,a]$ to S, and V is in OA then the Riemann-Stieltjes integral $\int_a^b Vf$ exists. In this section it will be shown that the member M in OM related to the member V in OA as in Theorems 2.2 and 3.2 is the only member M of OM satisfying $M(x,y)P=P+\int_x^y VM(\cdot,y)P$ for all $x\geqq y$ and all P in S.

THEOREM 5.1. *Suppose that M is in OM, V is in OA, M and V are related as in Theorem 3.2, $a>b$, and P is in S. Then $M(a,b)P=P+\int_a^b VM(\cdot,b)P$.*

Indication of proof. With the supposition of the theorem, suppose that β is as in condition 4M and $\varepsilon>0$. By Lemma 4.2, if t is a subdivision of $\{a,b\}$ then there is a refinement $\{s_p\}_0^m$ of t such that if k is an integer in $[1,m]$ then

$$|[M(s_{k-1},s_k)-1]M(s_k,b)P-V(s_{k-1},s_k)M(s_k,b)P| \leqq [\beta(s_{k-1})-\beta(s_k)]\varepsilon.$$

Then

$$\left|M(a,b)P-P-\sum_s VM(\cdot,b)P\right|$$

$$=\left|\sum_{p=1}^{n} M(s_{p-1},b)P-M(s_p,b)P-V(s_{p-1},s_p)M(s_p,b)P\right|$$

$$\leqq [\beta(a)-\beta(b)]\varepsilon.$$

LEMMA 5.1. *Suppose that $a>b$, V is in OA, β is as in condition 4A, M is in OM, P is in S, and $\varepsilon>0$. There is a positive number d such that if $a\geqq x\geqq y\geqq b$ and $x-y<d$ then $|\int_x^y VM(\cdot,y)P-V(x,y)P|\leqq[\beta(x)-\beta(y)]\varepsilon$.*

Indication of proof. With the supposition of the lemma, let B be the bounded set consisting of only the point P and α be as in condition 3M. Corresponding to

$\{a, b\}$, P, and ε, let δ be as in 4A. There is a positive number d such that if $a \geqq x \geqq y \geqq b$ and $x-y<d$ then $|M(x, y)P-P| \leqq \alpha(x)-\alpha(y)<\delta$ and, hence,

$$\left|\int_x^y VM(\cdot, y)P - V(x, y)P\right| < [\beta(x)-\beta(y)]\varepsilon.$$

THEOREM 5.2. *If V is in OA, M is in OM, and $M(x, y)P=P+\int_x^y VM(\cdot, y)P$ for $x>y$ and P in S then V and M are related as in Theorem 3.2.*

Indication of proof. Suppose that P is in S, each of U and V is in OA, M is in OM, and, for $x \geqq y$, both the following hold: $M(x, y)P=P+\int_x^y VM(\cdot, y)P$ and $U(x, y)P={}_x\sum^y [M-1]P$. If $a>b$ and t is a subdivision of $\{a, b\}$ then

$$\left|\sum_t [M-1]P-V(a, b)P\right| = \left|\sum_{p=1}^n \int_{t(p-1)}^{t(p)} VM(\cdot, b)P-V(t_{p-1}, t_p)P\right|.$$

The previous lemma gives that $U=V$.

6. **Examples.** In view of the proof for Proposition 3 to Theorem 3.2, an alternate characterization of OA may be obtained by changing 1A to (see also [12, Example 2])

1A′. There is a continuous function ρ which is of bounded variation on each finite interval such that if $a>b$ and $c>0$ then

$$\{1-c[\rho(a)-\rho(b)]\}|P-Q| \leqq |[1-cV(a, b)]P-[1-cV(a, b)]Q|.$$

EXAMPLE 1. The class OA described in this paper contains the continuous members of the class OA described in [10]. That is, a sufficient condition that U be in OA is that if $a \geqq b$ then $U(a, b)$ is a function from S to S and

1. there is a nondecreasing, continuous function ρ such that if $a \geqq b$ and P and Q are in S, then $|U(a, b)P-U(a, b)Q| \leqq [\rho(a)-\rho(b)]\,|P-Q|$,
2. if $x \geqq y \geqq z$ and P is in S then $U(x, y)P+U(y, z)P=U(x, z)P$, and
3. if $x \geqq y$ then $U(x, y)0=0$.

