

CONCERNING A BOUND PROBLEM IN KNOT THEORY

BY

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Abstract. In a recent paper Treybig shows that if two knot functions f, g determine equivalent knots, then f, g are the ends of a simple sequence x of knot functions. In an effort to bound the length of x in terms of f and g (1) a bound is found for the moves necessary in moving one polyhedral disk onto another in the interior of a tetrahedron and (2) it is shown that two polygonal knots K, L in regular position can "essentially" be embedded as part of the 1-skeleton of a triangulation T of a tetrahedron, where (1) all 3 cells which are unions of elements of T can be shelled and (2) the number of elements in T is determined by K, L . A number of "counting" lemmas are proved.

Introduction. The work in this paper is a continuation of research carried on in [3], [7], [9], [12], [13], [14], [15]. For a more detailed explanation of the ideas behind the problem attacked here see [15].

The theorems proved here are results of attempts by the author to settle a conjecture of the following type: Suppose a piece of rope in the shape of a simple closed curve is moved in space so as to form at different times two knots whose projections have no more than n crossings. Then, the rope can be moved in small steps from the first position to the second one so that its projection never has more than $2n$ crossings.

This type of problem can be considered as a special case of the more general problem: "Suppose K_1 and K_2 are finite complexes whose elements are rectilinear and lie in the interior of an n -simplex S , and $f: S \rightarrow S$ is an onto piecewise linear homeomorphism such that (1) $f(x) = x$ if $x \in \text{Bd}(S)$, and (2) $f(|K_1|) = |K_2|$. Then, determine a bound B stated in advance in terms of $\text{card}(K_1 \cup K_2)$ such that there is a triangulation T of S and a simplicial isotopy $h: S \times [0, 1] \rightarrow S$ such that (1) $h_t(x) = x$ if $t = 0$ or $x \in \text{Bd}(S)$, (2) $h_1(|K_1|) = |K_2|$, (3) h_t is affine on each simplex in T for each t in $[0, 1]$, and (4) $\text{card}(T) \leq B$." A major difficulty here is that f , when thought of as the final stage of an isotopy, may take $|K_1|$ and then "twist" things quite a bit more than necessary, and then finally place $|K_1|$ on $|K_2|$.

One type of problem in which a bound has been successfully located is the word problem for certain types of finitely presented groups [4], [5], [6]. In a forthcoming paper the author uses results analogous to those of Greendlinger [6] to show that, if K_1 and K_2 are polygonal knots in regular position in E^3 which are equivalent

Received by the editors February 10, 1970.

AMS 1969 subject classifications. Primary 1188; Secondary 1184.

Key words and phrases. Polyhedron, polygonal knot, piecewise linear homeomorphism, simplicial isotopy, free cell, shelling order.

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under an orientation preserving homeomorphism of E^3 onto E^3 , then there is a finitely presented group $|X:R|$ such that (1) the word problem is solvable in $|X:R|$, (2) K_1 and K_2 are represented as words w_1 and w_2 , respectively, (3) w_1 is a reduction of a word $(\prod_{p=1}^n g_p^{-1} r_p g_p)w_2$, where if the reduction is carried out in certain "small" steps, then K_2 is moved to K_1 in certain "small" steps. The group is constructed using (1) results in this paper and in [15] to obtain a special triangulated 3-cell and (2) certain methods analogous to those of Neuwirth [8] to relate the algebra and the geometry.

§3 contains several lemmas which count the number of simplexes in certain triangulations. In §4 a special case of the above general problem is handled; namely the one where the $|K_i|$ are polyhedral disks embedded in the interior of a tetrahedron. Also in §4 a theorem is proved which "essentially" shows how two polygonal knots can be embedded as a union of 1-simplexes of a triangulation T of a tetrahedron M , where (1) the number of 3-simplexes in T is stated in terms of the given knots and (2) every 3-cell which is the union of two or more 3-simplexes of T has two such simplexes that are free.

2. Definitions. Most of the terms used in this paper are defined in one of [1], [2], [10].

All of the spaces considered are subsets of Euclidean 3-space E^3 , and all triangulations of such spaces will be locally finite and have closed simplexes which are rectilinear (i.e., are points, straight line intervals, triangular disks or two cells, or solid tetrahedra).

A subset S of E^3 will be called a polyhedron, or be said to be polyhedral, if it has a rectilinear triangulation. A mapping $f: S \rightarrow T$ between polyhedra will be said to be piecewise linear (p.l.) if there is a rectilinear triangulation W of S such that f is affine on each simplex of W . If $f, f': S \rightarrow T$ are p.l. homeomorphisms, then we say $K: S \times [0, 1] \rightarrow T$ is a p.l. isotopy between f and f' if (0) K is continuous, (1) $K(s, 0) = f(s)$ and $K(s, 1) = f'(s)$ for $s \in S$, and (2) for each $t \in [0, 1]$ K_t is a p.l. homeomorphism. Furthermore, K will be said to be a simplicial isotopy if some fixed triangulation W of S can be found so that K_t is affine on each $w \in W$ for all $t \in [0, 1]$.

