

NONCOMMUTATIVE JORDAN ALGEBRAS OF CAPACITY TWO⁽¹⁾

BY
KIRBY C. SMITH

Abstract. Let J be a noncommutative Jordan algebra with 1. If J has two orthogonal idempotents e and f such that $1 = e + f$ and such that the Peirce 1-spaces of each are Jordan division rings, then J is said to have capacity two. We prove that a simple noncommutative Jordan algebra of capacity two is either a Jordan matrix algebra, a quasi-associative algebra, or a type of quadratic algebra whose plus algebra is a Jordan algebra determined by a nondegenerate symmetric bilinear form.

We refer the reader to [1] for all the basic concepts used in this paper as well as for all definitions of terms with the exception of quasi-associative algebra which may be found in [3].

Introduction. Recently much work has been done on the structure of simple noncommutative Jordan algebras J over a field Φ of characteristic not 2 which satisfy the following axioms:

- (i) J has an identity element 1.
- (ii) $1 = e_1 + e_2 + \cdots + e_n$, where the e_i 's are mutually orthogonal idempotents.
- (iii) The Peirce 1-space with respect to e_i , $J_1(e_i)$, is a Jordan division ring for $i = 1, 2, \dots, n$.

An algebra satisfying axioms (i)–(iii) is said to have *capacity* n .

N. Jacobson [2] proved that those simple commutative Jordan algebras of capacity n , where $n \geq 3$, are Jordan matrix algebras. K. McCrimmon and R. D. Schafer in [5] proved that the simple, not commutative Jordan algebras of capacity n ($n \geq 3$) are all quasi-associative. In [6] J. M. Osborn found that a simple Jordan algebra of capacity two is either a 2×2 Jordan matrix algebra or a Jordan algebra determined by a nondegenerate symmetric bilinear form. It is the goal of this paper to characterize the simple noncommutative Jordan algebras of capacity two.

For any idempotent e in a noncommutative Jordan algebra J we have the Peirce decomposition of J with respect to e ; namely $J = J_0(e) + J_{1/2}(e) + J_1(e)$ where $J_1 = \{x_i \mid \frac{1}{2}(ex_i + x_ie) = ix_i\}$. Throughout this work J will denote a noncommutative

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Jordan algebra of capacity two and e and f will be the two orthogonal idempotents in J such that $1 = e + f$ and J_0 (with respect to e) and J_1 are Jordan division rings. If $J_{1/2} \neq \{0\}$, which will be the case if J is simple, then it is not hard to show that any Jordan ring of capacity two may be considered an algebra of capacity two over an appropriate field.

1. **Osborn's results.** If $J = J_1 + J_{1/2} + J_0$ is simple then $J^+ = J_1^+ + J_{1/2}^+ + J_0^+$ will be a simple Jordan algebra of capacity two [5, p. 2]. From Osborn's work [6] we know that if J^+ is not a Jordan algebra determined by a symmetric bilinear form, then J^+ is a Jordan matrix algebra $H(E_2, \gamma)$, the set of symmetric 2×2 matrices under the canonical involution determined by the invertible diagonal matrix γ and where E is the associative algebra with involution $*$ generated by

$$S = \{2\bar{R}_{x_1}^+ \mid x_1 \in J_1^+ \text{ and } \bar{R}_{x_1}^+ \in \text{Hom}_\Phi(J_{1/2}, J_{1/2})\}$$

defined by $\bar{R}_{x_1}^+(y_{1/2}) = \frac{1}{2}(x_1 y_{1/2} + y_{1/2} x_1)$.

The nonzero generators of E are symmetric and invertible and moreover every symmetric element of E is a generator. E is one of the following two types:

- (a) An associative division algebra with involution which is not commutative and which is not a quaternion algebra over its center with the standard involution.
- (b) A direct sum of a division algebra (which is not commutative) with its anti-isomorphic copy and the involution switches components.

Finally $\gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$ where each γ_i is a symmetric, invertible element of E .

