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ON THE ORDER OF A STARLIKE FUNCTION

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Abstract. It is shown that if $f \in S$, the class of normalised starlike functions in the unit disc Δ, then

(i) $$\lim_{r\to 1-} \frac{\log P_\lambda(r)}{-\log(1-r)} = \alpha\lambda \quad \text{for } \lambda > 0;$$

(ii) $$\lim_{r\to 1-} \frac{\log \|f_r\|_p}{-\log(1-r)} = \alpha p - 1 \quad \text{for } \alpha p > 1;$$

and

(iii) $$\lim_{r\to 1-} \frac{\log \|f_r'\|_p}{-\log(1-r)} = (1+\alpha)p - 1 \quad \text{for } (1+\alpha)p > 1,$$

where $P_\lambda(r) = \sum_{n=1}^{\infty} n^{\lambda-1}|a_n|^\lambda r^n$, (a_n) is the sequence of coefficients and α the order of f, and where

$$\|f_r\|_p = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.$$

The results extend work of Pommerenke.

The methods of the paper yield various other results, one in particular being

$$\limsup_{n\to\infty} \frac{\log^+ n|a_n|}{\log n} = \alpha,$$

a result which has an analogy in the theory of entire functions.

1. **Introduction.** Let μ be a probability measure on the unit circle Γ, and define the function f on the unit disc Δ by

$$f(z) = z \exp\left\{-2\int \log(1-z\bar{\gamma})\, d\mu(\gamma)\right\}, \qquad z \in \Delta. \tag{1.1}$$

Then f is regular and *starlike* on Δ, that is, f is univalent and maps Δ onto a domain in the complex plane that is starshaped with respect to the origin.

Following Pommerenke [7] we call

$$\alpha_f = 2\max\{\mu(\{\gamma\}) : \gamma \in \Gamma\}$$

the *order* of f. Since, by hypothesis, μ is positive and $\int d\mu = 1$, it follows that $0 \leqq \alpha_f \leqq 2$. Further, $\alpha_f = 0$ if and only if μ is continuous; and $\alpha_f = 2$ if and only if f is (a rotation of) the Koebe function. If μ is discontinuous, then $\alpha_f > 0$, and μ has at least one maximum jump of height $\alpha_f/2$.

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In [7], Pommerenke showed that if $M(r,f)=\max\{|f(z)| : |z|=r, (0\leq r<1)\}$ then α_f is connected to $M(r,f)$ by the relationship

$$(\log M(r,f))/\log(1-r)^{-1}\to\alpha_f \quad \text{as } r\to 1, \tag{1.2}$$

a result which has an analogy in theory of entire functions. Using (1.2), Pommerenke [8] was then able to show that

$$\alpha_f = \lim_{r\to 1}\frac{(1-r)M'(r,f)}{M(r,f)}, \tag{1.3}$$

where M' denotes the left derivative.

The geometrical significance of α_f is as follows. If $\alpha_f>0$, then $f(\Delta)$ contains at least one sector of opening $\pi\alpha_f$ and no sector of larger opening. Thus the area of $f(\Delta)$ is infinite if $\alpha_f>0$. On the other hand, $\alpha_f=0$ does not necessarily imply that the area of $f(\Delta)$ is finite. In the light of this observation and Pommerenke's results (loc. cit.), it is natural to study the connection between α_f and the rate of growth of $\pi A(r,f)$, the area of the image of the disc $\Delta_r=\{z : |z|\leq r\}$ under f.

The present investigation stems from an attempt to extend (1.2) and (1.3) to $A(r,f)$. More specifically, we sought to prove that

$$(\log A(r,f))/\log(1-r)^{-1}\to 2\alpha_f \quad \text{as } r\to 1, \tag{1.4}$$

and

$$2\alpha_f = \lim_{r\to 1}\frac{(1-r)A'(r,f)}{A(r,f)}. \tag{1.5}$$

It is clear that (1.5) implies (1.4). In this paper, a simple proof of (1.4) is given. (1.5) seems to be very much deeper and a proof will be given in [6].