If T is a triangulation of a polyhedron S , we let $n(T_p) = \text{card}(T_p)$, where T_p denotes the collection of p -simplexes of T . Also, for any $T' \subset T$, let $|T'| = \bigcup T'$. Given $M \subset S$ let $\text{st}(M, T)$ denote the collection of all simplexes of T which contain M .

If M is an n -manifold with boundary then $\text{Bd}(M)$ will denote the set of all points of M which do not have a neighborhood in M homeomorphic to E^n . If M is an n -cell, then an n -cell $N \subset M$ will be said to be free in M provided $M = N$ or $N \cap \text{Bd}(M)$ is an $n-1$ cell. If $P \in E^n$ and $M \subset E^n$ then the cone over M from P will denote the union of all intervals Pm , where $m \in M$. Denote this cone by PM . Unless otherwise stated, by interval we mean straight line interval.

A polygonal knot K is a simple closed curve in E^3 which is the union of a finite collection $\{a_1a_2, a_2a_3, \dots, a_na_1\}$ of intervals. The knot K will be said to be in regular position provided the intervals can be selected so that the xy projection πa_i of each a_i is not the xy projection of any other point of K , and no point is the xy projection of more than two points of K . The points that are the projection of two such points will be called the double points or crossing points of the projection.

3. Preliminary lemmas. In this section we prove several lemmas which are necessary for the main theorems in §4.

LEMMA 0. *Suppose D is a polyhedral 2-cell with rectilinear triangulation T . Then, if $n(T_2) > 1$, there exist two elements t_1, t_2 of T_2 such that (1) t_i ($i=1, 2$) is free in D , and (2) if t_i ($i=1, 2$) has a 0-simplex v_i in $\text{Int}(D)$, then $|\text{st}(v_i, T)|$ is free in D , and $v_1 \neq v_2$ unless $T_0 \cap \text{Int}(D) = \{v_1\}$.*

Proof. By Lemma 2 of [10] there exist two elements s, s' of T_2 that are free in D . Suppose that $|\text{st}(v, T)|$ is not free in D for some vertex v of s in $\text{Int}(D)$. Each component C of $D - |\text{st}(v, T)|$ has the property that \bar{C} is a 2-cell and $\bar{C} \cap |\text{st}(v, T)|$ is an arc. Let C_1 and D_1 denote two such components. By Lemma 3 of [10] \bar{C}_1 contains an element s_1 of T_2 that is free in D . If $|\text{st}(v_1, T)|$ is not free in D for some 0-simplex v_1 of s_1 in $\text{Int}(D)$, then select C_2 a component of $D - |\text{st}(v_1, T)|$ not intersecting D_1 . Now there exists $s_2 \in T_2$ so that $s_2 \subset \bar{C}_2$ and is free in D . This process must stop eventually to produce a cell t_1 with the desired property. Analogously, we find $t_2 \subset \bar{D}_1$.

If, at the beginning of the argument all of the conclusion follows but the last part of (2), then Lemma 3 of [10] and the argument above may be applied to the component of $D - |\text{st}(v_1, T)|$.

LEMMA 1. *Suppose that T is a finite rectilinear triangulation of the boundary of the bounded, connected, simply connected open subset D of the plane. Then, there is a triangulation S of \bar{D} such that (1) $T \subset S$, and (2) $n(S_2) \leq a + 2b - 2$, where b is the number of elements w of T_1 such that $\text{seg } w \subset \text{Int}(\bar{D})$ and $a = n(T_1) - b$.*

Proof. The proof is by induction on $n = n(T_1)$, the case $n=3$ being obvious. Suppose for all cases $i=3, \dots, n-1$.

Let v belong to T_0 and the convex hull of \bar{D} and let U denote the union of all intervals vp such that $\text{seg } vp \subset D$ and $p \in \text{Bd}(D)$.

If for no such vp is $p \in T_0$ then a subset of T_1 may be expressed as vp_1, \dots, vp_m where $M = \bigcup_{i=1}^{m-1} (\text{seg } p_i p_{i+1} \cup \text{Int } vp_i p_{i+1}) \subset U$ and $vp_1 \cup vp_m \subset \text{Bd}(\bar{D})$. The desired triangulation is generated by the 2-cells $vp_i p_{i+1}$ ($i=1, \dots, m-1$) together with those cells obtained by applying the induction hypothesis to $D - \bar{M}$ and the obvious triangulation of its boundary.

