

DELETED PRODUCTS OF SPACES WHICH ARE UNIONS OF TWO SIMPLEXES⁽¹⁾

BY

W. T. WHITLEY⁽²⁾

Abstract. If X is a space, the deleted product space, X^* , is $X \times X - D$, where D is the diagonal. If Y is a space and f is a continuous map from X to Y , then X_f^* is the inverse image of Y^* under the map $f \times f$ taking $X \times X$ into $Y \times Y$. In this paper, we investigate the following questions: "What maps f are such that X_f^* is homotopically equivalent to X^* ?", and "What maps f are such that X_f^* is homotopically equivalent to $f(X)^*$?" If X is the union of two nondisjoint simplexes and f is a simplicial map from $X \times X$ such that $f|f(X)$ is one-to-one, we obtain necessary and sufficient conditions for X_f^* and $f(X)^*$ to be homotopically equivalent. If X is the union of nondisjoint simplexes A and B with $\dim B = 1 + \dim(A \cap B)$, we obtain necessary and sufficient conditions for X^* and X_f^* to be homotopically equivalent if f is in the class of maps mentioned.

1. Introduction and notation. If X is a space, the deleted product space, X^* , of X is $X \times X - D$, where D is the diagonal. If Y is a space and f is a continuous map from X to Y , then X_f^* is the inverse image of Y^* under the map $f \times f$ taking $X \times X$ into $Y \times Y$. In [1, p. 236], Brahana asks the question, "What maps f are such that there is a homotopy equivalence between X_f^* and X^* ?"

In this paper, we investigate Brahana's question and the related question, "What maps f are such that there is a homotopy equivalence between X_f^* and $f(X)^*$?" If X is a finite polyhedron, let $F(X)$ denote the class of simplicial maps f such that $f(X)$ is a subset of X and such that $f|f(X)$ is one-to-one. If X is the union of two nondisjoint simplexes, we are able to obtain complete answers to the second question for all maps f in $F(X)$, and we are able to obtain complete answers to Brahana's question for all maps f in $F(X)$ for a certain subcollection of such spaces.

If X and Y are finite polyhedra and f is a simplicial map from X to Y , let $P(X^*) = \bigcup \{r \times s \mid r \text{ and } s \text{ are simplexes in } X \text{ and } r \cap s = \emptyset\}$, and let $P(X_f^*) = \bigcup \{r \times s \mid r \text{ and } s \text{ are simplexes in } X \text{ and } f(r) \cap f(s) = \emptyset\}$. In [2, pp. 351–352] Hu has shown that X^* and $P(X^*)$ are homotopically equivalent, and in [3, p. 183] Patty has observed that X_f^* and $P(X_f^*)$ are homotopically equivalent.

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⁽²⁾ The author's present address is Department of Mathematics, Marshall University, Huntington, West Virginia.

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The symbol $\langle v_0, \dots, v_n \rangle$ will denote the n -simplex whose vertices are v_0, \dots, v_n , and we shall use the circumflex \hat{v}_i to indicate that the vertex v_i has been deleted. That is, $\langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle$ will denote the $(n-1)$ -face of $\langle v_0, \dots, v_n \rangle$ with the vertex v_i omitted. The symbol I will denote the closed unit interval $[0, 1]$, and $S^n = \{x \mid x \text{ is in } E^{n+1} \text{ with } |x| = 1\}$. If f is a continuous simplicial map on a finite polyhedron X to a finite polyhedron Y , η_f will be the map $f \times f$ on $P(X^*)$ (or a specified subspace of $P(X^*)$) to $Y \times Y$. If X is a polyhedron, $H_k(X)$, k a positive integer, will be the k th homology group of X over the integers \mathbb{Z} . If G and H are groups, $G + H$ will denote the direct sum of G and H . If X and Y are spaces, $X \simeq Y$ will mean that X is homotopically equivalent to Y .

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2. Some preliminary results.

THEOREM 2.1. *Let X be a finite simplicial complex. Let $g: X \rightarrow X$ and $f: g(X) \rightarrow X$ be simplicial maps with the following properties: (1) $f^2 = f$ and $g^2 = g$, (2) if x is in X , then x and $g(x)$ lie in a common simplex in X , and (3) if x is in $g(X)$, then x and $f(x)$ lie in a common simplex in X . Then $P(X_{fg}^*)$ is homotopically equivalent to $P(fg(X)^*)$.*

Proof. We first show that $(fg)^2 = fg$. Let v be a vertex of X . Since $fg(v)$ is a vertex in $g(X)$, $gfg(v) = fg(v)$ by (1). Then since $f^2 = f$, it follows that $fgfg(v) = ffg(v) = fg(v)$.

It now follows that $P(fg(X)^*)$ is a subset of $P(X^*)$, and that if

$$\eta_{fg} = fg \times fg: P(X_{fg}^*) \rightarrow P(fg(X)^*),$$

then $\eta_{fg}|_{P(fg(X)^*)}$ is the identity. Hence, if $\eta_i: P(fg(X)^*) \rightarrow P(X_{fg}^*)$ is the injection, $\eta_{fg}\eta_i$ is the identity. We shall show that η_{fg} and η_i are homotopy inverses.

Let (x, y) be a point in $P(X_{fg}^*)$. Let r and s be the smallest closed simplexes in X containing x and y , respectively. Then $r \times s$ is contained in both $P(X_g^*)$ and $P(X_{fg}^*)$. Let r' be a simplex in X containing both x and $g(x)$, and let s' be a simplex in X containing both y and $g(y)$. Let r_1 be the face of r' consisting of the vertices of r together with the vertices of $g(r) \cap r'$. Construct s_1 in a similar manner from s, s' , and $g(s)$. Then

$$\begin{aligned} fg(r_1) &\subset fg(r) \cup [fg(g(r)) \cap fg(r')] \\ &= fg(r) \cup [fg(r) \cap fg(r')] = fg(r). \end{aligned}$$

Similarly, $fg(s_1) = fg(s)$, so $r_1 \times s_1$ is a subset of $P(X_{fg}^*)$. Hence, the line joining (x, y) and $(g(x), g(y))$ is a subset of $P(X_{fg}^*)$.