Some by-products of our efforts to prove (1.4) and (1.5) are presented in §3, where, amongst other things, we derive results similar to (1.2) for the integral means of f and f'. In §4, we study analogous problems for certain means of the coefficients (a_n) of f. In particular, we prove that

$$\alpha_f = \limsup_{n\to\infty}\frac{\log^+ n|a_n|}{\log n}.$$

Notation. Throughout the paper, μ will denote a fixed probability measure on Γ, and f a function defined by (1.1). In order to simplify the writing a little, we shall write α in place of α_f, $M(r)$ in place of $M(r,f)$, etc. Also ω will denote a point on Γ such that $\alpha=2\mu(\{\omega\})$. We define the function F on Δ by

$$F(z) = zf'(z)/f(z) \qquad (z\in\Delta),$$

so that F is regular and Re $F>0$. Finally by σ we shall mean normalised Lebesgue measure on Γ, and we will adopt the convention that

$$\int g(rt)\,d\sigma(t) = \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\,d\theta.$$

2. Preliminaries.

We begin by proving a general lemma.

LEMMA 1. *If* $2\beta > 1$, *then*

$$\lim_{r\to 1}(1-r)^{2\beta-1}\int\frac{d\sigma(t)}{|1-rt|^{2\beta}} = \frac{\Gamma(\beta-\frac{1}{2})}{2\sqrt{\pi}\Gamma(\beta)}. \tag{2.1}$$

Proof. Set

$$(1-z)^{-\beta} = \sum_{n=0}^{\infty} c_n(\beta)z^n \qquad (z\in\Delta),$$

then [10, p. 58]

$$\begin{aligned} c_n(\beta) &= \frac{\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n+1)}, &\quad n = 0, 1, 2, \ldots, \\ &\sim \frac{n^{\beta-1}}{\Gamma(\beta)} &\quad \text{as } n\to\infty. \end{aligned}$$

Thus

$$\begin{aligned} \int\frac{d\sigma(t)}{|1-rt|^{2\beta}} &= \sum_{n=0}^{\infty}|c_n(\beta)|^2 r^{2n} &\quad (0 \leqq r < 1), \\ &\sim \frac{1}{\Gamma^2(\beta)}\sum_{n=1}^{\infty} n^{2\beta-2}r^{2n} &\quad \text{as } r\to 1, \\ &\sim \frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)(1-r^2)^{2\beta-1}} &\quad \text{as } r\to 1 \end{aligned}$$

[10, p. 225]. This and the duplication formula for the gamma function gives (2.1).

Next we use (1.1) to derive lower bounds for $|f|$ and $\operatorname{Re} F$.

LEMMA 2. *For all* $z\in\Delta$,

$$|f(z)| \geqq |z|2^\alpha/4|1-z\bar{\omega}|^\alpha, \tag{2.2}$$

and

$$\operatorname{Re} F(z) \geqq \alpha(1-|z|^2)/2|1-z\bar{\omega}|^2. \tag{2.3}$$

Moreover, equality can occur in each of (2.2) *and* (2.3) *if, and only if, f is* (*a rotation of*) *the Koebe function.*

Proof. We shall only prove (2.2), the proof of (2.3) is similar. Fix $z\in\Delta$, $z\neq 0$ and observe from (1.1) that

$$\log|f(z)| - \log|z| = -2\int\log|1-z\bar{t}|\,d\mu(t). \tag{2.4}$$

Now for $t\in\Gamma$, $\log 2 > \log|1-z\bar{t}|$, and μ is positive. Thus from (2.4)

$$\begin{aligned} \log 4|f(z)| - \log|z| &= 2\int\log\frac{2}{|1-z\bar{t}|}\,d\mu(t) \\ &\geqq 2\mu(\{\omega\})\log\frac{2}{|1-z\bar{\omega}|} = \alpha\log\frac{2}{|1-z\bar{\omega}|} \end{aligned}$$

and (2.2) follows. Clearly the inequality is strict, unless μ is concentrated at ω, in which case $\alpha=2$, and f is the Koebe function.

A straightforward application of (2.2) and (2.3), together with Lemma 1, will yield the following result, the proof of which we omit.

LEMMA 3. *If a and b are nonnegative and such that $\alpha a+2b>1$, then*

$$\liminf_{r\to 1}(1-r)^{\alpha a+b-1}\int |f(rt)|^a(\operatorname{Re} F(rt))^b\,d\sigma(t) \geqq \frac{\alpha^b 2^{\alpha a-1}\Gamma(b+(\alpha a-1)/2)}{4^a\sqrt{\pi}\,\Gamma(b+\alpha a/2)}. \tag{2.5}$$

For ease of reference, we include the following theorem, a proof of which may be found in [5].