If $\gamma_1 \neq 1$ we change the involution in E to be $\bar{\alpha} = \gamma_1^{-1} \alpha^* \gamma_1$ where $*$ is the old involution on E and at the same time we change γ to be $\begin{pmatrix} 1 & 0 \\ 0 & \gamma_1^{-1} \gamma_2 \end{pmatrix}$. Under the new involution, $\gamma_1^{-1} \gamma_2$ is symmetric and invertible. An easy calculation shows that using the new involution on E and the new involution on E_2 determined by the new γ , the set $H(E_2, \gamma)$ remains the same as before. So we may assume that γ is of the form $\begin{pmatrix} 1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$ in $H(E_2, \gamma)$ and this we will do for the remainder of this work. We note that if $(E, *)$ is of type (a) or (b) then $(E, -)$ will be also.

Products in J as well as those in E will be denoted by juxtaposition.

2. **Preliminary results.** We define $\bar{R}_{x_1} \in \text{Hom}_\Phi(J_{1/2}, J_{1/2})$ by

$$\bar{R}_{x_1}(y_{1/2}) = [y_{1/2} x_1]_{1/2},$$

i.e. $\bar{R}_{x_1}(y_{1/2})$ is the component of $y_{1/2} x_1$ in $J_{1/2}$. (Likewise we define $\bar{L}_{x_1}(y_{1/2}) = [x_1 y_{1/2}]_{1/2}$.) Then $2\bar{R}_{x_1}^+ = \bar{R}_{x_1} + \bar{L}_{x_1}$ and E is generated by $\{(\bar{R}_{x_1} + \bar{L}_{x_1}) \mid x_1 \in J_1\}$. Note that the identity 1 of E is $2\bar{R}_e^+ = \bar{R}_e + \bar{L}_e$.

$J_{1/2}$ is a left E -module by

$$(\bar{R}_{x_1} + \bar{L}_{x_1})a_{1/2} = a_{1/2}x_1 + x_1a_{1/2}$$

where $a_{1/2} \in J_{1/2}$. $J_{1/2}$ as an E -module is isomorphic to E as an E -module. Since $\text{Hom}_E(E, E)$ is anti-isomorphic to E , so is $\text{Hom}_E(J_{1/2}, J_{1/2})$. We know that $E_x F_e = F_e E_x$ if $x \in J_0 + J_1$ and $E, F = R, L$ [4, p. 188] and so \bar{R}_e and \bar{L}_e are in the center of $\text{Hom}_E(J_{1/2}, J_{1/2})$ and can be regarded as elements in the center of E .

K. McCrimmon [4, p. 189] has shown that

$$\begin{aligned}
 (1) \quad & \bar{L}_e(\bar{L}_{y_1} + \bar{R}_{y_1}) = \bar{L}_{y_1}, \\
 & \bar{R}_e(\bar{L}_{y_1} + \bar{R}_{y_1}) = \bar{R}_{y_1}; \\
 (2) \quad & \bar{L}_{x_1 y_1} = \bar{L}_{x_1} \bar{L}_{y_1} + \bar{L}_{y_1} \bar{R}_{x_1}, \\
 & \bar{R}_{x_1 y_1} = \bar{R}_{x_1} \bar{L}_{y_1} + \bar{R}_{y_1} \bar{R}_{x_1}.
 \end{aligned}$$

The equations of (1) show that \bar{L}_{y_1} and \bar{R}_{y_1} are in E for every $y_1 \in J_1$.

Define $\nu: J_1 \rightarrow S$ by $\nu(x_1) = \bar{R}_{x_1} + \bar{L}_{x_1}$. ν is onto and one-to-one. Using (1) and (2) we have

$$\begin{aligned}
 \bar{L}_e \nu(x_1) \nu(y_1) + \bar{R}_e \nu(y_1) \nu(x_1) &= \bar{L}_e(\bar{R}_{x_1} + \bar{L}_{x_1})(\bar{R}_{y_1} + \bar{L}_{y_1}) + \bar{R}_e(\bar{R}_{y_1} + \bar{L}_{y_1})(\bar{R}_{x_1} + \bar{L}_{x_1}) \\
 &= (\bar{R}_{x_1} + \bar{L}_{x_1})\bar{L}_e(\bar{R}_{y_1} + \bar{L}_{y_1}) + (\bar{R}_{y_1} + \bar{L}_{y_1})\bar{R}_e(\bar{R}_{x_1} + \bar{L}_{x_1}) \\
 &= (\bar{R}_{x_1} + \bar{L}_{x_1})\bar{L}_{y_1} + (\bar{R}_{y_1} + \bar{L}_{y_1})\bar{R}_{x_1} \\
 &= \bar{L}_{x_1 y_1} + \bar{R}_{x_1 y_1} = \nu(x_1 y_1).
 \end{aligned}$$