Now since $(g(x), g(y))$ is in $P(X_{fg}^*)$, there exist simplexes u and w in X such that $(g(x), g(y))$ is in $u \times w \subset P(X_{fg}^*)$. We may assume that u and w are the smallest closed simplexes in X containing $g(x)$ and $g(y)$ respectively. Let u' be a simplex in X containing both $g(x)$ and $fg(x)$, and let w' be a simplex in X containing both $g(y)$ and $fg(y)$. Let u_1 denote the face of u' consisting of the vertices of u together with the vertices of $fg(u) \cap u'$. Construct w_1 in a similar manner from w, w' , and $fg(w)$. Then as above, $fg(u_1) = fg(u)$ and $fg(w_1) = fg(w)$ since $(fg)^2 = fg$. Then since

$u \times w$ is a subset of $P(X_{fg}^*)$, $u_1 \times w_1$ is a subset of $P(X_{fg}^*)$ containing both $(g(x), g(y))$ and $(fg(x), fg(y))$. Hence, the line joining $(g(x), g(y))$ and $(fg(x), fg(y))$ is contained in $P(X_{fg}^*)$.

Define $H: P(X_{fg}^*) \times I \rightarrow P(X_{fg}^*)$ by

$$H(x, y, t) = (1-2t)(x, y) + 2t\eta_g(x, y) \quad \text{if } 0 \leq t \leq \frac{1}{2},$$

and

$$H(x, y, t) = (2-2t)\eta_g(x, y) + (2t-1)\eta_i\eta_{fg}(x, y) \quad \text{if } \frac{1}{2} \leq t \leq 1.$$

From the preceding remarks, it follows that H is a homotopy between $\eta_i\eta_{fg}$ and the identity on $P(X_{fg}^*)$. This completes the proof.

COROLLARY 2.1. *Let X be a finite simplicial complex. Let $f: X \rightarrow X$ be a simplicial map such that (1) $f^2=f$, and (2) if x is a point of X , x and $f(x)$ lie in a common simplex of X . Then $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$.*

Proof. Let $f=g$ in Theorem 2.1.

THEOREM 2.2. *Let X be a finite simplicial complex, and let f be a map in $F(X)$. Then there is a unique simplicial map $h: X \rightarrow X$ such that (1) $h^2=h$, (2) $h(X)=f(X)$, and (3) $P(X_h^*)=P(X_f^*)$. Furthermore, $f=fh$.*

Proof. Since $f|f(X)$ is one-to-one, there is a simplicial map $g: f(X) \rightarrow f(X)$ defined by $g(v)=v'$, where v is a vertex in $f(X)$ and v' is the unique vertex in $f(X)$ such that $f(v')=v$. Note that $g=(f|f(X))^{-1}$, so fg is the identity on $f(X)$. Let $h=gf$. We shall show that h is the required map.

Since $h^2=gfgf=gf=h$, (1) holds.

Let v be a vertex of $f(X)$. Let $v'=f(v)$. Then $v=g(v')=gf(v)$, so $f(X)$ is a subset of $gf(X)=h(X)$. Clearly $gf(X)$ is a subset of $f(X)$ by definition of g , so $h(X)=f(X)$.

Let r and s be simplexes in X . Suppose $f(r) \cap f(s) = \emptyset$. If $gf(r) \cap gf(s) \neq \emptyset$, then $fgf(r) \cap fgf(s) = f(r) \cap f(s) \neq \emptyset$, which is a contradiction. Hence, $P(X_f^*)$ is contained in $P(X_h^*)$. If $gf(r) \cap gf(s) = \emptyset$, then $fgf(r) \cap fgf(s) = f(r) \cap f(s) = \emptyset$ since $f|f(X)=f|h(X)$ is one-to-one. Hence, $P(X_h^*)$ is a subset of $P(X_f^*)$, so (3) holds.

Since $h=gf$, it follows that $fh=fgf=f$.

To prove uniqueness, suppose h_1 and h_2 are two simplicial maps satisfying conditions (1), (2), and (3). Let v be a vertex of X . Suppose $h_1(v) \neq h_2(v)$. Now $h_1(v)$ is in $h_1(X)=f(X)=h_2(X)$. Then since $h_2^2=h_2$, we have $h_2h_1(v)=h_1(v)$. Since $h_2(v) \neq h_1(v)$, $(v, h_1(v))$ is in $P(X_{h_2}^*)=P(X_{h_1}^*)$. However, since $h_1^2=h_1$, we must have $h_1(v)=h_1h_1(v)$, which is a contradiction. Thus, $h_1=h_2$, so the proof is complete.

Observe that if X, f , and h are as in Theorem 2.2, and if v and v' are vertices of X , then $f(v)=f(v')$ if and only if $h(v)=h(v')$.

Patty has shown in [4, Corollary 1] that if A is a p -simplex, $p \geq 1$, then $P(A^*)$ has the homotopy type of S^{p-1} . This fact, together with the preceding two theorems, yields the following result.

COROLLARY 2.2. *Let X be an n -simplex. Let f be a map in $F(X)$. Then $P(X^*)$ is homotopically equivalent to $P(X_f^*)$ if and only if $f(X) = X$. In any case, $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$.*

Proof. Let h be the map of Theorem 2.2 associated with f . By Theorem 2.1, $P(X_h^*)$ is homotopically equivalent to $P(h(X)^*)$. Hence, $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$. The result then follows from Corollary 1 of [4].

3. The main results. The main results are Theorems 3.1, 3.2, 3.3, and 3.4, the proofs of which require a sequence of lemmas. In this section, we shall let $X = A \cup B$, where $A = \langle v_0, \dots, v_k, v_{k+1}, \dots, v_n \rangle$ and $B = \langle v_0, \dots, v_k, w_{k+1}, \dots, w_m \rangle$ are n - and m -simplexes, respectively, such that $A \cap B = \langle v_0, \dots, v_k \rangle$, $k \geq 0$, is a proper k -face of each.

THEOREM 3.1. *Let f be a map in $F(X)$ such that $f(X) \cap (A - B)$ is not empty and $f(X) \cap (B - A)$ is not empty. Then $P(X_f^*)$ and $P(f(X)^*)$ have the same homotopy type.*

Proof. By Theorem 2.2, we may assume $f^2 = f$. Let us suppose w_m is a vertex in $f(X) \cap (B - A)$ and v_n is a vertex in $f(X) \cap (A - B)$. Then since $f^2 = f$, it follows that $f(w_m) = w_m$ and $f(v_n) = v_n$. Suppose $f(v)$ is in $A - B$ for some vertex v in B . Then $\langle v, w_m \rangle$ is a simplex in X , but $\langle f(v), f(w_m) \rangle = \langle f(v), w_m \rangle$ is not a simplex in X . Hence, $f(v)$ is in B if v is in B , so $f(B)$ is a subset of B . Similarly, $f(A)$ is a subset of A . The theorem now follows from Corollary 2.1.