THEOREM A. *If $p>1$, then*

$$\lim_{r\to 1}(1-r)^{p-1}\int |F(rt)|^p\,d\sigma(t) = \frac{\Gamma(p/2-1)}{2\sqrt{\pi}\,\Gamma(p/2)}\sum_{t\in\Gamma}|\mu(\{t\})|^p. \tag{2.6}$$

3. **Integral means.** In this section, if $p>0$ and g is regular in Δ, we set

$$\|g_r\|_p = \int |g(rt)|^p\,d\sigma(t) \qquad (0 \leqq r < 1).$$

With this notation we have

THEOREM 1. *If $\alpha p>1$, then*

$$\lim_{r\to 1}\frac{\log \|f_r\| p}{-\log(1-r)} = \alpha p-1. \tag{3.1}$$

Proof. Choosing $a=p$ and $b=0$ in (2.5), we have

$$\liminf_{r\to 1}(1-r)^{\alpha p-1}\|f_r\|_p \geqq \frac{2^{\alpha p-1}\Gamma((\alpha p-1)/2)}{4^p\sqrt{\pi}\,\Gamma(\alpha p/2)},$$

and so

$$\liminf_{r\to 1}\frac{\log \|f_r\|_p}{-\log(1-r)} \geqq \alpha p-1. \tag{3.2}$$

On the other hand, for $t\in\Gamma$,

$$r\,\partial \log |f(rt)|/\partial r = \operatorname{Re} F(rt),$$

and therefore

$$r\frac{d}{dr}\|f_r\|_p = p\int |f(rt)|^p \operatorname{Re} F(rt)\,d\sigma(t)$$

$$\leqq p(M(r))^p\int \operatorname{Re} F(rt)\,d\sigma(t) = p(M(r))^p,$$

since $\operatorname{Re} F(z)>0$, $z\in\Delta$. Thus

$$\|f_r\|_p \leqq p\int_0^r \frac{(M(s))^p}{s}\,ds. \tag{3.3}$$

But, it follows from (1.2), that for every $\varepsilon > 0$,

$$M(r) = O(1)(1-r)^{-\alpha-\varepsilon} \quad \text{as } r \to 1,$$

and thus (3.3) gives

$$\|f_r\|_p = O(1)(1-r)^{-\alpha p-\varepsilon p+1} \quad \text{as } r \to 1$$

giving

$$\limsup_{r\to 1} \frac{\log \|f_r\|_p}{-\log(1-r)} \leqq \alpha p - 1. \tag{3.4}$$

Theorem 1 now follows from (3.2) and (3.4).

We require slightly different techniques to deal with the next theorem, which gives a similar estimate for f'.

THEOREM 2. *If* $(1+\alpha)p-1>0$, *then*

$$\lim_{r\to 1} \frac{\log \|f_r'\|_p}{-\log(1-r)} = (1+\alpha)p-1. \tag{3.5}$$

Proof. We have $zf'(z)=f(z)F(z)$, $z \in \Delta$, and so, if $0 \leqq r < 1$,

$$r^p\|f_r'\|_p = \int |f(rt)|^p|F(rt)|^p\, d\sigma(t) \geqq \int |f(rt)|^p(\operatorname{Re} F(rt))^p\, d\sigma(t).$$

Taking $a=b=p$ in (2.5), it now follows easily that

$$\liminf_{r\to 1} \frac{\log \|f_r'\|_p}{-\log(1-r)} \geqq (1+\alpha)p-1. \tag{3.6}$$

To obtain the lim sup variant of (3.6), we treat separately the cases: (i) $p>1$; (ii) $p=1$ and (iii) $0<p<1$.

Case (i). $p>1$. Here

$$\begin{aligned} r^p\|f_r'\|_p &= \int |f(rt)|^p|F(rt)|^p\, d\sigma(t) \\ &\leqq (M(r))^p\|F_r\|_p = O(1)(M(r))^p(1-r)^{1-p} \quad \text{as } r \to 1 \end{aligned}$$

by Theorem A. Consequently, using (1.2) we have

$$\limsup_{r\to 1} \frac{\log \|f_r'\|_p}{-\log(1-r)} \leqq p \lim_{r\to 1} \frac{\log M(r)}{-\log(1-r)} + p - 1 = \alpha p + p - 1, \tag{3.7}$$

and the result follows from this and (3.6).