Thus,

$$(3) \quad \nu(x_1 y_1) = \bar{L}_e \nu(x_1) \nu(y_1) + \bar{R}_e \nu(y_1) \nu(x_1).$$

We are now assuming that J^+ is isomorphic to $H(E_2, \gamma)$ and so we can consider $J_0, J_{1/2}, J_1$ as sets of matrices of the form $\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1_\delta \\ \gamma_2^{-1} \alpha & 0 \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix}$, respectively, where $\alpha, \delta \in E$ with α having the property that $\gamma_2^{-1} \alpha^* \gamma_2 = \alpha$ and where

$$\beta \in S = \{\beta \in E \mid \beta^* = \beta\} = \{(\bar{R}_{x_1} + \bar{L}_{x_1}) \mid x_1 \in J_1\}.$$

We are ready to describe the multiplication in J_1 . For notation we will let αe denote the element $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$ in J_1 . If $x_1 = \alpha e$ and $y_1 = \beta e$, then (3) implies $(\alpha e)(\beta e) = (\bar{L}_e \alpha \beta + \bar{R}_e \beta \alpha) e$ where $\bar{L}_e \alpha \beta + \bar{R}_e \beta \alpha \in S$. We have therefore proved

LEMMA 1. *If $\alpha e, \beta e \in J_1$, then $(\alpha e)(\beta e) = (\bar{L}_e \alpha \beta + \bar{R}_e \beta \alpha) e$.*

LEMMA 2. *The involution $*$ on E has the property that $\bar{L}_{x_1}^* = \bar{R}_{x_1}$ for every $x_1 \in J_1$.*

Proof. From Lemma 1 we know that

$$(4) \quad (\bar{L}_e \alpha \beta + \bar{R}_e \beta \alpha)^* = \bar{L}_e \alpha \beta + \bar{R}_e \beta \alpha.$$

Since $\bar{L}_e + \bar{R}_e = 1$, (4) implies

$$\alpha \beta - \beta \alpha = (\bar{L}_e^* + \bar{L}_e)(\alpha \beta - \beta \alpha)$$

or

$$[1 - (\bar{L}_e^* + \bar{L}_e)](\alpha \beta - \beta \alpha) = 0,$$

for every $\alpha, \beta \in S$. Since E is not commutative there exists $\alpha, \beta \in S$ such that $\alpha \beta - \beta \alpha \neq 0$ and since $1 - (\bar{L}_e^* + \bar{L}_e) \in S$ it is either 0 or invertible. So from the above equation we conclude that $1 - (\bar{L}_e^* + \bar{L}_e) = 0$ and this means $\bar{L}_e^* + \bar{L}_e = 1$ and hence $\bar{L}_e^* = \bar{R}_e$. Using the equations of (1) we get $\bar{L}_{x_1}^* = \bar{R}_{x_1}$ for every $x_1 \in J_1$.

Let " \cdot " be the Jordan product in J^+ . We define a new multiplication " \times " on the set J by $a \times b = a \cdot b - ab$ where ab is the multiplication in J . Then $a \cdot b = \frac{1}{2}(ab + ba)$ implies

$$(5) \quad a \times b = -b \times a$$

and using the fact that J satisfies the flexible law we obtain

$$(6) \quad (a \times b) \cdot a + (a \cdot b) \times a = 0.$$

Linearization of (6) gives

$$(7) \quad (c \times b) \cdot a + (a \times b) \cdot c + (c \cdot b) \times a + (a \cdot b) \times c = 0.$$

By Lemma 1 $(\alpha e)(\beta e) = (\bar{L}_e \alpha \beta + \bar{R}_e \beta \alpha)e$, so, from the definition of the \times -multiplication, $\alpha e \times \beta e = m(\alpha \beta - \beta \alpha)e$ where $m = \frac{1}{2} - \bar{L}_e$.

LEMMA 3. If $\alpha e \in J_1$ and $\beta f \in J_0$, then $(\alpha e)(\beta f) = 0$.