LEMMA 1. *Let f be a map in $F(X)$ such that $f|A$ is nonconstant, $f|B$ is nonconstant, and $f(X)$ is a subset of A . Then $P(A_f^*)$ is homotopically equivalent to $P(f(A)^*)$, which has the homotopy type of S^p ; and $P(B_f^*)$ is homotopically equivalent to $P(f(B)^*)$, which has the homotopy type of S^q , where $p + 1 = \dim f(A)$, and $q + 1 = \dim f(B)$.*

Proof. By Theorem 2.2, we may assume $f^2 = f$. Then since $f(A)$ is a subset of A , $P(A_f^*)$ is homotopically equivalent to $P(f(A)^*)$ by Corollary 2.1.

Since $f(B)$ is a face of A , there is a face of B , say $B' = \langle u_0, \dots, u_q \rangle$ such that $f(u_i) \neq f(u_j)$, $i \neq j$, and such that $f(B) = f(B')$. Define $h: f(B) \rightarrow B$ to be the simplicial map $hf(u_i) = u_i$, $i = 0, \dots, q$. Then $h: f(B) \rightarrow B'$ is a homeomorphism. Then $hf|B: B \rightarrow B$ is a continuous simplicial map, and since h is one-to-one, $P(B_f^*) = P(B_{hf}^*)$. It is clear that $hf|B' = hf|hf(B)$ is one-to-one. Since $hf(B)$ is contained in B , $(hf|B)^2 = (hf|B)$. Hence, $P(B_{hf}^*)$ is homotopically equivalent to $P(hf(B)^*)$ by Corollary 2.1. Then since $hf(B)$ is homeomorphic to $f(B)$, it follows that $P(B_f^*)$ is homotopically equivalent to $P(f(B)^*)$.

The remainder of the lemma follows from Corollary 1 of [4] since $f(A)$ and $f(B)$ are simplexes.

LEMMA 2. *Let $f: X \rightarrow X$ be a simplicial map such that (1) $f^2 = f$, (2) $f(X)$ is a subset of A , (3) $f(v_n)$ is not in $f(B)$, and (4) $f(B) \neq f(A \cap B)$. Let $D_1 = \bigcup \{r \times s \mid r \text{ and}$*

s are simplexes in X , $f(r) \cap f(s) = \emptyset$, $r \subset A$, and $s \subset B$. Then the sets D_1 and $D_1 \cap P(A_f^*)$ are contractible.

Proof. We first show that D_1 is contractible to (v_n, v_0) .

Let $g: D_1 \rightarrow D_1$ be the linear map defined by $g(v, w) = (v_n, w)$ if $v \in f^{-1}f(v_0)$, and $g(v, w) = (v, w)$ otherwise, where v and w are vertices in X . Let $(x, y) \in r \times s \subset D_1$. Then $r \subset A$ and $s \subset B$. If $r \cap f^{-1}f(v_0) = \emptyset$, then $g(r \times s) = r \times s \subset D_1$. Suppose $r \cap f^{-1}f(v_0) \neq \emptyset$. Since $f(v_n)$ is not in $f(B)$, $f(v_n)$ is not in $f(s)$. Hence, $r \times s \subset r' \times s \subset D_1$, where r' is the face of A consisting of the vertices of r together with v_n . Then clearly $g(r \times s) = g(r' \times s) \subset r' \times s \subset D_1$. Hence, in any case, if (x, y) is in D_1 , the line joining (x, y) and $g(x, y)$ is contained in D_1 .

Now we shall show that if (x, y) is in $g(D_1)$, the line joining (x, y) and (v_n, v_0) is contained in D_1 . Let $(x, y) \in r \times s \subset g(D_1)$, r and s simplexes in X . By definition of g , it is clear that $r \cap f^{-1}f(v_0) = \emptyset$. Thus, $r \times s \subset r' \times s' \subset D_1$, where r' is the face of A consisting of the vertices of r together with v_n , and s' is the face of B consisting of the vertices of s together with v_0 . Then $\{(x, y), (v_n, v_0)\} \subset r' \times s' \subset D_1$, which proves our assertion.

Let $H: D_1 \times I \rightarrow D_1$ be the map defined by

$$\begin{aligned} H(x, y, t) &= (1-2t)(x, y) + 2tg(x, y), & 0 \leq t \leq \frac{1}{2}, \\ &= (2-2t)g(x, y) + (2t-1)(v_n, v_0), & \frac{1}{2} \leq t \leq 1. \end{aligned}$$

Then in view of the preceding remarks, it is clear that H is a homotopy between the constant map (v_n, v_0) and the identity on D_1 . Therefore, D_1 is contractible.

Note that $D_1 \cap P(A_f^*) = \bigcup \{r \times s \mid r \text{ and } s \text{ are simplexes in } X, f(r) \cap f(s) = \emptyset, r \subset A, s \subset A \cap B\}$. Then by the same arguments as above, $D_1 \cap P(A_f^*)$ is contractible to (v_n, v_0) . This proves the lemma.

LEMMA 3. Let f and D_1 satisfy the hypotheses of Lemma 2. Let $D_2 = \bigcup \{s \times r \mid r \text{ and } s \text{ are simplexes in } X, f(r) \cap f(s) = \emptyset, r \subset A, s \subset B\}$. Then $P(A_f^*)$ is homotopically equivalent to $P(A_f^*) \cup D_1 \cup D_2$.

Proof. By arguments similar to those of Lemma 2, we have that D_2 and $D_2 \cap P(A_f^*)$ are contractible. Since D_1 and $D_1 \cap P(A_f^*)$ are contractible, $P(A_f^*) \simeq P(A_f^*) \cup D_1$. It is clear that $D_1 \cap D_2 = P(A_f^*) \cap D_1 \cap D_2 = P((A \cap B)_f^*)$. Hence, $(P(A_f^*) \cup D_1) \cap D_2 = P(A_f^*) \cap D_2$, which is contractible. Thus, we have

$$P(A_f^*) \cup D_1 \cup D_2 \simeq P(A_f^*) \cup D_1 \simeq P(A_f^*).$$

LEMMA 4. Let $f: X \rightarrow X$ be a simplicial map such that (1) $f^2 = f$, (2) $f(X)$ is a subset of A , and (3) $f(w_m)$ is not in $f(A \cap B)$, and let D_1 and D_2 be as in Lemma 2 and Lemma 3. Then $(D_1 \cup D_2) \cap P(B_f^*)$ is homotopically equivalent to

$$P(\langle v_0, \dots, v_k, w_m \rangle_f^*).$$

(Note that we are not assuming quite all of the conditions of Lemma 2 on f .)

