TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 157, June 1971

ADDENDA TO "A VARIATIONAL PROBLEM RELATED TO AN OPTIMAL FILTER PROBLEM WITH SELF-CORRELATED NOISE"

BY
LEONARD D. BERKOVITZ[1] AND HARRY POLLARD

Abstract. The explicit solution is given of a nonclassical variational problem that is related to an optimal filter problem.

1. **Introduction.** In [1] we showed that the following variational problem has a unique solution and we gave a partial characterization of the solution. We also discussed the relationship of the variational problem to an optimal filter problem.

PROBLEM. Let Y denote the class of functions y that are in $L_1[0, \infty]$, that are absolutely continuous, that satisfy

$$y(0) = 0, \tag{1.1}$$

and have the property that the function G defined by

$$G(u) = -y'(u)+e^{-u}+e^{-u}\int_0^u e^t y'(t)\,dt \tag{1.2}$$

is in $L_2[0, \infty]$. Minimize the functional

$$J(y) = \left(\int_0^\infty |y|\,dt\right)^2+\int_0^\infty G^2\,dt \tag{1.3}$$

in the class Y.

In this paper we shall give the explicit solution of the variational problem. As in [1] we denote the unique minimizing function by z and we denote by F the function obtained by taking $y'=z'$ in the right-hand side of (1.2). We denote the value of the minimum of the functional J by m and let

$$I = \int_0^\infty |z|\,dt.$$

In [1] we showed that z has compact support $(0, \lambda)$ and that on its support z is composed of a finite number of contiguous quartic arches whose signs alternate. In this paper we show that on its support z consists of precisely *one* quartic arch. The principal result of this paper is the following theorem.

Received by the editors April 10, 1970.

AMS 1969 *subject classifications*. Primary 4900, 9340; Secondary 9410.

Key words and phrases. Nonclassical variational problem, optimal min-max filter.

[1] The work of this author was supported by NSF Grant GP-11477.

THEOREM 3. *The minimizing function is*

$$z(u) = q(u), \qquad 0 \leqq u \leqq \lambda,$$
$$= 0, \qquad u \geqq \lambda,$$

where

$$q(u) = -\frac{Iu^4}{24} + \frac{(1-\nu)^2u^3}{12} + \frac{(2I+\nu^2-1)u^2}{4} + \nu u$$

and $\nu = (1-2m)^{1/2}$. *The numerical values of the constants, correct to six decimal places, are*

$$m = 0.478318, \qquad I = 0.106953,$$
$$\nu = 0.208234, \qquad \lambda = 2.732889.$$

Note that since z and z' are continuous (the absolute continuity of z' is proved in Theorem 2 of [1]), the quartic q has a double zero at $u = \lambda$.

The remainder of this paper is a continuation of [1]. We assume that the reader has [1] at hand. We use the notation and numbering of [1]. Thus, the next section of this paper will be numbered 10 since the last section of [1] is numbered 9.

10. Further characterization of z.

LEMMA 11. *Let I be given by* (5.3), *let m be the value of the minimum, and let*

$$\nu = (1-2m)^{1/2}. \tag{10.1}$$

Then

$$z'(0) = (1-2m)^{1/2} = \nu, \tag{10.2}$$

$$z''(0) = I - m = (2I+\nu^2-1)/2, \tag{10.3}$$

$$z'''(0) = 1 - m - (1-2m)^{1/2} = (1-\nu)^2/2. \tag{10.4}$$

Since $0 < m < 1/2$ (Lemma 7), the quantity ν defined in (10.1) is real. Equation (10.2) is an immediate consequence of (9.1), (9.2), and (10.1). Equation (10.3) follows from Corollary 2 of Lemma 9 and from (10.1). To establish (10.4) we proceed as follows. From (8.10) and (8.7) we get

$$z'''(u) = -\Psi'(u) = F(u) - \Psi(u).$$

If we set $u = 0$ in the preceding equation and use (8.1), (5.8) and (5.7), we get

$$z'''(0) = F(0) - m = F^2(0)/2. \tag{10.5}$$

If we now use (9.2) and (10.2) we obtain the first equality in (10.4). The second follows from (10.1).

Another useful relationship is

$$1 - 2m = \int_0^\lambda e^{-t} z'(t)\, dt, \tag{10.6}$$

which is established as follows. Substitute (1.2), with y' replaced by z' and G replaced by F, into (5.8) and interchange the order of integration in the iterated integral.

As a corollary of the next lemma we shall obtain an inequality involving I and ν that will be used to show that z has only one arch.