EXAMPLE 2. Suppose that A is a function with values in S and that A has the following properties: (compare [18, Theorem 3])

(a) if t is a number then $A(t, \cdot)$ has domain all of S,

(b) if P is in S then $A(\cdot, P)$ is continuous,

(c) if $a>b$ and B is a bounded subset of S then A is bounded on $[b, a] \times B$,

(d) if $a>b$, P is in S, and $\varepsilon>0$ then there is a positive number δ having the property that if $a \geqq u \geqq b$ and Q is in S such that $|Q-P|<\delta$ then $|A(u, Q)-A(u, P)| <\varepsilon$, and

(e) there is a continuous function ρ such that if t is a number, P and Q are in S, and $c>0$ then $|[P-cA(t, P)]-[Q-cA(t, Q)]| \geqq [1-c\rho(t)]\,|P-Q|$.

THEOREM 6.1. *If $a>b$ and Q is in S then $\int_b^a A(\cdot, Q)\,dI$ exists and, if $V(x, y)P$ is defined to be $\int_y^x A(\cdot, P)\,dI$ for $x \geqq y$ and P in S, then V is in OA.*

By the usual arguments, it can be shown that if $a>b$ and P is in S then the Riemann-Stieltjes integral $\int_b^a A(\cdot, P)\, dI$ exists. That this integral generates a member of OA will be proved in the next sequence of lemmas.

Let S^* be the dual space of S and $|\cdot|$ denote the norm on S^*. As in [8] if x is in S, denote by Fx the set of all functions f in S^* such that $f(x)=|x|^2=|f|^2$. As in [11], if x is in S, denote by Gx the set of all functions g in S^* such that $g(x)=|x|$ and $|g|=1$. Note that g is in Gx only in case $|x|\cdot g$ is in Fx.

LEMMA 6.1. *If x and y are in S and k is a number then these are equivalent*:

(i) *if* $c>0$ *then* $(1-ck)|x|\leqq|x+cy|$, *and*

(ii) *there is a member f of Fx such that* $\operatorname{Re} f(y)\geqq -k|x|^2$.

Indication of proof. The proof of [8, Lemma 1.1] may be used with only minor modifications.

LEMMA 6.2 (MARTIN, [12, REMARK 4]). *Suppose that k is a number and L is a function which is continuous from S to S. These are equivalent: for all P and Q*

(1) *if* $c>0$ *then* $|[P-cL(P)]-[Q-cL(Q)]|\geqq[1-ck]|P-Q|$, *and*

(2) *if g is in $G(P-Q)$ then* $\operatorname{Re} g(L(P)-L(Q))\leqq k|P-Q|$.

Indication of proof. With the supposition of the lemma, consider the following statements, for all P and Q is S:

(i) if $c>0$ then

$$|[P-Q]-c[L(P)-L(Q)]| \geqq [1-ck]|P-Q|,$$

(ii)
$$\lim_{h\to 0-} \frac{|[P-Q]+h[L(P)-L(Q)]|-|P-Q|}{h} \leqq k|P-Q|,$$

(iii)
$$\lim_{h\to 0+} \frac{|[P-Q]+h[L(P)-L(Q)]|-|P-Q|}{h} \leqq k|P-Q|,$$

(iv) if g is in $G(P-Q)$ then

$$\operatorname{Re} g(L(P)-L(Q)) \leqq k|P-Q|.$$

That (i) implies (ii) can be seen by rearranging the inequality in (i). That (ii) implies (iii) follows from [12, Remark 4]. That (iii) and (iv) are equivalent follows from [12, Example 1] and [11, Corollary 2.2]. That (iv) implies (i) follows from Lemma 6.1 and the preceding remarks and definitions.

Indication of proof for Theorem 6.1. Let A have the properties (a)–(e) and V be as defined in the theorem. It is not difficult to show that V has properties 2A–4A. Suppose that $a>b$ and $\{t_p\}_0^{2n}$ is a nondecreasing sequence such that $t(0)=b$ and $t(2n)=a$. Let P and Q be in S and g be a member of $G(P-Q)$. Then

$$\operatorname{Re} g\left(\sum_{p=1}^{n} A(t_{2p-1}, P)\cdot(t_{2p}-t_{2p-2})-\sum_{p=1}^{n} A(t_{2p-1}, Q)\cdot(t_{2p}-t_{2p-2})\right)$$
$$\geqq \sum_{p=1}^{n} (t_{2p}-t_{2p-2})\rho(t_{2p-1})|P-Q|.$$

Hence, if $c>0$ then

$$\left|\left[P-c\int_b^a A(\cdot,P)\,dI\right]-\left[Q-c\int_b^a A(\cdot,Q)\,dI\right]\right| \geqq \left[1-c\int_b^a \rho\,dI\right]|P-Q|$$

or

$$|[1-cV(a,b)]P-[1-cV(a,b)]Q| \geqq \left[1-c\int_b^a \rho\,dI\right]|P-Q|.$$

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