Case (ii). $p=1$. Since $\operatorname{Re} F>0$,

$$\|F_r\| = \int |F(rt)|\, d\sigma(t) = O(1)\log(1-r)^{-1} \quad \text{as } r \to 1,$$

and so

$$\|f_r'\| = O(1)M(r)\log(1-r)^{-1} \quad \text{as } r \to 1,$$

from which (3.7) follows with $p=1$.

Case (iii). $0<p<1$. Set $\lambda=1/p$, $\lambda'=1/(1-p)$. As before,

$$\begin{aligned} r^p\|f_r'\|_p &= \int |f(rt)|^p|F(rt)|^p\,d\sigma(t) \\ &\leqq \left(\int |f(rt)|^{p\lambda'}\,d\sigma(t)\right)^{1/\lambda'}\left(\int |F(rt)|^{p\lambda}\,d\sigma(t)\right)^{1/\lambda} \\ &= \|f_r\|_{p\lambda'}^{1/\lambda'}\cdot\|F_r\|_{p\lambda}^{1/\lambda}, \end{aligned} \tag{3.8}$$

by Hölder's inequality. Now since $p\lambda=1$,

$$\|F_r\|_{p\lambda}^{1/\lambda} = O(1)(\log(1-r)^{-1})^{1/\lambda} \quad \text{as } r\to 1. \tag{3.9}$$

Also by Theorem 1 since $\alpha p\lambda'>1$ by hypothesis,

$$\lim_{r\to 1}\frac{\log\|f_r\|_{p\lambda'}}{-\log(1-r)} = \alpha p\lambda'-1. \tag{3.10}$$

Combining (3.9) and (3.10), we deduce from (3.8) that

$$\limsup_{r\to 1}\frac{\log\|f_r'\|_p}{-\log(1-r)} \leqq \frac{\alpha p\lambda'-1}{\lambda'} = (1+\alpha)p-1,$$

and this together with (3.6) again gives (3.5). This completes the proof.

4. **Coefficient means.** The function f defined by (1.1) has an expansion of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad z\in\Delta.$$

In this section, we shall study the growth of the sequence (a_n) and the means

$$P_\lambda(r) = \sum_{n=1}^{\infty} n^{\lambda-1}|a_n|^\lambda r^n, \qquad \lambda>0, \quad 0\leqq r<1.$$

We begin by stating an important lemma, due essentially to Clunie and Keogh [1].

LEMMA 4. *If* $0\leqq r<1$, *then* $(n+1)|a_n|r^n\leqq 2M(r)$, $n=1,2,\ldots.$

We now establish

THEOREM 3. *If* $\lambda>0$, *then, with the above notation,*

$$\lim_{r\to 1}\frac{\log P_\lambda(r)}{-\log(1-r)} = \alpha\lambda. \tag{4.1}$$

Proof. To begin with, by Lemma 4 we have, for $0\leqq r<1$,

$$P_\lambda(r^{\lambda+1}) = \sum_{n=1}^{\infty}(n|a_n|r^n)^\lambda\frac{r^n}{n} \leqq (2M(r))^\lambda\log(1-r)^{-1},$$

and so by (1.2)

$$\limsup_{r\to 1}\frac{\log P_\lambda(r)}{-\log(1-r)} \leqq \alpha\lambda. \tag{4.2}$$

Next, if $\lambda \geqq 1$, a direct application of Hölder's inequality shows that

$$P_\lambda(r)(\log(1-r)^{-1})^{\lambda-1} \geqq (M(r))^\lambda,$$

thus

$$\liminf_{r\to 1} \frac{\log P_\lambda(r)}{-\log(1-r)} \geqq \alpha\lambda,$$

and so (4.1) is proved for $\lambda \geqq 1$.

If $0<\lambda<1$, then for $0 \leqq r < 1$, we have, again using Hölder's inequality,

$$\begin{aligned} P_1(r) &= \sum_{n=1}^{\infty} |a_n| r^n = \sum_{n=1}^{\infty} (n^{\lambda-1}|a_n|^\lambda r^n)^\lambda (n^\lambda |a_n|^{1+\lambda} r^n)^{1-\lambda} \\ &\leqq \left(\sum_{n=1}^{\infty} n^{\lambda-1}|a_n|^\lambda r^n\right)^\lambda \left(\sum_{n=1}^{\infty} n^\lambda |a_n|^{1+\lambda} r^n\right)^{1-\lambda} \\ &= (P_\lambda(r))^\lambda (P_{1+\lambda}(r))^{1-\lambda}. \end{aligned}$$