Proof. In (6) with $a = \alpha e$, $b = \beta f$ we get $(\alpha e \times \beta f) \cdot \alpha e + (\alpha e \cdot \beta f) \times \alpha e = 0$. So $(\alpha e \times \beta f) \cdot \alpha e = 0$ and if $\alpha e \times \beta f = \xi e + \delta_{12} + \epsilon f$ where $\delta_{12} = \begin{pmatrix} 0 & 1 \\ \gamma_2^{-1} \delta & 0 \end{pmatrix}$, then

$$\begin{aligned} 0 &= (\alpha e \times \beta f) \cdot \alpha e = (\xi e + \delta_{12} + \epsilon f) \cdot \alpha e \\ &= \frac{1}{2}(\alpha \xi + \xi \alpha)e + \frac{1}{2}(\alpha \delta)_{12}. \end{aligned}$$

So $\alpha \xi + \xi \alpha = 0$ and $\alpha \delta = 0$. But $\alpha \in S$ and every nonzero element of S is invertible so $\delta = 0$ and $\alpha e \times \beta f$ has zero component in $J_{1/2}$. Letting $\alpha = 1$ shows $e \times \beta f$ has zero component in J_1 .

Linearization of $(\alpha e \times \beta f) \cdot \alpha e = 0$ gives $(\alpha e \times \beta f) \cdot \eta e + (\eta e \times \beta f) \cdot \alpha e = 0$. Letting $\eta = 1$ we get

$$(8) \quad (\alpha e \times \beta f) \cdot e + (e \times \beta f) \cdot \alpha e = 0$$

and since the second term of (8) has zero component in J_1 ($e \times \beta f$ has zero component in J_1) we conclude that $\alpha e \times \beta f$ does also.

With $a = \beta f$ and $b = \alpha e$ in (6) we get $(\beta f \times \alpha e) \cdot \beta f = 0$. We recall that β has the property that $\gamma_2^{-1} \beta^* \gamma_2 = \beta$. Hence $\beta^* \gamma_2 = \gamma_2 \beta = (\beta^* \gamma_2)^*$ and so $\gamma_2 \beta \in S$. This means $\gamma_2 \beta$ is invertible if $\beta \neq 0$ and we conclude that β is invertible also. So now we can repeat the argument used in the first part of the proof of this lemma to get that the J_0 and $J_{1/2}$ components of $\beta f \times \alpha e = -\alpha e \times \beta f$ are zero. Hence $\alpha e \times \beta f = 0$. But $\alpha e \times \beta f = \alpha e \cdot \beta f - (\alpha e)(\beta f)$, so $(\alpha e)(\beta f) = 0$ and the proof of the lemma is complete.

For notation let $\lambda = \bar{L}_e$ and let $\delta_{12} = \begin{pmatrix} 0 & 1 \\ \gamma_2^{-1} \delta & 0 \end{pmatrix}$. So $\lambda^* = \bar{R}_e$ and elements of $J_{1/2}$ now have the form δ_{12} where $\delta \in E$.

LEMMA 4. For $\alpha e \in J_1$ and $\delta_{12} \in J_{1/2}$, $(\alpha e)\delta_{12} = (\lambda \alpha \delta)_{12}$.

Proof. Since $\bar{L}_{x_1}, \bar{R}_{x_1} \in E$ we have that $J_1 J_{1/2} \subset J_{1/2}$ and $J_{1/2} J_1 \subset J_{1/2}$, so $\alpha e \times \delta_{12} \in J_{1/2}$ for every $\alpha \in S, \delta \in E$. In particular $e \times l_{12} \in J_{1/2}$ so $e \times l_{12} = n_{12}$ for some $n \in E$.

Substitution in (7) of αe , βe , δ_{12} for a , b , c respectively gives

$$(9) \quad \alpha e \cdot (\beta e \times \delta_{12}) + \frac{1}{2}(m(\beta\alpha - \alpha\beta)\delta)_{12} + \frac{1}{2}(\alpha e) \times (\beta\delta)_{12} = \frac{1}{2}((\alpha\beta + \beta\alpha)e) \times \delta_{12}.$$

Letting $\alpha = 1$ in (9) gives

$$\frac{1}{2}(\beta e \times \delta_{12}) + \frac{1}{2}e \times (\beta\delta)_{12} = \beta e \times \delta_{12}, \quad \beta \in S, \delta \in E,$$

or

$$(10) \quad \beta e \times \delta_{12} = e \times (\beta\delta)_{12}, \quad \beta \in S, \delta \in E.$$