Proof. Let $g: X \rightarrow X$ be the simplicial map defined by $g(w_i) = w_m$, $k+1 \leq i \leq m$, $g(v_i) = v_i$, $0 \leq i \leq n$. We shall show that $\eta_g: (D_1 \cup D_2) \cap P(B^*) \rightarrow P(\langle v_0, \dots, v_k, w_m \rangle_f^*)$ is the required homotopy equivalence.

Clearly $P(\langle v_0, \dots, v_k, w_m \rangle_f^*)$ is a subset of $(D_1 \cup D_2) \cap P(B_f^*)$. It is straightforward to check that $\eta_g((D_1 \cup D_2) \cap P(B_f^*))$ is contained in $P(\langle v_0, \dots, v_k, w_m \rangle_f^*)$ by virtue of condition (3).

Let $\eta_i: P(\langle v_0, \dots, v_k, w_m \rangle_f^*) \rightarrow (D_1 \cup D_2) \cap P(B_f^*)$ be the injection. Then clearly $\eta_g \eta_i$ is the identity.

Let $(x, y) \in r \times s$, where r and s are simplexes in X such that

$$r \times s \subset (D_1 \cup D_2) \cap P(B_f^*).$$

If both r and s are contained in $A \cap B$, then $\eta_g(r \times s) = r \times s$. Suppose s is not contained in $A \cap B$. Then r is in $A \cap B$. Hence, if s' is the face of B consisting of the vertices of s together with w_m , we must have $r \times s \subset r \times s' \subset D_1 \cap P(B_f^*)$. Similarly, if r is not contained in $A \cap B$, then $r \times s \subset r' \times s \subset D_2 \cap P(B_f^*)$ where r' is r with w_m adjoined. Clearly $\eta_g(r \times s') \subset r \times s'$ (or $\eta_g(r' \times s) \subset r' \times s$). Hence, in any case, if (x, y) is a point in $(D_1 \cup D_2) \cap P(B_f^*)$, the line joining (x, y) and $\eta_g(x, y)$ is contained in $(D_1 \cup D_2) \cap P(B_f^*)$. Thus, the map

$$H: (D_1 \cup D_2) \cap P(B_f^*) \times I \rightarrow (D_1 \cup D_2) \cap P(B_f^*)$$

defined by $H(x, y, t) = (1-t)(x, y) + t\eta_i\eta_g(x, y)$ is a homotopy between $\eta_i\eta_g$ and the identity. This proves the lemma.

LEMMA 5. *Let f be as in Lemma 4. Let w_m be a vertex of B such that $f(w_m)$ is not in $f(A \cap B)$. Let $B' = \langle v_0, \dots, v_k, w_m \rangle$. If $f(B) = f(B')$, then $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$.*

Proof. Let $X' = A \cup B'$. Since $f(B) = f(B')$, it is clear that $f(X') = f(X)$. We first show that $P(X_f'^*)$ has the homotopy type of $P(f(X)^*)$.

Let $D'_1 = \bigcup \{r \times s \mid r \text{ is a simplex in } A, s \text{ is a simplex in } B', \text{ and } f(r) \cap f(s) = \emptyset\}$. Let $D'_2 = \bigcup \{s \times r \mid r \text{ is a simplex in } A, s \text{ is a simplex in } B', \text{ and } f(r) \cap f(s) = \emptyset\}$. Then since $P(B_f'^*)$ is a subset of $D'_1 \cup D'_2$, we have $P(X_f'^*) = P(A_f^*) \cup D'_1 \cup D'_2$. By Lemma 3, $P(A_f^*)$ is homotopically equivalent to $P(X_f'^*)$. Since $f^2 = f$ and $f(A)$ is contained in A , it follows that $P(A_f^*)$ is homotopically equivalent to $P(f(A)^*)$ by Corollary 2.1. However, $f^2 = f$ and $f(X)$ a subset of A imply that $f(X) = f(A)$. Hence $P(X_f'^*) \simeq P(A_f^*) \simeq P(f(X)^*)$.

To complete the proof, we need to show that $P(X_f^*)$ and $P(X_f'^*)$ have the same homotopy type. Note that $P(X_f'^*)$ is clearly a subset of $P(X_f^*)$. Let u_0, \dots, u_q be vertices of $A \cap B$ such that $f(u_i) \neq f(u_j)$, $i \neq j$, and $f(\langle u_0, \dots, u_q \rangle) = f(A \cap B)$. Then if v is a vertex of B , v is in exactly one of the sets $f^{-1}f(u_0), \dots, f^{-1}f(u_q), f^{-1}f(w_m)$. Let $g: X \rightarrow X$ be the simplicial map defined by $g(v) = u_i$, $v \in f^{-1}f(u_i) \cap B$, $0 \leq i \leq q$, $g(v) = w_m$, $v \in f^{-1}f(w_m) \cap B$, and $g(v) = v$ otherwise, where v is a vertex in X . Note that $g^2 = g$.

Suppose v and v' are vertices of X with (v, v') in $P(X_f^*)$. Then since $fg(v)=f(v)$ and $fg(v')=f(v')$, we see that $(g(v), g(v')) \in P(X_f'^*) \subset P(X_f^*)$. Thus, we have $\eta_g: P(X_f^*) \rightarrow P(X_f'^*)$. Note that we also have $\eta_g: P(X_f^*) \rightarrow P(X_f'^*) \subset P(X_f^*)$ by the above argument. We shall show that $\eta_g: P(X_f^*) \rightarrow P(X_f'^*)$ and $\eta_g: P(X_f'^*) \rightarrow P(X_f^*)$ are homotopy inverses.

Let $(x, y) \in r \times s \subset P(X_f^*)$, r and s simplexes in X . If r is in A , then $g(r)$ is in A ; and if r is in B , then $g(r)$ is in B . Let r' be the simplex in X consisting of the vertices of r together with the vertices of $g(r)$, and let s' be the simplex in X consisting of the vertices of s together with the vertices of $g(s)$. Then since $fg(r)=f(r)$ and $fg(s)=f(s)$, $r' \times s' \subset P(X_f^*)$ since $r \times s \subset P(X_f^*)$. Hence, if (x, y) is in $P(X_f^*)$, the line joining (x, y) and $(g(x), g(y))$ is contained in $P(X_f^*)$. Therefore, the map $H: P(X_f^*) \times I \rightarrow P(X_f^*)$, defined by $H(x, y, t) = (1-t)(x, y) + t\eta_g\eta_g(x, y)$, is a homotopy between $\eta_g\eta_g$ and the identity on $P(X_f^*)$.