Hence

$$\frac{\log P_1(r)}{-\log(1-r)} \leqq \lambda \frac{\log P_\lambda(r)}{-\log(1-r)} + (1-\lambda)\frac{\log P_{1+\lambda}(r)}{-\log(1-r)}.$$

Since $M(r) \leqq P_1(r)$, we obtain from this and (1.2) that

$$\alpha \leqq \lambda \liminf_{r\to 1} \frac{\log P_\lambda(r)}{-\log(1-r)} + (1-\lambda)(1+\lambda)\alpha.$$

(In the last expression we have used (4.1) for the case $\lambda \geqq 1$, already proved.) Thus

$$\alpha\lambda \leqq \liminf_{r\to 1} \frac{\log P_\lambda(r)}{-\log(1-r)}. \tag{4.3}$$

From (4.2) and (4.3) we obtain (4.1) for $0<\lambda<1$.

Corollary. *If $\pi A(r)$ denotes the area of the image of Δ_r under f, then*

$$\lim_{r\to 1} \frac{\log A(r)}{-\log(1-r)} = 2\alpha.$$

Proof. It is well known that, for $0 \leqq r < 1$,

$$A(r) = \sum_{n=1}^{\infty} n|a_n|^2 r^{2n} = P_2(r^2),$$

and so the corollary follows immediately from (4.1).

Remark. If $\lambda = 1$, then Theorem 3 shows that

$$\lim \frac{\log P_1(r)}{-\log(1-r)} = \alpha,$$

where $P_1(r) = \sum_{n=1}^{\infty} |a_n| r^n$, $0 \leqq r < 1$.

The following theorem, which we state without proof, can be proved using similar arguments to those used in Theorem 3.

THEOREM 4. *If* $\lambda>0$ *and* $\alpha\lambda-\lambda+1>0$, *then*

$$\lim_{r\to 1}\frac{\log\sum_{n=1}^{\infty}|a_n|^{\lambda}r^n}{-\log(1-r)}=\alpha\lambda-\lambda+1.$$

The next result connects the sequence of coefficients (a_n), with the order α, and has an analogy in the theory of entire functions.

THEOREM 5.

$$\limsup_{n\to\infty}\frac{\log^+ n|a_n|}{\log n}=\alpha. \tag{4.4}$$

Proof. Since the Bieberbach conjecture holds for starlike functions [4] $n|a_n|\leqq n^2$, $n=1,2,\ldots$, thus

$$O\leqq\frac{\log^+ n|a_n|}{\log n}\leqq 2,\qquad n=1,2,\ldots.$$

Set

$$\tau=\limsup_{n\to\infty}\frac{\log^+ n|a_n|}{\log n},$$

and let $\varepsilon>0$ be given. Then there exists N such that $n|a_n|<n^{\tau+\varepsilon}$ for all $n>N$. Hence

$$\begin{aligned}M(r)&\leqq\sum_{n=1}^{\infty}|a_n|r^n=\sum_{n=1}^{N}|a_n|r^n+\sum_{n=N+1}^{\infty}|a_n|r^n\\&\leqq\sum_{n=1}^{N}|a_n|r^n+\sum_{n=N+1}^{\infty}n^{\tau+\varepsilon-1}r^n=O(1)(1-r)^{-\tau-\varepsilon}\quad\text{as } r\to 1,\end{aligned}$$

which gives, since ε is arbitrary,

$$\limsup_{r\to 1}\frac{\log M(r)}{-\log(1-r)}\leqq\tau,$$

and so by (1.2) $\alpha\leqq\tau$.

On the other hand, there is an increasing sequence of integers $\{n_k\}$, such that $n_k|a_{n_k}|>n_k^{\tau-\varepsilon}$, $k=1,2,\ldots$. With $r_k=1-1/n_k$, we deduce from Lemma 4, that

$$2M(r_k)\geqq(1-1/n_k)^{n_k}(1-r_k)^{-\tau+\varepsilon},\qquad k=1,2,\ldots,$$

which gives, again since ε is arbitrary,

$$\limsup_{r\to 1}\frac{\log M(r)}{-\log(1-r)}\geqq\tau,$$

and so $\alpha\geqq\tau$. Hence (4.4) is proved.