LEMMA 12. *Let*

$$K = 2\int_0^\infty e^{-u}g'(u)\,du+\int_0^\infty (1+u)e^{-u}z(u)\,du, \tag{10.7}$$

where g is defined by (6.7). *Then*

$$K = (9\nu^2-4\nu+1)/2.$$

In the course of showing that F is absolutely continuous we showed that g is absolutely continuous and that $|g'(u)|\leqq I$ [1, p. 166, first sentence]. Hence the first integral in (10.7) is finite. Moreover, from the inequality $|g'(u)|\leqq I$ and the inequality $(1+u)\exp(-u)\leqq 1$ for $u\geqq 0$, we obtain the following consequence of Lemma 12:

$$[9\nu^2-4\nu+1]/6 \leqq I. \tag{10.8}$$

We now proceed to establish Lemma 12. From (7.8), (6.7), and (8.5), we get that

$$g'(u) = z''(u)+e^u\int_u^\infty e^{-t}F(t)\,dt.$$

If we substitute this expression for g' into the right-hand side of (10.7), we get

$$K = 2\int_0^\infty e^{-u}z''(u)\,du+2\int_0^\infty du\int_u^\infty e^{-t}F(t)\,dt-\int_0^\infty z(u)\,d((u+2)e^{-u}). \tag{10.9}$$

Since z' is absolutely continuous (Theorem 2 of [1]) we may write $z''(u)$ as $dz'(u)$ in the first integral in (10.9). We now develop the right-hand side of (10.9) as follows. Interchange the order of integration in the second integral and perform the resulting integration with respect to t. In the first and third integrals, integrate by parts, taking into account $z(0)=z(\infty)=z'(\infty)=0$ and (10.2). The result is

$$K = -2\nu+4\int_0^\infty e^{-u}z'(u)\,du+2\int_0^\infty te^{-t}F(t)\,dt+\int_0^\infty ue^{-u}z'(u)\,du.$$

If we now use (10.6), we get

$$K = 4-2\nu-8m+\int_0^\infty ue^{-u}(2F(u)+z'(u))\,du. \tag{10.10}$$

Let L denote the integral on the right-hand side of (10.10). If we use (1.2) we get

$$L = -\int_0^\infty e^{-u}z'(u)\,du+\int_0^\infty (2u)e^{-2u}\,du+2\int_0^\infty ue^{-2u}\,du\int_0^u e^tz'(t)\,dt.$$

The value of the second integral is $\frac{1}{2}$. In the last integral we interchange the order of integration and perform the integration with respect to u. The result is

$$\int_0^\infty e^{-t}(t+\tfrac{1}{4})z'(t)\,dt.$$

If we relabel the variable of integration in the last integral as u and substitute into the defining relation for L we get

$$2L = 1+\int_0^\infty e^{-u}z'(u)\,du.$$

From this and from (10.6) we get $L=1-m$. Substituting this into (10.10) gives $K=5-2\nu-9m$. If we now use

$$m = (1-\nu^2)/2, \tag{10.12}$$

which is a consequence of (10.1), we find that $K=(9\nu^2-4\nu+1)/2$ as asserted.

The next lemma characterizes the behavior of z''' *under the assumption that there is more than one quartic arch.* Other results of this kind were given in §9 of [1]. The consequences of these results and the inequality (10.8) will lead to a contradiction.

LEMMA 13. *With the possible exception of the last component, z''' changes sign in each component. In fact*

$$\text{signum } z'''(\alpha_i) = (-1)^i, \qquad i = 0, 1, 2, \ldots, N-1. \tag{10.13}$$

From (10.5) we get that $z'''(0)\geqq 0$ with equality holding if and only if $F(0)=0$. From (9.1) and Lemma 7 we see that $F(0)=0$ is impossible. Hence $z'''(0)>0$. If z''' did not change sign in the interior of the first component $(0, \alpha_1)$, then since z''' is linear we would have $z'''(\alpha_1)\geqq 0$. On the next component (α_1, α_2), $z(u)<0$, and so $z^{(iv)}(u)=I$. Hence $z'''(u)>0$ on (α_1, α_2). Since $z'(\alpha_1)<0$ it follows from Lemma 9 that $z''(\alpha_1+0)\geqq 0$. Since $z'''(u)>0$ on (α_1, α_2) we have $z''(u)>0$ on (α_1, α_2). Hence z' is strictly increasing on (α_1, α_2). Since $z(\alpha_1)=z(\alpha_2)$ and $z(u)<0$ on (α_1, α_2), z' must have a zero in the interior of (α_1, α_2). Therefore, $z'(\alpha_2)>0$ and there exists a third component (α_2, α_3) on which $z(u)>0$. Hence by (5.5)

$$z''(\alpha_2+0) = z''(\alpha_2-0)+2I > 0.$$

Since $z'(\alpha_2)>0$, this contradicts Lemma 9. Therefore, $z'''(\alpha_1)<0$.

By repeating the argument we can next show that z''' must change sign in the second component. We proceed inductively in this fashion up to the next to the last component. There we conclude by noting that if z''' does not change sign we have $z'(\alpha_N)\neq 0$, a contradiction.

11. **Proof that** $N=1$. In this section we shall show that the minimizing function has only one quartic arch. The formula for q given in Theorem 3 then follows from Lemma 11. This leaves only the numerical determination of the constants to be found in §12.

