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ON THE EXISTENCE OF STRONGLY SERIES SUMMABLE MARKUSCHEVICH BASES IN BANACH SPACES

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Abstract. The main result is: *Let X be a complex separable Banach space. If the identity operator on X^* is the limit in the strong operator topology of a uniformly bounded net of linear operators of finite rank, then X admits a strongly series summable Markuschevich basis.*

I. **Introduction.** Let X be a separable Banach space. A biorthogonal sequence $\{x_i, f_i\}_{i=1}^{\infty}$ in (X, X^*) is called a Markuschevich basis (M-basis) for X provided $\{x_i\}_{i=1}^{\infty}$ is fundamental in X and $\{f_i\}_{i=1}^{\infty}$ is total over X. Following Ruckle [8], we say that an M-basis $\{x_i, f_i\}_{i=1}^{\infty}$ for X is strongly series summable (s.s.s.) provided there exists a set $\{\lambda_{i,n} : i=1, 2, \ldots, n; n=1, 2, \ldots\}$ of scalars (called a summation matrix for $\{x_i, f_i\}_{i=1}^{\infty}$) such that, for each x in X, $x=\lim_{n\to\infty} \sum_{i=1}^{n} \lambda_{i,n} f_i(x) x_i$. Note that a Schauder basis is a s.s.s. M-basis for which each $\lambda_{i,n}$ can be chosen to be 1.

The results of [8], [9], and [4] indicate that the duality theory of a space which has a s.s.s. M-basis is essentially the same as that of a space which admits a Schauder basis. The reason for this appears to be that if $\{x_i, f_i\}_{i=1}^{\infty}$ is a s.s.s. M-basis for X with summation matrix $(\lambda_{i,n})$ then, for each f in the coefficient space(1) of the basis, f is the norm limit of $\{\sum_{i=1}^{n} \lambda_{i,n} f(x_i) f_i\}_{n=1}^{\infty}$. Thus the adjoints of the "partial sum" operators defined by $T_n(x)=\sum_{i=1}^{n} \lambda_{i,n} f_i(x) x_i$ also act like partial sum operators. In this respect s.s.s. M-bases behave more like Schauder bases than do such weaker structures as generalized summation bases (see [3]).

In this paper we prove the following rather strong existence theorem for s.s.s. M-bases:

THEOREM 1. *Let X be a separable complex Banach space such that X^* has the λ-metric approximation property for some $\lambda \geqq 1$. If Y is a separable subspace of X^*, then there exists a strongly series summable Markuschevich basis for X whose coefficient space contains Y.*

If $\lambda \geqq 1$, we say that the Banach space X has the λ-metric approximation property (λ-m.a.p.) if there is a net $\{S_d : d \in D\}$ of linear operators of finite rank on X

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(1) The coefficient space of an M-basis $\{x_i, f_i\}_{i=1}^{\infty}$ for X is the norm closure in X^* of the linear span of $\{f_i\}_{i=1}^{\infty}$.

uniformly bounded by λ which converges pointwise (i.e., in the strong operator topology) to the identity operator on X. Equivalently, X has the λ-m.a.p. provided that, for each finite-dimensional subspace F of X and positive number ε, there is an operator S of finite rank on X such that $\|S\| \leqq \lambda$ and $\|S(x)-x\| \leqq \varepsilon\|x\|$ for each $x \in F$.

The 1-m.a.p. was introduced by Grothendieck [2] under the name metric approximation property. Grothendieck showed that if X is reflexive and has the (topological) approximation property, then in fact both X and X^* have the 1-m.a.p. This result together with Theorem 1 implies that every separable, reflexive complex Banach space which has the approximation property also admits a s.s.s. M-basis. Of course, it may be that Theorem 1 is always applicable, for it is not even known that there exists a Banach space which does not have the 1-m.a.p.

We use the following notation: X represents a complex Banach space and X^* is the dual to X. The complex assumption is used in an essential way in Lemma 4, and we do not know whether the real version of Theorem 1 is true. I denotes the identity operator on either X or X^*. "Operator" means "bounded linear operator". The range space and null space of an operator, L, are denoted by, respectively, $\mathscr{R}(L)$ and $\ker L$. If L is an operator on X and S is a subspace of X, $L_{|S}$ denotes the restriction of L to S. The linear span of a subset, A, of a linear space is denoted by sp A. The canonical embedding of X into X^{**} is denoted by "$\hat{\ }$".