REMARK. If, in place of Lemma 4, we use Theorem 1 of [2], the above proof shows that even for a close-to-convex function f, we have

$$\limsup_{r\to 1}\frac{\log M(r,f)}{-\log(1-r)}=\limsup_{n\to\infty}\frac{\log^+ n|a_n|}{\log n}.$$

5. **The growth of** $A(r)$. In this section we present a number of results concerning the growth of the area function $A(r)$.

THEOREM 6. *The map*

$$r \to (1-r)^4 A(r)/r^2 \tag{5.1}$$

is decreasing on the interval $(0, 1)$. *Furthermore,*

$$\lim_{r\to 1}(1-r)^4 A(r) = 0$$

unless f *is (a rotation of) the Koebe function, in which case the limit is* $3/8$.

Proof. Since, by definition $F(z) = zf'(z)/f(z)$, $z \in \Delta$, (1.1) gives

$$F(z) = \int \frac{1+z\bar{\gamma}}{1-z\bar{\gamma}}\, d\mu(\gamma), \qquad z \in \Delta.$$

Therefore, by the Schwarz inequality,

$$|F(z)|^2 \leqq \int \frac{(1+|z|)^2}{|1-z\bar{\gamma}|^2}\, d\mu(\gamma) = \frac{1+|z|}{1-|z|}\,\mathrm{Re}\, F(z).$$

Thus, for $0 \leqq r < 1$,

$$\int |f(rt)|^2 |F(rt)|^2\, d\sigma(t) \leqq \frac{1+r}{1-r}\int |f(rt)|^2\, \mathrm{Re}\, F(rt)\, d\sigma(t).$$

But $\pi A(r)$ represents the area of the image of Δ_r under f, and so

$$A(r) = \sum_{n=1}^{\infty} n|a_n|^2 r^{2n} = \int rtf'(rt)\overline{f(rt)}\, d\sigma(t)$$

$$= \int |f(rt)|^2\, \mathrm{Re}\, F(rt)\, d\sigma(t),$$

which gives

$$rA'(r) \leqq 2\,\frac{1+r}{1-r}\, A(r).$$

Hence

$$\frac{d}{dr}\log A(r) \leqq 2\,\frac{1+r}{r(1-r)} = \frac{d}{dr}\log\frac{r^2}{(1-r)^4},$$

and the first part of the theorem is now obvious.

Let $\beta = \lim_{r\to 1}(1-r)^4 A(r)$, and suppose that $\beta \neq 0$. Then [3, p. 170] $A'(r) \sim 4\beta/(1-r)^5$ as $r \to 1$. Thus $\lim_{r\to 1}(1-r)A'(r)/A(r) = 4$. From (1.4) and (1.5) we deduce therefore that when $\beta \neq 0$, $\alpha = 2$ so that μ is concentrated at ω, that is, f is (a rotation of) the Koebe function. If f is a Koebe function then $|a_n| = n$, $n = 1, 2, \ldots$, and

$$\lim_{r\to 1}(1-r)^4 A(r) = \lim_{r\to 1}(1-r)^4 \sum_{n=1}^{\infty} n^3 r^{2n} = 3/8.$$

This completes the proof.

The following corollary is sometimes quite useful:

COROLLARY. $rA(\sqrt{r}) < 16A(r)$, $0 < r < 1$.

Proof. $(1-\sqrt{r})^4 A(\sqrt{r})/r < (1-r)^4 A(r)/r^2$, and the result is then obvious.

A similar result, with a worse constant, can be proved for any *univalent* function using Theorem 1.3 [4].

THEOREM 7. *If* $\lambda \geqq 2$ *and* $\|f_r\|_\lambda = \int |f(rt)|^\lambda \, d\sigma(t)$, $0 \leqq r < 1$, *then*

$$\liminf_{r\to 1} \frac{\|f_r\|_\lambda}{(1-r)(A(r))^{\lambda/2}} \geqq \frac{\lambda}{2\lambda-1}. \tag{5.2}$$

The inequality is sharp for $\lambda = 2$.