To complete the proof, note that if r and s are simplexes in X with $r \times s$ in $P(X_f'^*)$, then r and s are in X' , which implies $g(r)$ and $g(s)$ are in X' . Hence, if r' and s' are constructed as above, $r' \times s'$ is contained in $P(X_f'^*)$. Then the map $H|P(X_f'^*) \times I$ is a homotopy between $\eta_g\eta_g$ and the identity on $P(X_f'^*)$. This proves the lemma.

LEMMA 6. *Let f be a map in $F(X)$ such that (1) $f(v_n)$ is not in $f(B)$, (2) $f(w_m)$ is not in $f(A \cap B)$, and (3) $f(X)$ is a subset of A . Then $P(X_f^*)$ and $P(f(X)^*)$ have the same homotopy type if and only if $f(B) = f(\langle v_0, \dots, v_k, w_m \rangle)$.*

Proof. By Theorem 2.2, we may assume $f^2=f$. Then since $f(A)$ is contained in A , it follows that $P(A_f^*)$ is homotopically equivalent to $P(f(A)^*)$ by Corollary 2.1. Since $f^2=f$ and $f(X)$ is a subset of A , we have $f(X)=f(A)$. Hence, $P(f(X)^*)$ has the homotopy type of $P(A_f^*)$. Let D_1 and D_2 be as in Lemma 2 and Lemma 3. Then we have $P(X_f^*) = P(A_f^*) \cup D_1 \cup D_2 \cup P(B_f^*)$.

If $f(B) = f(\langle v_0, \dots, v_k, w_m \rangle)$, then $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$ by Lemma 5.

Suppose $f(B) \neq f(\langle v_0, \dots, v_k, w_m \rangle)$. Then $P(B_f^*)$ is not a subset of $D_1 \cup D_2$. By Lemma 1, $P(A_f^*)$ has the homotopy type of S^q , and $P(B_f^*)$ has the homotopy type of S^t , where $q+1 = \dim f(A)$ and $t+1 = \dim f(B)$. By conditions (1) and (3) and the fact that $f^2=f$, we must have $q > t$. By Lemma 3, $P(A_f^*) \simeq P(A_f^*) \cup D_1 \cup D_2 \simeq S^q$. Note that $P(A_f^*) \cap P(B_f^*) = P((A \cap B)_f^*)$, which is a subset of $D_1 \cap P(B_f^*)$. Thus,

$$\begin{aligned} [P(A_f^*) \cup D_1 \cup D_2] \cap P(B_f^*) &= (P(A_f^*) \cap P(B_f^*)) \cup [(D_1 \cup D_2) \cap P(B_f^*)] \\ &= (D_1 \cup D_2) \cap P(B_f^*) \\ &\simeq P(\langle v_0, \dots, v_k, w_m \rangle)_f^*, && \text{by Lemma 4,} \\ &\simeq S^r, && \text{by Lemma 2,} \end{aligned}$$

where $r+1 = \dim f(\langle v_0, \dots, v_k, w_m \rangle)$. Then since $f(B) \neq f(\langle v_0, \dots, v_k, w_m \rangle)$, we must have $r < t$. Hence, $H_t(P(A_f^*) \cup D_1 \cup D_2 \cup P(B_f^*)) = H_t(P(X_f^*))$ is isomorphic

to Z if $r+1 < t$, and is isomorphic to $Z+Z$ if $r+1=t$. However, $H_t(P(f(X)^*))$ is trivial since $P(f(X)^*)$ has the homotopy type of S^q , $q > t$. Hence, $P(X_f^*)$ is not homotopically equivalent to $P(f(X)^*)$, so the proof is complete.

LEMMA 7. *Let $f: X \rightarrow X$ be a simplicial map such that (1) $f^2=f$, (2) $f(X)$ is a subset of A , (3) $f(X)=f(A)=f(B) \neq f(A \cap B)$, and (4) $f(A \cap B)$ is contained in $A \cap B$. Let $u_0, \dots, u_p, \dots, u_q$ be the vertices of A such that $f(u_i) \neq f(u_j)$, $i \neq j$, $0 \leq i, j \leq p$, u_i is in $A \cap B$ for $0 \leq i \leq p$, $f(A \cap B)=f(\langle u_0, \dots, u_p \rangle)$, and $f(A)=f(A')=A'$, where $A'=\langle u_0, \dots, u_p, \dots, u_q \rangle$. Let $t_0, \dots, t_p, \dots, t_q$ be vertices of B such that $t_i=u_i$ for $0 \leq i \leq p$, $f(t_i) \neq f(t_j)$ for $i \neq j$, $0 \leq i, j \leq q$, and $f(B)=f(B')=f(A)=f(A')=A'$, where $B'=\langle t_0, \dots, t_p, \dots, t_q \rangle$. Let $W=A' \cup B'$. Then $P(W_f^*)$ is homotopically equivalent to $P(X_f^*)$.*

Proof. Note that clearly $P(W_f^*)$ is a subset of $P(X_f^*)$. Further, if v is a vertex of A , there is a unique vertex u_i in A' such that $f(v)=f(u_i)$, and if v is a vertex in B , there is a unique vertex t_i in B' such that $f(v)=f(t_i)$.

Let $g: X \rightarrow X$ be the simplicial map defined by $g(v)=A' \cap f^{-1}f(v)$ if v is a vertex in A , and $g(v)=B' \cap f^{-1}f(v)$ if v is a vertex in B . Then clearly $g^2=g$, $g(A)$ is in A , and $g(B)$ is in B . Further, since $fg(v)=f(v)$ for each vertex v in X , it is easy to see that $\eta_g(P(X_f^*))=P(W_f^*)$. We shall show that η_g is a homotopy equivalence.

Let $\eta_i: P(W_f^*) \rightarrow P(X_f^*)$ be the injection. Since $g|_W$ is the identity, $\eta_g \eta_i$ is the identity on $P(W_f^*)$.

Let (x, y) be in $P(X_f^*)$. Let r and s be simplexes in X such that $(x, y) \in r \times s \subset P(X_f^*)$. If r is in A , then $g(r)$ is in A , and if r is in B , then $g(r)$ is in B . Let r' denote the simplex in X consisting of the vertices of r together with the vertices of $g(r)$, and let s' denote the simplex in X consisting of the vertices of s together with the vertices of $g(s)$. Since $fg=f$, it is easy to see that $f(r)=f(r')$ and $f(s)=f(s')$. Hence, the line joining (x, y) and $\eta_g(x, y)$ is contained in $r' \times s'$, which, in turn, is contained in $P(X_f^*)$. Hence, the map $H: P(X_f^*) \times I \rightarrow P(X_f^*)$, defined by $H(x, y, t) = (1-t)(x, y) + t\eta_i\eta_g(x, y)$, is a homotopy between $\eta_i\eta_g$ and the identity. This completes the proof.