With $\beta = 1$ in (9) we get

$$\alpha e \cdot (e \times \delta_{12}) + \frac{1}{2}\alpha e \times \delta_{12} = \alpha e \times \delta_{12}, \quad \alpha \in S, \delta \in E,$$

or

$$(11) \quad 2\alpha e \cdot (e \times \delta_{12}) = \alpha e \times \delta_{12}, \quad \alpha \in S, \delta \in E.$$

So for $\delta = 1$, (10) and (11) imply

$$(12) \quad \begin{aligned} \alpha e \times 1_{12} &= e \times \alpha_{12} = 2\alpha e \cdot (e \times 1_{12}) \\ &= 2\alpha e \cdot n_{12} = (\alpha n)_{12}, \quad \alpha \in S. \end{aligned}$$

Using (10), (11), and (12) we have for $\alpha, \beta \in S$

$$(13) \quad \alpha e \times \beta_{12} = e \times (\alpha\beta)_{12} = 2\alpha e \cdot (e \times \beta_{12}) = 2\alpha e \cdot (\beta n)_{12} = (\alpha\beta n)_{12}.$$

Now we assume that for any k elements $\alpha_1, \alpha_2, \dots, \alpha_k \in S$ the following is valid:

$$(14) \quad \alpha_k e \times (\alpha_{k-1} \cdots \alpha_1)_{12} = (\alpha_k \alpha_{k-1} \cdots \alpha_1 n)_{12}.$$

Then for $\alpha_{k+1}, \alpha_k, \dots, \alpha_1 \in S$ we have

$$(15) \quad \begin{aligned} \alpha_{k+1} e \times (\alpha_k \alpha_{k-1} \cdots \alpha_1)_{12} &= e \times (\alpha_{k+1} \alpha_k \cdots \alpha_1)_{12} = 2\alpha_{k+1} e \cdot (e \times (\alpha_k \cdots \alpha_1)_{12}) \\ &= 2\alpha_{k+1} e \cdot (\alpha_k \cdots \alpha_1 n)_{12} = (\alpha_{k+1} \cdots \alpha_1 n)_{12}. \end{aligned}$$

So since S generates E , (15) implies that

$$(16) \quad \alpha e \times \delta_{12} = (\alpha \delta n)_{12}, \quad \alpha \in S, \delta \in E.$$

Finally, using (16) in (9) and remembering that E is not commutative will show that $m = n = \frac{1}{2} - \bar{L}_e = \frac{1}{2} - \lambda$.

The \times -multiplication relating J_1 with $J_{1/2}$ is now known so

$$\begin{aligned} (\alpha e)(\delta_{12}) &= (\alpha e) \cdot (\delta_{12}) - (\alpha e) \times (\delta_{12}) \\ &= \frac{1}{2}(\alpha \delta)_{12} - (\alpha \delta (\frac{1}{2} - \lambda))_{12} \\ &= \frac{1}{2}(\alpha \delta)_{12} - \frac{1}{2}(\alpha \delta)_{12} + (\alpha \delta \lambda)_{12} = (\lambda \alpha \delta)_{12}. \end{aligned}$$

LEMMA 5. If $\delta_{12}, \eta_{12} \in J_{1/2}$, then the J_0 -component of $\delta_{12}\eta_{12}$ is

$$(\lambda \gamma_2^{-1} \delta^* \eta + \lambda^* \gamma_2^{-1} \eta^* \delta) f.$$

Proof. In (7) let $a=e$, $b=\eta_{12}$ and $c=\delta_{12}$ to get

$$(17) \quad (\delta_{12} \times \eta_{12}) \cdot e + (e \times \eta_{12}) \cdot \delta_{12} + (\delta_{12} \cdot \eta_{12}) \times e + (e \cdot \eta_{12}) \times \delta_{12} = 0.$$

Since $\delta_{12} \cdot \eta_{12} \in J_1 + J_0$ the third term of (17) is zero using Lemmas 1 and 3. Also, $e \times \eta_{12} = e \cdot \eta_{12} - e\eta_{12} = \frac{1}{2}\eta_{12} - \lambda\eta_{12} = (\frac{1}{2} - \lambda)\eta_{12}$ using Lemma 4. So (17) reduces to

$$(18) \quad (\delta_{12} \times \eta_{12}) \cdot e + ((\frac{1}{2} - \lambda)\eta_{12})_{12} \cdot \delta_{12} + \frac{1}{2}\eta_{12} \times \delta_{12} = 0.$$