Proof. For $0 \leqq r < 1$,

$$\begin{aligned} r\frac{d}{dr}\|f_r\|_\lambda &= \lambda \int |f(rt)|^\lambda \operatorname{Re} F(rt)\, d\sigma(t) \\ &\geqq \lambda\left(\int |f(rt)|^2 \operatorname{Re} F(rt)\, d\sigma(t)\right)^{\lambda/2} = \lambda (A(r))^{\lambda/2}, \end{aligned}$$

where we have used Hölder's inequality and the fact that $\operatorname{Re} F(z) > 0$ for $z \in \Delta$. Hence, using the monotonicity of (5.1) we deduce that

$$\|f_r\|_\lambda \geqq \lambda \int_0^r \frac{(A(s))^{\lambda/2}}{s}\, ds \geqq \lambda \left(\frac{(1-r)^4 A(r)}{r^2}\right)^{\lambda/2} \int_0^r \frac{s^{\lambda-1}}{(1-s)^{2\lambda}}\, ds.$$

Consequently

$$\liminf_{r\to 1} \frac{\|f_r\|_\lambda}{(1-r)(A(r))^{\lambda/2}} \geqq \lambda \lim_{r\to 1} (1-r)^{2\lambda-1} \int_0^r \frac{s^{\lambda-1}}{(1-s)^{2\lambda}}\, ds = \frac{\lambda}{2\lambda-1},$$

and this is (5.2).

If $\lambda = 2$ and f is the Koebe function, then as $r \to 1$, $\|f_r\|_2 \sim 2(1-r^2)^{-3}$ and $A(r) \sim 6(1-r^2)^{-4}$, giving

$$\lim_{r\to 1} \frac{\|f_r\|_2}{(1-r)A(r)} = 2/3,$$

which shows that (5.2) is sharp when $\lambda = 2$.

Our next result extends Theorem 2 [8] in two directions.

THEOREM 8. *Let* $\alpha > 0$ *and* $p \geqq 1$, *then*

$$\|f_r'\|_p = O(1)(A(r))^{p/2}(1-r)^{1-p} \quad \text{as } r \to 1. \tag{5.3}$$

Proof. In what follows, K will denote a positive constant depending on α, but will not necessarily be the same at each occurrence.

In view of (1.3) there is an r_0, such that $M(r) \leqq K(1-r)M'(r,f)$, if $0 < r_0 < r < 1$. Now

$$M'(r,f) \leqq M(r,f') \leqq \sum_{n=1}^{\infty} n|a_n| r^{n-1}, \qquad 0 \leqq r < 1,$$

and so, if $0<r_0<r<1$,

$$M(r) \leqq K(1-r)\sum_{n=1}^{\infty} n|a_n|r^{n-1} \leqq K(1-r)\sqrt{\left(\sum_{n=1}^{\infty} n|a_n|^2 r^n\right)}\sqrt{\left(\sum_{n=1}^{\infty} nr^n\right)}.$$

Thus by the corollary to Theorem 6

$$M(r) \leqq K\sqrt{(A(r))}. \tag{5.4}$$

If $p=1$, Theorem 2 of [8] together with (5.4) gives

$$\|f_r'\|_1 = O(1)M(r) = O(1)\sqrt{(A(r))} \quad \text{as } r \to 1,$$

which is (5.3) when $p=1$. On the other hand, if $p>1$, we deduce from Theorem A and (5.4) that

$$r^p\|f_r'\|_p = \int |f(rt)|^p |F(rt)|^p \, d\sigma(t) \leqq K(A(r))^{p/2}(1-r)^{1-p},$$

and this is (5.3) when $p>1$.

REMARK. In connection with (5.4), we mention that Sheil-Small, in his thesis [9] considered, inter alia,

$$\lim_{r\to 1} \begin{matrix}\sup\\ \inf\end{matrix} A(r)/M^2(r)$$

and found the sharp bounds for these expressions in terms of α.

In conclusion we prove

THEOREM 9. *If* $0 \leqq r < 1$, *then*

$$\int |F(rt)-1|^2 \, d\sigma(t) \leqq \frac{r^2 \log(16r^{-2}A(r))}{(1-r^2)\log 2/(1-r^2)}. \tag{5.5}$$

Proof. Define $\tilde{\mu}$ on the Borel sets E of Γ by $\tilde{\mu}(E)=\mu(\bar{E})$, and let ν denote the convolution of $\tilde{\mu}$ and μ. Then ν is a positive measure on Γ, and $\int d\nu=1$.