LEMMA 8. *Suppose $n=m$. Let $f: X \rightarrow X$ be a simplicial map such that (1) $f^2=f$, and (2) $f(X)=f(A)=f(B)=A$. Let $D_1 = \bigcup \{r \times s \mid r \text{ is a simplex in } A, s \text{ is a simplex in } B, \text{ and } f(r) \cap f(s) = \emptyset\}$. Then $D_1 \cap P(A_f^*)$ is contractible.*

Proof. We shall show that $D_1 \cap P(A_f^*)$ contracts to (v_n, v_0) . Note that under the conditions of the lemma, $f|_A$ is the identity. Further, it is easy to see that $D_1 \cap P(A_f^*) = \bigcup \{r \times s \mid r \text{ is a simplex in } A \text{ and } s \text{ is a simplex in } A \cap B \text{ with } r \cap s = \emptyset\}$. Hence, if r and s are simplexes in X with $r \times s$ in $D_1 \cap P(A_f^*)$, then v_n is not in s , so $r \times s \subset r' \times s \subset D_1 \cap P(A_f^*)$, where r' is the face of A consisting of the vertices of r together with v_n . To complete the proof, we now use exactly the same procedure as in the proof of Lemma 2, contracting $D_1 \cap P(A_f^*)$ to (v_n, v_0) .

LEMMA 9. Let f be a map in $F(X)$ such that (1) $f(X)$ is contained in A , (2) $f(X) = f(A) = f(B)$, and (3) $f(X) \neq f(A \cap B)$. Then $P(X_f^*)$ is not homotopically equivalent to $P(f(X)^*)$.

Proof. By Theorem 2.2, we may assume $f^2 = f$. Then since $f(X)$ is contained in A , we have $P(A_f^*) \simeq P(f(A)^*) = P(f(X)^*)$ by Corollary 2.1. Clearly

$$P(X_f^*) = P(A_f^*) \cup D_1 \cup D_2 \cup P(B_f^*),$$

where $D_1 = \bigcup \{r \times s \mid r \text{ is a simplex in } A, s \text{ is a simplex in } B, \text{ and } f(r) \cap f(s) = \emptyset\}$, and $D_2 = \emptyset \{s \times r \mid r \text{ is a simplex in } A, s \text{ is a simplex in } B, \text{ and } f(r) \cap f(s) = \emptyset\}$.

Case I. $f|A$ is one-to-one and $f|B$ is one-to-one. Note that in this case we must have that $f|A$ is the identity and $f(X) = A$ since $f^2 = f$. We must also assume $n = m$. Then if $n = 1$, it is easy to see that $P(X_f^*)$ has the homotopy type of four points, while $P(f(X)^*)$ has the homotopy type of two points. Hence, we may assume $n > 1$ in what follows.

To complete the proof for this case, we first show that D_1 and D_2 are homeomorphic to $P(A_f^*)$ by the map $\eta_f: D_1 \rightarrow P(A_f^*)$. Since $f^2 = f$, it is clear that $\eta_f(D_1) \subset P(f(A)^*) = P(A^*) = P(A_f^*)$. Let v and v' be vertices of A with (v, v') in $P(A_f^*)$. There is a unique vertex w' in B such that $f(w') = v'$. Hence, $\eta_f(v, w') = (v, v')$, so η_f is a surjection. Since $f|A$ is the identity and $f|B$ is one-to-one, it is easy to see that $\eta_f|D_1$ is an injection, so $\eta_f: D_1 \rightarrow P(A_f^*)$ is a homeomorphism. Since D_2 is homeomorphic to D_1 , we have $D_1 \simeq D_2 \simeq P(A_f^*)$.

Now by Lemma 1 and the fact that $f(X) = f(A) = f(B) = A$, we have $D_1 \simeq D_2 \simeq P(A_f^*) \simeq P(B_f^*) \simeq S^{n-1}$. By Lemma 8, $D_1 \cap P(A_f^*)$ is contractible, and thus $D_2 \cap P(A_f^*)$ is also contractible.

If $m = n = k + 1$, then $P(B_f^*)$ is contained in $D_1 \cup D_2$, so $P(X_f^*) = P(A_f^*) \cup D_1 \cup D_2$. In this case, $\dim f(A \cap B) = k = n - 1$, so by the proof of Lemma 1, $P((A \cap B)_f^*)$ has the homotopy type of S^{n-2} . Then since

$$\begin{aligned} (P(A_f^*) \cup D_1) \cap D_2 &= (P(A_f^*) \cap D_2) \cup (D_1 \cap D_2) \\ &= (P(A_f^*) \cap D_2) \cup P((A \cap B)_f^*) \\ &= P(A_f^*) \cap D_2, \end{aligned}$$

which is contractible, we have $H_{n-1}(P(X_f^*)) = H_{n-1}(P(A_f^*) \cup D_1 \cup D_2)$ is isomorphic to $Z + Z + Z$.

Suppose $n > k + 1$. Then

$$\begin{aligned} (P(A_f^*) \cup D_1 \cup D_2) \cap P(B_f^*) &= (P(A_f^*) \cap P(B_f^*)) \cup ((D_1 \cup D_2) \cap P(B_f^*)) \\ &= P((A \cap B)_f^*) \cup ((D_1 \cup D_2) \cap P(B_f^*)) \\ &= (D_1 \cup D_2) \cap P(B_f^*) \\ &\simeq P(\langle v_0, \dots, v_k, w_n \rangle_f^*) \simeq S^k \end{aligned}$$

by Lemma 4 and Lemma 1. Then since $k < n - 1$, we still have $H_{n-1}(P(X_f^*)) = H_{n-1}(P(A_f^*) \cup D_1 \cup D_2 \cup P(B_f^*))$ is isomorphic to a direct summand of $Z + Z + Z$.

However, since $H_{n-1}(P(f(X)^*)) = H_{n-1}(P(A_f^*))$ is isomorphic to Z , we conclude that $P(X_f^*)$ is not homotopically equivalent to $P(f(X)^*)$.

Case II. We now drop the assumption that $f|A$ is one-to-one and $f|B$ is one-to-one.