$(\delta_{12} \times \eta_{12}) \cdot e$ has zero component in J_0 so (18) says

$$[\delta_{12} \times \eta_{12}]_0 = [\eta_{12} \cdot \delta_{12}]_0 - [(2\lambda\eta)_{12} \cdot \delta_{12}]_0.$$

But

$$\begin{aligned} [\delta_{12}\eta_{12}]_0 &= [\delta_{12} \cdot \eta_{12}]_0 - [\delta_{12} \times \eta_{12}]_0 = [(2\lambda\eta)_{12} \cdot \delta_{12}]_0 \\ &= (\gamma_2^{-1}\delta^*\lambda\eta + \gamma_2^{-1}(\lambda\eta)^*\delta)f = (\lambda\gamma_2^{-1}\delta^*\eta + \lambda^*\gamma_2^{-1}\eta^*\delta)f. \end{aligned}$$

LEMMA 6. If $\delta_{12}, \eta_{12} \in J_{1/2}$, then the $J_{1/2}$ -component of $\delta_{12}\eta_{12}$ is 0.

Proof. In (7) let $a=\alpha e$, $b=\eta_{12}$, $c=\delta_{12}$, and comparing components in $J_{1/2}$ will give

$$(19) \quad 2[(\delta_{12} \times \eta_{12}) \cdot \alpha e]_{1/2} = [\delta_{12} \times (\alpha\eta)_{12}]_{1/2}, \quad \alpha \in S, \delta, \eta \in E.$$

Again in (7) let $c=\delta_{12}$, $b=\alpha e$, $a=\eta_{12}$ and get

$$(20) \quad (\alpha\delta)_{12} \times \eta_{12} = \delta_{12} \times (\alpha\eta)_{12}, \quad \alpha \in S, \delta, \eta \in E.$$

For $\alpha \in S$ we have from (19) that $[1_{12} \times \alpha_{12}]_{1/2} = 2[(1_{12} \times 1_{12}) \cdot \alpha e]_{1/2} = 0$. For $\alpha, \beta \in S$, (20) and (19) say $[\beta_{12} \times \alpha_{12}]_{1/2} = [1_{12} \times (\beta\alpha)_{12}]_{1/2} = 2[(1_{12} \times \alpha_{12}) \cdot \beta e]_{1/2} = 0$. Using an inductive process as in Lemma 4 and recalling that S generates E will give $[\delta_{12} \times \eta_{12}]_{1/2} = 0$ for every $\delta, \eta \in E$. This implies $[\delta_{12}\eta_{12}]_{1/2} = 0$.

3. The first structure theorem. With the aid of the lemmas it is now quite easy to prove our first theorem.

THEOREM 1. Let J be a simple noncommutative Jordan algebra of capacity two such that J^+ is not a Jordan algebra determined by a symmetric bilinear form. Then J is either commutative Jordan or quasi-associative.

Proof. Let “ \circ ” denote ordinary associative matrix multiplication in E_2 . If $x_1, y_1 \in J_1$, Lemma 1 says $x_1 y_1 = \lambda \circ x_1 \circ y_1 + (1 - \lambda) \circ y_1 \circ x_1$. If $x_1 \in J_1, y_{1/2} \in J_{1/2}$, Lemma 4 says $x_1 y_{1/2} = \lambda \circ x_1 \circ y_{1/2} + (1 - \lambda) \circ y_{1/2} \circ x_1$.

Since $f\delta_{12} = (1 - e)\delta_{12} = \delta_{12} - e\delta_{12} = \delta_{12} - (\lambda\delta)_{12} = ((1 - \lambda)\delta)_{12} = \delta_{12}e$ we have that $\bar{L}_f = (1 - \lambda) = \lambda^* = \bar{R}_e$. Symmetry of the arguments used in the lemmas with respect to f will give corresponding multiplications with J_0 and the analogue of Lemma 5 together with Lemma 6 will show that $\delta_{12}\eta_{12} = \lambda \circ \delta_{12} \circ \eta_{12} + (1 - \lambda) \circ \eta_{12} \circ \delta_{12}$. If $\lambda = \lambda^*$, J is commutative Jordan and if $\lambda \neq \lambda^*$, J is quasi-associative.