Since, by (1.1)

$$\begin{aligned} F(z) &= \int \frac{1+z\bar{\gamma}}{1-z\bar{\gamma}} \, d\mu(\gamma), \qquad z \in \Delta, \\ &= 1+2\sum_{n=1}^{\infty} z^n \int \bar{\gamma}^n \, d\mu(\gamma), \end{aligned}$$

we have

$$\int \bar{t}^n \operatorname{Re} F(rt) \, d\sigma(t) = r^n \int \bar{\gamma}^n \, d\mu(\gamma), \qquad n = 0, 1, 2, \ldots.$$

It follows that

$$\int \overline{g(rt)} \operatorname{Re} F(rt) \, d\sigma(t) = \int g(r^2 t) \, d\mu(t), \qquad 0 \leqq r < 1,$$

for a variety of g, and certainly if g is the real part of a regular function. In particular, if $0 \leqq r < 1$,

$$\begin{aligned}\int \log (r^{-2}|f(r^2t)|)\, d\mu(t) &= \int \log (r^{-1}|f(rt)|) \operatorname{Re} F(rt)\, d\sigma(t)\\ &= 2 \iint \log \frac{1}{|1-rt\bar{\gamma}|}\, d\mu(\gamma) \operatorname{Re} F(rt)\, d\sigma(t)\\ &= 2 \operatorname{Re} \sum_{n=1}^{\infty} \frac{r^n}{n} \int \bar{\gamma}^n\, d\mu(\gamma) \int t^n \operatorname{Re} F(rt)\, d\sigma(t)\\ &= 2 \operatorname{Re} \sum_{n=1}^{\infty} \frac{r^{2n}}{n} \int \bar{\gamma}^n\, d\mu(\gamma) \int \gamma^n\, d\mu(\gamma)\\ &= 2 \operatorname{Re} \sum_{n=1}^{\infty} \frac{r^{2n}}{n} \int \gamma^n\, d\nu(\gamma)\\ &= 2 \int \log \frac{1}{|1-r^2\gamma|}\, d\nu(\gamma).\end{aligned}$$

Since $\int d\mu = \int d\nu = 1$, we deduce that

$$\int \log (4r^{-2}|f(r^2t)|)\, d\mu(t) = 2 \int \log \frac{2}{|1-r^2\gamma|}\, d\nu(\gamma).$$

But, for each $\gamma \in \Gamma$,

$$\log \frac{2}{|1-r^2\gamma|} \Big/ \log \frac{2}{1-r^2}$$

is a decreasing function of r, and so

$$r \to \int \log (4r^{-2}|f(r^2t)|)\, d\mu(t) \Big/ \log \frac{2}{1-r^2}$$

is likewise decreasing on (0, 1). Consequently, the derivative of this last displayed function is nonpositive, and we deduce that

$$\int (\operatorname{Re} F(r^2t)-1)\, d\mu(t) \leqq r^2 \frac{\int \log (4r^{-2}|f(r^2t)|)\, d\mu(t)}{(1-r^2) \log 2/(1-r^2)};$$

that is,

$$\int (\operatorname{Re} F(rt)-1) \operatorname{Re} F(rt)\, d\sigma(t) \leqq \frac{r^2 \int \log (4r^{-1}|f(rt)|) \operatorname{Re} F(rt)\, d\sigma(t)}{(1-r^2) \log 2/(1-r^2)}$$

whenever $0 \leqq r < 1$. Now

$$2 \int (\operatorname{Re} F(rt)-1) \operatorname{Re} F(rt)\, d\sigma(t) = \int |F(rt)-1|^2\, d\sigma(t),$$

and the convexity of the exponential function implies that

$$2 \int \log (4r^{-1}|f(rt)|) \operatorname{Re} F(rt)\, d\sigma(t) \leqq \log (16r^{-2}A(r)),$$

thus (5.5) follows, and the proof is complete.

We remark finally that, in view of (1.3), (1.5), and [6], the following problems suggest themselves:

Show

(i) $$\lim_{r\to 1}(1-r)\frac{P'_\lambda(r)}{P_\lambda(r)} = \alpha\lambda \quad \text{for } \lambda > 0;$$

(ii) $$\lim_{r\to 1}\frac{(1-r)\,d\|f_r\|_p/dr}{\|f_r\|_p} = \alpha p - 1 \quad \text{for } \alpha p > 1;$$

(iii) $$\lim_{r\to 1}\frac{(1-r)\,d\|f'_r\|_p/dr}{\|f'_r\|_p} = (1+\alpha)p - 1 \quad \text{for } (1+\alpha)p > 1.$$

All these questions are open ones.

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