Suppose that $N>1$. Then from Theorem 2 we have that $z(u)>0$ on the first component $(0, \alpha_1)$, $z(u)<0$ on the second component (α_1, α_2), and $z'(\alpha_1)<0$. Since $z'(\alpha_1)<0$ it follows from Lemma 9 that $z''(\alpha_1+0)\geqq 0$. From (5.5) we get

$$z''(\alpha_1+0) = z''(\alpha_1-0)-2I.$$

Hence

$$z''(\alpha_1-0) > 2I. \tag{11.1}$$

From Lemma 13 it follows that there exists a point β in $(0, \alpha_1)$ such that $z'''(\beta)=0$. Since z'' is a quadratic on $(0, \alpha_1)$ and $(z'')''=z^{\text{iv}}=-I$, it follows that z'' has a maximum at β. Since $z''(\beta)>z''(\alpha_1-0)$ it follows from (11.1) that

$$z''(\beta) > 2I. \tag{11.2}$$

On the interval $(0, \alpha_1)$ we have

$$z''(u) = -Iu^2/2+z'''(0)u+z''(0), \tag{11.3}$$

$$z'''(u) = -Iu+z'''(0). \tag{11.4}$$

Since $z'''(\beta)=0$ it follows from (11.4) that $\beta=z'''(0)/I$. Substituting this value of β into (11.3) and using (11.2) gives

$$(z'''(0))^2/2I > 2I-z''(0).$$

Using (10.3) and (10.4) we may rewrite this inequality in the following equivalent form:

$$8I^2+4I(1-\nu^2)-(1-\nu)^4 < 0. \tag{11.5}$$

From (11.1) and from the fact that z'' is a quadratic on $(0, \alpha)$ with maximum at β it follows that $z''(u)$ is positive on (β, α_1). Therefore z' is strictly increasing on (β, α_1). Since $z'(\alpha_1)<0$ it follows that $z'(\beta)<0$. If we substitute $u=\beta=z'''(0)/I$ into

$$z'(u) = -\frac{Iu^3}{6}+\frac{z'''(0)}{2}u^2+z''(0)u+z'(0), \qquad 0 \leqq u \leqq \alpha_1,$$

we get

$$3I^2z'(\beta) = (z'''(0))^3+3Iz''(0)z'''(0)+3z'(0)I^2.$$

If we now use (10.2)–(10.4) we see that $z'(\beta)<0$ implies

$$12I^2(1+\nu^2)-6I(1-\nu)^3(1+\nu)+(1-\nu)^6 < 0. \tag{11.6}$$

We now multiply (11.5) by $(1-\nu)^2$, add the resulting inequality to (11.6), and then divide by I. After some elementary simplifications we get

$$I < (1+\nu)(1-\nu)^3/2(5\nu^2-4\nu+5).$$

We now eliminate I between this inequality and the inequality (10.8) to obtain, after simplification,

$$24\nu^4-31\nu^3+33\nu^2-9\nu+1 < 0. \tag{11.7}$$

The left-hand side of (11.7) can be rewritten as follows:

$$\left(1-\frac{9}{2}\nu\right)^2+\frac{51}{4}\nu^2\left(1-\frac{62}{51}\nu\right)^2+\frac{263}{51}\nu^4.$$

This expression is clearly nonnegative for all ν. Hence (11.7) cannot be satisfied and the assumption that $N>1$ leads to a contradiction.

12. **Numerical determination of constants.** The quartic q of Theorem 3 has the following properties:

$$q(0) = 0, \qquad q^{\text{iv}}(u) = -I, \qquad q(\lambda) = q'(\lambda) = 0.$$

Hence q has another positive root $r>\lambda$ and we can write

$$q(u) = Iu(\lambda-u)^2(r-u)/24. \tag{12.1}$$

If we expand the right-hand side of (12.1) and compare coefficients of the powers of u with those in the representation of q given in Theorem 3, we obtain the equations

$$\begin{aligned} I\lambda^2 r &= 24\nu, \\ I(\lambda^2+2r\lambda+12) &= 6(1-\nu^2), \\ I(r+2\lambda) &= 2(1-\nu)^2. \end{aligned} \tag{12.2}$$

Since $z(u)=0$ for $u\geqq\lambda$ and $q(u)>0$ on $(0, \lambda)$, it follows that $I=\int_0^\lambda q(u)\,du$. Substituting (12.1) into the right-hand side of this equation and performing the integration gives

$$24/\lambda^4+\lambda/30 = r/12. \tag{12.3}$$

Equations (12.2) and (12.3) were solved numerically to obtain the values of I, λ, ν given in Theorem 3. The value of m is obtained from the value of ν by means of (10.1).

References

1. L. D. Berkovitz and H. Pollard, *A variational problem related to an optimal filter problem with self-correlated noise*, Trans. Amer. Math. Soc. **142** (1969), 153–175.

Purdue University,
Lafayette, Indiana 47907