If $E = \Delta \oplus \bar{\Delta}$, a direct sum of a division ring Δ with its anti-isomorphic copy $\bar{\Delta}$, then J is an algebra over $\Phi = \{(\alpha, \bar{\alpha}) \mid \alpha \in \text{center of } \Delta\}$. Any skew element λ in

$\Delta \oplus \bar{\Delta}$ is of the form $\lambda = (\beta, -\bar{\beta})$. Hence if λ is skew and central then $\lambda^2 = (\beta, -\bar{\beta})^2 = (\beta^2, \bar{\beta}^2)$ which has a square root in Φ . So by a theorem of McCrimmon [3, p. 1458], J is split quasi-associative.

If E is a division ring with a central skew element λ then λ^2 does not have a square root in Φ , the basefield of E . Hence by McCrimmon's theorem, J is nonsplit quasi-associative.

4. The second structure theorem. For the rest of this work we will assume J is a noncommutative Jordan algebra such that J^+ is a Jordan algebra determined by a symmetric bilinear form. Then $J^+ = \Phi + V$ where Φ is a field, V is a vector space over Φ , and $f: V \times V \rightarrow \Phi$ is a symmetric bilinear form on V . If we denote the elements of J^+ by (α, v) where $\alpha \in \Phi, v \in V$ then the multiplication in J^+ is given by $(\alpha, v)(\beta, w) = (\alpha\beta + f(v, w), \alpha w + \beta v)$. It can easily be shown that if f is nondegenerate, then J^+ is simple. The elements of J will also be denoted by ordered pairs of elements from Φ and V . If J^+ has an idempotent $e \neq (1, 0)$, then J^+ has capacity two. An easy calculation will show that such an idempotent exists if and only if $f(u, u) = \frac{1}{4}$ for some vector $u \in V$.

If f is not identically 0 there exists an element $w \in V$ such that $f(w, w) = \beta \neq 0$. If $\alpha = \beta^{1/2} \in \Phi$, then $f(\frac{1}{2}\alpha^{-1}w, \frac{1}{2}\alpha^{-1}w) = \frac{1}{4}$ and J^+ (and J) contains a nontrivial idempotent. If $\beta^{1/2} \notin \Phi$ we may take a quadratic extension of Φ to obtain an idempotent. So after a possible quadratic extension of the base field Φ we may assume J has capacity two.

Henceforth u will denote the vector in V such that $f(u, u) = \frac{1}{4}$. Then $e = (\frac{1}{2}, u)$ and $f = (\frac{1}{2}, -u)$ are two orthogonal idempotents such that $1 = e + f$. With respect to e we have

$$\begin{aligned} J_0 &= \{(\tfrac{1}{2}\alpha, -\alpha u) \mid \alpha \in \Phi\}, \\ J_{1/2} &= \{(0, w) \mid f(u, w) = 0\}, \\ J_1 &= \{(\tfrac{1}{2}\alpha, \alpha u) \mid \alpha \in \Phi\}. \end{aligned}$$

We recall that E is the associative ring generated by $\{2\bar{R}_{\alpha e}^+ \mid \alpha e = (\frac{1}{2}\alpha, \alpha u) \in J_1\}$. Since $\bar{R}_{\alpha e}^+ \bar{R}_{\beta e}^+ = \bar{R}_{\beta e}^+ \bar{R}_{\alpha e}^+$, E is a field isomorphic to Φ . Since $\nu(x_1)\nu(y_1) = \nu(y_1)\nu(x_1)$, equation (3) says that $\nu(x_1 y_1) = \nu(x_1)\nu(y_1)$. So J_1 is a field isomorphic to Φ .

We will have occasions to identify the element $(0, v)$ with the vector $v \in V$, so we will make this identification whenever necessary. J induces a multiplication on the vectors of V as follows: for $w, y \in V$ define

$$(21) \quad w \times y = [(0, w)(0, y)]_{1/2},$$

i.e. $w \times y$ is the $J_{1/2}$ -component of the product $(0, w)(0, y)$ in J . Clearly we have

$$w \times y = -y \times w.$$

For $a_{1/2} = (0, w) \in J_{1/2}$ we know $ea_{1/2} \in J_{1/2}$, so $e(0, w) = (\frac{1}{2}, u)(0, w) = (\frac{1}{2}, 0)(0, w) + (0, u)(0, w) = (0, \frac{1}{2}w) + (0, u)(0, w)$. Hence $(0, u)(0, w) \in J_{1/2}$, so $(0, u)(0, w) = (0, u \times w)$.