II. The existence theorem.

Our first lemma is both a generalization and a special case of Helly's theorem [11, p. 103].

LEMMA 1. *Let F be a finite-dimensional Banach space, S a finite-dimensional subspace of X^*, L an operator from X^* into F, and $\varepsilon>0$. There exists a weak*-continuous operator T from X^* into F such that $T_{|S}=L_{|S}$ and $\|T\| \leqq \|L\|+\varepsilon$.*

Proof. We use the notation of [10] in this proof. We identify the weak*-continuous operators from X^* to F with $X \otimes_\lambda F$ and the operators from X^* to F with $X^{**} \otimes_\lambda F$ [10, p. 30]. Since F is finite dimensional, $X^{**} \otimes_\lambda F$ is thereby identified with $(X \otimes_\lambda F)^{**}$. Now $S \otimes F^*$ is identified with a (finite-dimensional) subspace of $(X \otimes_\lambda F)^*$, so by Helly's theorem [11, p. 103], there is T in $X \otimes_\lambda F$ such that $\|T\| \leqq \|L\|+\varepsilon$ and $f(T(s))=f(L(s))$ for each $s \in S$ and $f \in F^*$. Since F^* is total over F, $T(s)=L(s)$ for each $s \in S$ and hence $T_{|S}=L_{|S}$. Q.E.D.

A Banach space X is said to have the λ duality metric approximation property ($\lambda \geqq 1$) provided there is a net $\{S_d : d \in D\}$ of operators of finite rank on X uniformly bounded by λ such that $\{S_d : d \in D\}$ converges pointwise to I and $\{S_d^* : d \in D\}$ converges pointwise to I. Equivalently, X has the λ duality m.a.p. provided that, for each $\varepsilon>0$ and each pair of finite-dimensional subspaces E of X and F of X^*, there is an operator L of finite rank on X such that $\|L\| \leqq \lambda$, $\|L(x)-x\| \leqq \varepsilon\|x\|$ for each $x \in E$, and $\|L^*(f)-f\| \leqq \varepsilon\|f\|$ for each $f \in F$.

LEMMA 2. *Suppose that X^* has the λ-m.a.p. Then X has the λ duality m.a.p.*

Proof. Using the hypothesis and Lemma 1, we can construct a net $\{S_d : d \in D\}$ of operators of finite rank on X uniformly bounded by λ such that $\{S_d^* : d \in D\}$ is pointwise convergent on X^* to I. For each $x \in X$, the net $\{S_d(x) : d \in D\}$ weakly converges to x, hence (cf., e.g., [1, p. 477]) there is a net $\{T_e : e \in E\}$ of operators on X such that $\{T_e : e \in E\}$ is pointwise convergent on X to I; each T_e is a convex combination of a subset $\{S_{e(i)}\}_{i=1}^{n_e}$ of $\{S_d : d \in D\}$; and for each $d \in D$ there is $e' \in E$ such that if $e \geqq e'$ then $e(i) \geqq d$ for $i=1, 2, \ldots, n_e$. Thus $\{T_e : e \in E\}$ is uniformly bounded by λ and $\{T_e^* : e \in E\}$ is pointwise convergent on X^* to I. Q.E.D.

The proof of the next lemma is suggested by the proof of Lemma 3.1 of [5].

LEMMA 3. *Suppose that X has the λ duality m.a.p., E is a finite-dimensional subspace of X, F is a finite-dimensional subspace of X^*, and $\varepsilon > 0$. Then there exists an operator L of finite rank on X such that $\|L\| \leqq \lambda + \varepsilon$, $L_{|E} = I_{|E}$, and $L_{|F}^* = I_{|F}$.*

Proof. Let $n = \dim E$ and $m = \dim F$. Choose $1 > \beta > 0$ small enough so that $\beta + \beta m(\lambda+\beta)/(1-\beta) \leqq \varepsilon$ and choose $1 > \alpha > 0$ small enough so that $(n\alpha/(1-\alpha))\lambda \leqq \beta/2$.

Let M be an operator of finite rank on X such that $\|M\| \leqq \lambda$ and, for each $x \in E$ and $f \in F$,

(1) $\|x - M(x)\| \leqq \alpha \|x\|$ and

(2) $\|f - M^*(f)\| \leqq \beta/2 \|f\|$.