Case II(i). $f(A \cap B)$ is a subset of $A \cap B$. Let $u_0, \dots, u_p, \dots, u_q$ be the vertices of A such that $f(u_i) \neq f(u_j)$ for $i \neq j$, $0 \leq i, j \leq q$, u_i is a vertex of $A \cap B$ for $0 \leq i \leq p$, $f(\langle u_0, \dots, u_p \rangle) = f(A \cap B)$, and $f(A) = f(A') = A'$, where $A' = \langle u_0, \dots, u_p, \dots, u_q \rangle$. Note that by conditions (2) and (3) of the lemma, $q > p$. Let $B' = \langle t_0, \dots, t_p, \dots, t_q \rangle$, where t_i is a vertex of B for $0 \leq i \leq q$, $t_i = u_i$ for $0 \leq i \leq p$, and $f(t_i) = u_i$ for $0 \leq i \leq q$. Let $W = A' \cup B'$. Then clearly $f|A'$ is one-to-one, $f|B'$ is one-to-one, and $f(W) = A'$. Then by Case I, $P(W_f^*)$ is not homotopically equivalent to $P(f(W)^*)$. Since $f(X) = f(A) = A'$, we have $f(W) = f(X)$. By Lemma 7, $P(X_f^*)$ is homotopically equivalent to $P(W_f^*)$, so in this case, $P(X_f^*)$ does not have the homotopy type of $P(f(X)^*)$.

Case II(ii). $f(A \cap B)$ is not a subset of $A \cap B$. Let u_0, \dots, u_p be vertices of $A \cap B$ such that $f(u_i) \neq f(u_j)$ for $i \neq j$, $0 \leq i, j \leq p$, and $f(A \cap B) = f(\langle u_0, \dots, u_p \rangle)$. Consider the simplicial map $g: X \rightarrow X$, defined by $g(v) = f(v)$ if v is not in $f^{-1}f(A \cap B)$, and $g(v) = u_i$ if v is in $f^{-1}f(u_i)$, v a vertex of X . Clearly $g^2 = g$, $P(X_f^*) = P(X_g^*)$, and $f(X)$ is homeomorphic to $g(X)$. By Case II(i), $P(X_g^*)$ is not homotopically equivalent to $P(g(X)^*)$, so the lemma follows.

Observe that we have shown in the preceding lemma that if $\dim f(X) = 1$, $P(X_f^*)$ has the homotopy type of four points, while $P(f(X)^*)$ has the homotopy type of two points. If $\dim f(X) = q > 1$, we have shown that $H_{q-1}(P(X_f^*))$ is a direct summand of $Z + Z + Z$.

LEMMA 10. *Let $f: X \rightarrow X$ be a simplicial map such that $f^2 = f$, $f(B) = f(A \cap B)$, and $f(X)$ is contained in A . Then $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$.*

Proof. Let u_0, \dots, u_p be vertices of $A \cap B$ such that $f(u_i) \neq f(u_j)$ for $i \neq j$, $0 \leq i, j \leq p$, $f(\langle u_0, \dots, u_p \rangle) = f(B)$, and $A \cap B \cap f(X) = \langle u_0, \dots, u_p \rangle \cap f(X)$. Then if v is a vertex of B , it follows that v is in exactly one of the sets $f^{-1}f(u_i)$, $0 \leq i \leq p$.

Let $g: X \rightarrow X$ be the simplicial map defined by $g(v) = u_i$ if v is a vertex in $f^{-1}f(u_i) \cap B$, and $g(v) = v$ if v is a vertex in $A - B$. Then clearly $g^2 = g$, $g(B)$ is a subset of $A \cap B$, and $g(A)$ is a subset of A .

Let $h: g(X) \rightarrow X$ be the simplicial map defined by $h(v) = f(v)$, v a vertex of $g(X)$. Then since $f^2 = f$, we must have $h^2 = h$. Since $g(X)$ contains $f(X)$, every point x of $g(X)$ is in some simplex in X containing $h(x)$.

Thus, g and h satisfy Theorem 2.1, so $P(X_{hg}^*)$ is homotopically equivalent to $P(hg(X)^*)$. Since $hg = f$, the lemma follows.

THEOREM 3.2. *Let f be a map in $F(X)$ such that $f(X)$ is contained in A . Then $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$ if and only if f satisfies at least one of the following conditions: (1) $f(B) = f(A \cap B)$, or (2) $f(B) = f(\langle v_0, \dots, v_k, w_i \rangle)$ for some i , $k+1 \leq i \leq m$ and $f(B) \neq f(A)$.*

Proof. By Theorem 2.2, we may assume $f^2=f$. If (1) holds, $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$ by Lemma 10.

Suppose (1) fails. Since $f^2=f$ and $f(X)$ is contained in A , $f(B)$ is contained in $f(A)$. If, in addition, $f(B) \neq f(A)$, there is a vertex v_j in A such that $f(v_j)$ is not in $f(B)$. Necessarily $k+1 \leq j \leq n$.

Now suppose (1) fails but (2) holds. Then $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$ by Lemma 6.

Finally, suppose both (1) and (2) fail. If $f(B) \neq f(A)$ and $f(B) \neq f(\langle v_0, \dots, v_k, w_i \rangle)$ for every i , $k+1 \leq i \leq m$, then since $f(B)$ is a subset of $f(A)$ with $f(B) \neq f(A)$, and $f(B) \neq f(A \cap B)$ by the failure of (1), $P(X_f^*)$ is not homotopically equivalent to $P(f(X)^*)$ by Lemma 6. If $f(B)=f(A)$, then $P(X_f^*)$ is not homotopically equivalent to $P(f(X)^*)$ by Lemma 9. This completes the proof.

Obviously dual conditions hold in Theorem 3.2 in case $f(X)$ is contained in B . Hence Theorems 3.1 and 3.2 give us all maps f in $F(X)$ such that $P(X_f^*)$ and $P(f(X)^*)$ have the same homotopy type. Also, in the cases where $P(X_f^*)$ and $P(f(X)^*)$ are not homotopically equivalent, the proofs of Lemmas 6 and 9 give us information about the homology groups of $P(X_f^*)$.

We are able to use Theorems 3.1 and 3.2 to answer Brahana's question for the maps f in $F(X)$ on a subcollection of the spaces which are unions of two simplexes.

The following lemma is a special case of Theorem 5 of [5].