If for some vp above $p \in T_0$, then $D - vp$ is the union of two mutually separated sets D_1 and D_2 . The induction hypothesis is then applied to D_1 and D_2 and the obvious triangulations of their boundaries.

LEMMA 2. Suppose that J_0, J_1, \dots, J_n ($n \geq 0$) is a collection of mutually exclusive polyhedral subsets of the plane E^2 such that (1) there is a bounded, connected, simply connected, open subset D_0 of E^2 such that $J_0 = \text{Bd}(D_0)$, (2) if $1 \leq p \leq n$, then J_p is a simple closed curve which is a subset of D_0 , and (3) if $0 \leq p \leq n$, then J_p has a triangulation T^p . Then, \bar{D}_0 has a triangulation T such that (1) $\bigcup_{p=0}^n T^p \subset T$ and (2) $n(T_2) \leq a + 2(b + \sum_{p=1}^n n(T_1^p)) - 2$, where b is the number of 1-simplexes w of T^0 such that $\text{seg } w \subset \text{Int}(\bar{D}_0)$ and $a = n(T_1^0) - b$.

Proof. The proof is by induction on n , the case $n=0$ reducing to Lemma 1. Suppose true for all cases $j=0, 1, \dots, n-1$.

The boundary of the convex hull of $\bigcup_{p=1}^n J_p$ may be denoted by $A_1A_2 \cdots A_sA_1$ where each of A_1A_2, \dots, A_sA_1 is an interval and $\{A_1, \dots, A_s\} \subset \bigcup_{p=1}^n T_p^0$.

Consider a typical A_p . Since $\bigcup_{q=1}^n J_q \subset D_0$, there is a point of J_0 in the interior of the angle vertical to angle $A_{p-1}A_pA_{p+1}$, which then implies there is an interval XA_p where $X \in T_0^0$ and $\text{seg } XA_p \subset D_0 - \bigcup_{p=1}^n J_p$.

Suppose, for simplicity of notation, that $A_p \in J_1$ and that J_2, \dots, J_t are the ones of J_0, \dots, J_n that are subsets of $\text{Int } J_1$, and that J_{t+1}, \dots, J_n are subsets of $D_0 - \text{Cl}(\text{Int } J_1)$. It is easily verified that $\{J_0 \cup XA_p \cup J_1, J_{t+1}, \dots, J_n\}$ and $\{J_1, \dots, J_t\}$ together with triangulations $\{T^0 \cup T^1 \cup \{XA_p\}, T^{t+1}, \dots, T^n\}$ and $\{T^1, \dots, T^t\}$, respectively, satisfy the induction hypothesis. The triangulations of $\text{Cl}(D_0 - \text{Int } J_1)$ and $\text{Cl}(\text{Int } J_1)$ thus obtained may be combined to give the desired collections.

LEMMA 3. Let K denote a polygonal knot in regular position in E^3 such that the xy projection πK , of K , has exactly n crossing points. Then, there is a polygonal knot L in regular position such that (1) there is a p.l. homeomorphism h from E^3 onto E^3 such that $h(K) = L$, and h takes lines parallel to the z -axis onto lines of that type, and (2) πL is the union of $\leq 2^{n+2} - 2$ intervals, no two of which have more than a common endpoint common, and L itself is the union of $\leq 2^{n+2} - 2$ intervals, where in this case no endpoint of one of the intervals is projected into a crossing point of πL .

Proof. In case $n=1$, it is straightforward to show that L may be constructed so that it is the union of 4 intervals and πL is the union of 6 intervals. Now suppose the theorem holds for all cases $j=1, \dots, n-1$.

First we note that an oriented knot M whose projection has exactly n double points has the property [14, Theorem 19] that if $a_i b_i$ ($i=1, 2$) are intervals which lie in M and whose projections cross at e , an interior point of each, and which contain no other double point of πM , then (assuming that the orientation of M goes from a_i to b_i ($i=1, 2$)) a new knot M' , whose projection has exactly $n-1$ crossings, can be made from M by picking $c_i, d_i \in a_i b_i$ in the order $a_i c_i e_i d_i b_i$ (where $\pi e_i = e$) and letting $M' = (M - (c_1 d_1 \cup c_2 d_2)) \cup c_1 c_2 \cup d_1 d_2$.

On the other hand, suppose we take a knot projection P with orientation O_1 induced by an orientation of the knot, and consider a complementary domain D of P . D has a boundary $a_1 a_2 \cup a_2 a_3 \cup \dots \cup a_n a_1$ (where each $a_p a_{p+1}$ denotes a straight

line interval and where $\text{Bd}(D) = \text{Bd}(\bar{D})$) (see [14, Theorem 19]) such that we may choose in D close to $\text{Bd}(D)$ a simple closed curve $J = b_1 b_2 \cup \dots \cup b_i b_1$ so that each b_p is close to a_p and each $b_p b_{p+1}$ is a straight line interval. Let J be given an orientation and let that orientation induce a circular orientation O_2 on $\text{Bd}(D)$.