LEMMA 7. If $\alpha e \in J_1$ and $(0, w) \in J_{1/2}$, then $(\alpha e)(0, w) = (0, \frac{1}{2}\alpha w + \alpha(u \times w))$.

Proof. The linearized flexible law says

$$(\alpha e)[e(0, w)] + (0, w)[e(\alpha e)] = [(\alpha e)e](0, w) + [(0, w)e](\alpha e)$$

or

$$(22) \quad (\alpha e)(0, \frac{1}{2}w + u \times w) + (0, w)(\alpha e) = (\alpha e)(0, w) + (0, \frac{1}{2}w + w \times u)(\alpha e).$$

Due to the fact that we know the multiplication in J^+ , (22) becomes

$$\begin{aligned} (\alpha e)(0, \frac{1}{2}w + u \times w) + (0, \alpha w) - (\alpha e)(0, w) \\ = (\alpha e)(0, w) + (0, \frac{1}{2}\alpha w + \alpha(w \times u)) - (\alpha e)(0, \frac{1}{2}w + w \times u). \end{aligned}$$

So $(\alpha e)(0, w) = (0, \frac{1}{2}\alpha w + \alpha(u \times w))$.

LEMMA 8. Let $v, w \in V$, then $(0, v)(0, w) = (f(v, w), v \times w)$.

Proof. The flexible law says

$$(0, v)[(0, w)(0, v)] - [(0, v)(0, w)](0, v) = 0,$$

and so we have

$$(0, v)[(0, w)(0, v)] + [(0, w)(0, v)](0, v) = 2(f(v, w), 0)(0, v).$$

Hence we know that the product in J^+ of $(0, v)$ with $(0, w)(0, v)$ is the same as that of $(0, v)$ with $(f(v, w), 0)$. This together with (21) implies that $(0, v)(0, w) = (f(v, w), v \times w)$.

Lemmas 7 and 8 imply the following theorem which was motivated by Theorem 1 of [7].

THEOREM 2. If J is a noncommutative Jordan algebra such that $J^+ = \Phi + V$ is a Jordan algebra determined by a nonzero symmetric bilinear form, then possibly after a quadratic extension of the field Φ the multiplication in J is given by

$$(\alpha, v)(\beta, w) = (\alpha\beta + f(v, w), \alpha w + \beta v + v \times w)$$

where " \times " is an anti-commutative multiplication on the vector space V and f is the symmetric bilinear form of J^+ such that $f(v \times w, y) = f(v, w \times y)$.

If we assume that V has a basis $\{w_i\}$ over Φ such that $f(w_i, w_j) = 0$ if $i \neq j$ and $f(w_i, w_i) = \beta_i$, then we can give more information about the \times -multiplication on V . Let $w_i \times w_j = \sum_k \gamma_{ijk} w_k$ where $\gamma_{ijk} \in \Phi$. Then $w_i \times w_j = -w_j \times w_i$ means that

$$(23) \quad \gamma_{ijk} = -\gamma_{jik}.$$

Also $f(w_i, w_j \times w_k) = f(w_i \times w_j, w_k)$ is equivalent to

$$(24) \quad f(w_k, w_k)\gamma_{ijk} = f(w_i, w_i)\gamma_{jki} = f(w_j, w_j)\gamma_{kij}.$$

So given any set of constants $\{\gamma_{ijk}\}$ satisfying (23) and (24), $w_i \times w_j = \sum_k \gamma_{ijk} w_k$ gives a multiplication on V such that J is a noncommutative Jordan algebra and J^+ is a

Jordan algebra determined by a symmetric bilinear form. Whenever V has an orthogonal basis, such a set of γ_{ijk} 's can always be found, for we can define γ_{ijk} arbitrarily for $i < j < k$ and then use (23) and (24) to find γ_{ijk} for the other permutations of i, j, k .

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UNIVERSITY OF MISSISSIPPI,
UNIVERSITY, MISSISSIPPI 38677