LEMMA 11. *If $X=A \cup B$, where A is an n -simplex and B is an m -simplex with $A \cap B$ an $(m-1)$ -face of each, $m \geq 1$, then X^* is homotopically equivalent to A^* .*

THEOREM 3.3. *Let $X=A \cup B$, where $A=\langle v_0, \dots, v_{m-1}, v_m, \dots, v_n \rangle$ is an n -simplex and $B=\langle v_0, \dots, v_{m-1}, w_m \rangle$ is an m -simplex such that $A \cap B=\langle v_0, \dots, v_{m-1} \rangle$ is an $(m-1)$ -face of each, with $n > m$. Let f be a map in $F(X)$. Then $P(X^*)$ is homotopically equivalent to $P(X_f^*)$ if and only if either (1) $f(X)=X$, or (2) $f(X)=A$.*

Proof. If (1) holds, the result is trivial. If (2) holds, we can infer from Theorem 3.2 that $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)=P(A^*)$. By Lemma 11, we have $P(A^*)$ homotopically equivalent to $P(X^*)$, so that the result follows.

Suppose neither condition holds. First suppose $f(X) \cap (A-B) \neq \emptyset$ and $f(X) \cap (B-A) \neq \emptyset$. Then by Theorem 3.1, it follows that $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$. But $P(f(X)^*) \simeq P((A \cap f(X))^*)$ by Lemma 11. Since $f(X) \neq X$ but w_m is in $f(X)$, it follows that $A \cap f(X)$ is a proper face of A . Then $P((A \cap f(X))^*) \not\simeq P(A^*) \simeq P(X^*)$ since A and $A \cap f(X)$ are simplexes. Thus, $P(X_f^*)$ is not homotopically equivalent to $P(X^*)$ in this case.

Next, suppose $f(X)$ is a subset of A . Then $f(X)$ is a proper face of A since $f(X) \neq A$. If either of the conditions of Theorem 3.2 holds, $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$. However, $P(f(X)^*) \not\simeq P(A^*) \simeq P(X^*)$ by Corollary 1 of [4]. Suppose both conditions of Theorem 3.2 fail. Note that we must then have $f(A)=f(B)$ in this case. Then by the proof of Lemma 9, either $P(X_f^*)$ has the homotopy type of four points, or $H_q(P(X_f^*))$ is isomorphic to a direct summand of $Z+Z+Z$

for some $q \geq 2$. However, since $P(A^*)$ has the homotopy type of S^{m-1} by Corollary 1 of [4], $H_q(P(A^*))$ is either trivial or isomorphic to \mathbb{Z} for each integer $q \geq 1$, and $P(A^*)$ does not have the homotopy type of four points. Hence, $P(X_f^*) \not\simeq P(A^*) \simeq P(X^*)$.

Finally, suppose $f(X)$ is a subset of B . The dual conditions of Theorem 3.2 applied to this situation state that (1') $f(A) = f(A \cap B)$, or (2') $f(A) = f(\langle v_0, \dots, v_{m-1}, v_j \rangle)$ for some j such that $m \leq j \leq n$, and $f(B) \neq f(A)$. If either (1') or (2') holds, $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$. Since $f(X)$ is a face of B and $\dim B < \dim A$, it follows from Corollary 1 of [4] that $P(f(X)^*)$ is not homotopically equivalent to $P(A^*)$. Since $P(A^*)$ is homotopically equivalent to $P(X^*)$, we conclude that $P(X_f^*)$ is not homotopically equivalent to $P(X^*)$. If both (1') and (2') fail, it is straightforward to show that we must have $f(A) = f(B)$ since $f(X)$ is contained in B . In this case, the proof that $P(X_f^*)$ is not homotopically equivalent to $P(X^*)$ is analogous to the case when $f(X)$ is contained in A .

The final theorem treats the case when $n = m$, which was omitted in Theorem 3.3.

THEOREM 3.4. *Let $X = A \cup B$, where*

$$A = \langle v_0, \dots, v_{n-1}, v_n \rangle \quad \text{and} \quad B = \langle v_0, \dots, v_{n-1}, w_n \rangle$$

are two n -simplexes such that $A \cap B = \langle v_0, \dots, v_{n-1} \rangle$ is an $(n-1)$ -face of each. Let f be a nonconstant map in $F(X)$. Then $P(X_f^)$ is homotopically equivalent to $P(X^*)$ if and only if f satisfies one of the following conditions: (1) $f(X) = X$, (2) $f(X) = A$ and $f(w_n)$ is in $f(A \cap B)$, or (3) $f(X) = B$ and $f(v_n)$ is in $f(A \cap B)$.*

Proof. By Theorem 2.2, we may assume $f^2 = f$. If (1) holds, the result is trivial. If either (2) or (3) holds, $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$ by condition (1) of Theorem 3.2. Since either $f(X) = A$ or $f(X) = B$, we can conclude that $P(X_f^*) \simeq P(A^*) \simeq P(B^*) \simeq P(X^*)$.

Suppose all three conditions of the corollary fail. First, suppose $f(X) \cap (A - B) \neq \emptyset$ and $f(X) \cap (B - A) \neq \emptyset$. Then by Theorem 3.1, $P(X_f^*)$ has the homotopy type of $P(f(X)^*)$. Also, $f(X)$ is the union of two k -simplexes with a common $(k-1)$ -face for some $k \geq 1$. Since $f(X) \neq X$, we must have $P(f(X)^*) \simeq S^{k-1} \not\simeq S^{n-1} \simeq P(X^*)$ by Lemma 11 and Corollary 1 of [4], so $P(X_f^*)$ is not homotopically equivalent to $P(X^*)$ in this case.

Next, suppose $f(X)$ is contained in A . If $f(w_n)$ is in $f(A \cap B)$, then we have $f(B) = f(A \cap B)$; so by condition (1) of Theorem 3.2, it follows that $P(X_f^*)$ is homotopically equivalent to $P(f(X)^*)$. Since $f(X) \neq A$, $f(X)$ is a proper face of A , so $P(f(X)^*)$ is not homotopically equivalent to $P(A^*)$ by Corollary 1 of [4]. Hence, we have $P(X_f^*) \simeq P(f(X)^*) \not\simeq P(A^*) \simeq P(X^*)$.

Suppose $f(X)$ is contained in A and $f(w_n)$ is not in $f(A \cap B)$. Since $f^2 = f$, we must have $f(A) = f(B)$. Then by the proof of Lemma 9, since $f(X) = f(A) = f(B) \neq f(A \cap B)$, either $P(X_f^*)$ has the homotopy type of four points, or $H_q(P(X_f^*))$ is isomorphic to a direct summand of $\mathbb{Z} + \mathbb{Z} + \mathbb{Z}$ for some integer $q \geq 1$. However, since

$P(X^*) \simeq P(A^*) \simeq S^{n-1}$, it is clear that $P(X_f^*)$ is not homotopically equivalent to $P(X^*)$.

If $f(X)$ is contained in B , the proof is analogous to the case when $f(X)$ is contained in A , so the proof is complete.

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VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY,
BLACKSBURG, VIRGINIA 24061