By (1), for each $x \in E$, $(1-\alpha)\|x\| \leqq \|M(x)\|$, hence $M_{|E}$ has an inverse, Q, satisfying $\|Q\| \leqq 1/(1-\alpha)$. Also, for each $y \in M[E]$, $\|Q(y) - y\| \leqq (\alpha/(1-\alpha))\|y\|$.

Let P be a projection of X onto $M[E]$ such that $\|P\| \leqq n$. Let $N = QPM + (I-P)M$. Clearly N has finite rank and $N_{|E} = I_{|E}$. Now if $x \in X$,

$$\|N(x) - M(x)\| = \|QPM(x) - PM(x)\| \leqq \frac{\alpha}{1-\alpha} \|P\| \, \|M\| \, \|x\| \leqq \frac{n\alpha\lambda}{1-\alpha} \|x\|.$$

Thus $\|N - M\| \leqq n\alpha\lambda/(1-\alpha)$, from which it follows that $\|N^*\| \leqq \lambda + \beta/2$ and $\|N^* - M^*\| \leqq \beta/2$. This last inequality and (2) imply that, for each $f \in F$, $\|N^*(f) - f\| \leqq \beta \|f\|$.

As in the first part of the proof, we have that $N_{|F}^*$ is an isomorphism with inverse, Q', satisfying $\|Q'(f) - f\| \leqq (\beta/(1-\beta))\|f\|$ for each $f \in N^*[F]$. Let P' be a projection of X^* onto $N^*[F]$ such that $\|P'\| \leqq m$ and let $L^* = Q'P'N^* + (I-P')N^*$. (Note that L^* is indeed weak*-continuous because N^* is weak*-continuous and has finite rank.) Then $L_{|F}^* = I_{|F}$ and, for each $f \in X^*$,

$$\|L^*(f) - N^*(f)\| \leqq \frac{\beta}{1-\beta} \|P'\| \, \|N^*\| \, \|f\| \leqq \frac{\beta m(\lambda+\beta)}{1-\beta} \|f\|.$$

Thus $\|L^*\| \leqq \lambda + \beta + \beta m(\lambda+\beta)/(1-\beta) \leqq \lambda + \varepsilon$.

Since $\|L\| = \|L^*\|$, it remains to be seen only that $L_{|E} = I_{|E}$. Let $x \in E$ and suppose that $f \in X^*$. Then using the fact that $x = N(x)$, we have

$$\begin{aligned} f(L(x)) &= L^*(f)(x) = L^*(f)(N(x)) \\ &= N^*Q'P'N^*(f)(x) + N^*(I-P')N^*(f)(x) \\ &= P'N^*(f)(x) + f(N(N(x))) - P'N^*(f)(N(x)) = f(x). \end{aligned}$$

Since X^* is total over E, $L(x) = x$. Q.E.D.

LEMMA 4. *Let $\{x_i, f_i\}_{i=1}^n$ be a finite biorthogonal set in (X, X^*), let T be an operator of finite rank on X such that $T(x_i)=x_i$ and $T^*(f_i)=f_i$ for $i=1, 2, \ldots, n$, and let $\varepsilon>0$. Then there exists a finite biorthogonal set $\{x_i, f_i\}_{i=n+1}^{n+m}$ in (X, X^*) and a set $\{\lambda_i\}_{i=1}^m$ of complex numbers such that $\{x_i, f_i\}_{i=1}^{n+m}$ is biorthogonal and $\|L-T\| \leqq \varepsilon$, where L is the operator on X defined by*

$$L(x)=\sum_{i=1}^{n} f_i(x)x_i+\sum_{i=n+1}^{n+m} \lambda_{i-n}f_i(x)x_i.$$

Proof. Define a projection U on X by $U(x)=\sum_{i=1}^n f_i(x)x_i$ and let $X_0=\mathscr{R}(I-U)$. Note that $TU=UT=U$, so $T[X_0]\subset X_0$ and $\ker T\subset X_0$. Let P be a projection of finite rank on X_0 such that $PT(I-U)=TP(I-U)=T(I-U)$. (For example, choose $\ker P$ to be a closed complement in $\ker T$ to $\mathscr{R}(T)\cap \ker T$ and choose $\mathscr{R}(P)$ to be a complement in X_0 to $\ker P$ which contains $\mathscr{R}(T)\cap X_0$.)

Let $m=\dim \mathscr{R}(P)$ and choose a basis $\{z_i\}_{i=1}^m$ for $\mathscr{R}(P)$ such that the matrix representation $(\alpha_{ij})_{i,j=1}^m$ of $T_{|\mathscr{R}(P)}$ with respect to $\{z_i\}_{i=1}^m$ is lower triangular—i.e., $\alpha_{ij}=0$ if $j>i$. Now pick a sequence $\{\lambda_i\}_{i=1}^m$ of pairwise distinct complex numbers sufficiently close to $\{\alpha_{ii}\}_{i=1}^m$ so that

$$\|Q-T_{|\mathscr{R}(P)}\| \leqq \varepsilon/\|P\| \, \|I-U\|,$$

where Q is the operator on $\mathscr{R}(P)$ whose matrix representation, (β_{ij}), with respect to $\{z_i\}_{i=1}^m$ is given by

$$\begin{aligned} \beta_{ij} &= \lambda_i \quad \text{if } i=j, \\ &= \alpha_{ij} \quad \text{if } i\neq j. \end{aligned}$$

Since (β_{ij}) is lower triangular, $\{\lambda_i\}_{i=1}^m$ is the set of eigenvalues for Q. The λ_i's are distinct, so there is a basis $\{x_i\}_{i=n+1}^{n+m}$ for $\mathscr{R}(P)$ such that $Q(x_i)=\lambda_{i-n}x_i$ for $i=n+1, \ldots, n+m$. Picking $\{f_i\}_{i=n+1}^{n+m}$ in $\mathscr{R}([P(I-U)]^*)$ biorthogonal to $\{x_i\}_{i=n+1}^{n+m}$, we have that, for each $x\in\mathscr{R}(P)$, $Q(x)=\sum_{i=n+1}^{n+m}\lambda_{i-n}f_i(x)x_i$.

Now $\{x_i, f_i\}_{i=1}^{n+m}$ is biorthogonal and if L is defined by

$$L(x) = \sum_{i=1}^{n} f_i(x)x_i+\sum_{i=n+1}^{n+m} \lambda_{i-n}f_i(x)x_i,$$

then clearly $L=TU+QP(I-U)$. Thus

$$\begin{aligned} \|L-T\| &= \|TU+QP(I-U)-TU-TP(I-U)\| \\ &\leqq \|Q-T_{|\mathscr{R}(P)}\| \, \|P\| \, \|I-U\| \leqq \varepsilon. \end{aligned}$$

Q.E.D.

Proof of Theorem 1. Let λ be such that X has the λ duality m.a.p. (Lemma 2). Let $\{z_i\}_{i=1}^\infty$ be fundamental in X and let $\{g_i\}_{i=1}^\infty$ be a subset of X^* such that Y is contained in the closure of the linear span of $\{g_i\}_{i=1}^\infty$. Assume, without loss of generality, that $\|z_1\|=\|g_1\|=g_1(z_1)=1$. We define the desired s.s.s. M-basis $\{x_i, f_i\}_{i=1}^\infty$ for X and a summability matrix $(\lambda_{i,n})$ for $\{x_i, f_i\}_{i=1}^\infty$ by induction. Set $k(1)=1$, $x_1=z_1$, $f_1=g_1$, $\lambda_{1,1}=1$. Now suppose $k(m)$, $\{x_i, f_i\}_{i=1}^{k(m)}$, and

$\{\lambda_{i,n} : i \leqq n; n=1, 2, \ldots, k(m)\}$ have been defined. Extend $\{x_i, f_i\}_{i=1}^{k(m)}$ to a biorthogonal set $\{x_i, f_i\}_{i=1}^{j}$ $(j=k(m), k(m)+1$, or $k(m)+2)$ so that $z_{m+1} \in \operatorname{sp}\{x_i\}_{i=1}^{j}$ and $g_{m+1} \in \operatorname{sp}\{f_i\}_{i=1}^{j}$ (cf., e.g., the proof of Theorem III.1 in [3]). Now by Lemma 3 and Lemma 4 there are a positive integer $k(m+1) \geqq j$, a biorthogonal set $\{x_i, f_i\}_{i=j+1}^{k(m+1)}$ in (X, X^*), and complex numbers $\{\alpha_i\}_{i=j+1}^{k(m+1)}$ such that $\{x_i, f_i\}_{i=1}^{k(m+1)}$ is biorthogonal and if T is defined on X by $T(x)=\sum_{i=1}^{j} f_i(x)x_i + \sum_{i=j+1}^{k(m+1)} \alpha_i f_i(x)x_i$, then $\|T\| \leqq \lambda + 1/m$. We complete the induction by defining

$$\begin{aligned} \lambda_{i,n} &= \lambda_{i,k(m)} && \text{if } i \leqq k(m) < n < k(m+1), \\ &= 0 && \text{if } k(m) < i \leqq n < k(m+1), \\ &= 1 && \text{if } i \leqq k(m) \text{ and } n = k(m+1), \\ &= \alpha_i && \text{if } k(m) < i \leqq k(m+1) = n. \end{aligned}$$

It is easy to check that $\{x_i, f_i\}_{i=1}^{\infty}$ has the desired properties. Q.E.D.

REMARK 1. Suppose that X is separable and X^* has the λ-m.a.p. for some λ. Theorem 1 shows that there are s.s.s. M-bases for X whose coefficient spaces are "arbitrarily large". One might guess that if Y is a separable subspace of X^* and Y contains a subspace which is the coefficient space of some s.s.s. M-basis for X, then Y is itself the coefficient space for some s.s.s. M-basis for X, because the corresponding statement for generalized summation bases is true (cf. [3, proof of Theorem IV.1]). This is not the case: Let $X=l_1$. It is a rather easy consequence of Theorem 4.3 of [6] that the coefficient space of any s.s.s. M-basis for l_1 is an $\mathscr{L}_\infty$ space in the sense of [6]. Simply pick Y to be a separable subspace of l_∞ $(=l_1^*)$ which contains c_0 but is not an $\mathscr{L}_\infty$ space. (For example, Y can be the closed span of $c_0 \cup K$, where K is a subspace of l_∞ isomorphic to l_2. It follows from Theorem 1 of [7] that Y is isomorphic to $c_0 \oplus l_2$ and is thus not an $\mathscr{L}_\infty$ space.)

Recall that an M-basis $\{x_i, f_i\}_{i=1}^{\infty}$ for X whose coefficient space is X^* is called shrinking (see [3]). Now if $\{x_i, f_i\}_{i=1}^{\infty}$ is a s.s.s. M-basis then the remarks in the introduction show that $\{f_i, \hat{x}_i\}_{i=1}^{\infty}$ is a s.s.s. M-basis for the coefficient space of the basis. Thus a shrinking M-basis $\{x_i, f_i\}_{i=1}^{\infty}$ which is s.s.s. is also shrinking as a s.s.s. M-basis in the sense that $\{f_i, \hat{x}_i\}_{i=1}^{\infty}$ is a s.s.s. M-basis for X^*. In view of Theorem 1, we thus have

COROLLARY 1. *If X^* is separable and has the λ-m.a.p. for some $\lambda \geqq 1$, then X admits a shrinking s.s.s. M-basis.*

Let us say that a s.s.s. M-basis $\{x_i, f_i\}_{i=1}^{\infty}$ is *boundedly complete* provided there is a summability matrix $(\lambda_{i,n})$ for $\{x_i, f_i\}_{i=1}^{\infty}$ such that for every sequence $\{t_i\}_{i=1}^{\infty}$ of scalars, if $\{\sum_{i=1}^{n} t_i \lambda_{i,n} x_i\}_{n=1}^{\infty}$ is bounded then it is convergent. A simple modification of Theorem II.3 of [3] shows that a s.s.s. M-basis is *boundedly complete* if and only if it is boundedly complete as an M-basis in the sense of [3] (and thus in the above definition of *boundedly complete* "there is a summability matrix" can be replaced by "for each summability matrix"). Thus using Corollary 1 and the results of [3] we have

THEOREM 2. *X admits a boundedly complete s.s.s. M-basis if and only if X has the λ-m.a.p. for some $\lambda \geqq 1$ and X is isomorphic to a separable conjugate Banach space.*

We conclude with a conjecture which, by Theorem 2, has an affirmative answer if X is a conjugate space:

CONJECTURE 1. If X is separable and has the λ-m.a.p. for some $\lambda \geqq 1$, then X admits a s.s.s. M-basis.

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