Now suppose that $a_r a_{r+1}$ and $a_s a_{s+1}$ are such that O_1 and O_2 agree on $a_r a_{r+1}$ if and only if they agree on $a_s a_{s+1}$. Noting that r may be s , we choose points $b, c \in \text{seg } a_r a_{r+1}$ and $d, e \in \text{seg } a_s a_{s+1}$ in the respective orders $a_r b c a_{r+1}$, $a_s d e a_{s+1}$; and if $r = s$ we also let c precede d . Let bxe and $cx d$ denote polygonal arcs which (1) lie in D except for their endpoints and (2) contain intervals which cross at x , but where $bxe \cap cx d = \{x\}$. The set $P' = (P - (bc \cup de)) \cup bxe \cup cx d$ is a knot projection having exactly one more crossing point than P . Using [10], [14] it can be shown that every knot projection of n crossings can be made from one of $n - 1$ crossings by the above construction. We now return to the original problem.

By the former construction construct a knot K' from K where $\pi K'$ has exactly $n - 1$ crossing points. By the induction hypothesis there is a homeomorphism f and a knot L' such that conditions (1) and (2) hold relative to K', L', f' and $n - 1$. Thus, $\pi L'$ is the union of $\leq 2^{n+1} - 2$ intervals, no two of which have more than a common endpoint common. The projection P of the required knot L is now constructed from $\pi L'$ from the latter construction described above.

In case $r = s$, then P is constructed from $\pi L'$ by adding a single loop to a side of a complementary domain. Such a P can be formed with no more than $6 + 2^{n+1} - 2$ and thus with no more than $2^{n+2} - 2$ intervals.

In case $r \neq s$, we observe that i is no more than $2^{n+1} - 2$. Also the arcs bxe and $cx d$ can be required to "travel" around $\text{Bd}(D)$ in the "shortest" direction. We find that P can be formed with no more than $2 + 2(2^{n+1} - 2) = 2^{n+2} - 2$ intervals.

The required knot L is then built above P so as to have the correct over and under crossings. The required map h is built by first finding a p.l. homeomorphism h' from E^2 onto E^2 taking πK onto πL , and then h is defined so as to take K onto L in such a way that $h'(\pi x) = \pi h(x)$ for $x \in E^3$.

The proof of Lemma 3 may now be altered slightly to yield

LEMMA 3'. *As in Lemma 3 except that the conclusion is (a) weakened by replacing $2^{n+2} - 2$ by 2^{n+2} , and (b) strengthened by requiring that πL be the union of intervals which are parallel to either the x -axis or the y -axis.*

LEMMA 4. *Suppose that M_i ($i = 1, 2, 3$) is a polyhedral p -cell ($p = 2$ or 3) in E^p which has a rectilinear triangulation T^i . Let $f_i: M_i \rightarrow M_{i+1}$ ($i = 1, 2$) be a p.l. homeomorphism which is affine on each $s \in T^i$. Then, there is a subdivision T of T^1 such that (1) $f_2 f_1$ is affine on each $s \in T$, and (2) $n(T_p) \leq (3 - p + 2^{3p-4})n(T_p^1)n(T_p^2)$.*

Proof for $p = 3$. For $t_1 \in T_3^1$ and $t_2 \in T_3^2$ consider the convex set $t = t_2 \cap f_1(t_1)$, if it has a nonvoid interior. Each flat face w of t is a subset of a face of $f_1(t_1)$ or t_2 and is a 2-cell bounded by q sides for $q \in \{3, 4, 5, 6\}$. For each such w if $q > 3$ we

use Lemma 1 to triangulate w into $q-2$ triangular disks. Now each such t as above is triangulated radially from some $P \in \text{Int}(t)$ using sets of the form $K = Pw'$ where w' is either a w with 3 sides or a 2-cell of the triangulation of w if $q > 3$. Since each t has at most 8 such faces w , we find that the subdivision of T^1 whose 3 simplexes are sets of the form $f_1^{-1}(K)$ satisfies the conclusion of the lemma.

LEMMA 5. In E^p ($p=2$ or 3) suppose M is a polyhedral p -cell with triangulation T and that $C \neq M$ is in T_p and free in M . Then, there is a subdivision T' of T such that (1) $n(T'_p) \leq (11p-18)n(T_p)$ and (2) there is a simplicial isotopy $h: M \times [0, 1] \rightarrow M$ such that (a) h_t is affine on each $s \in T'$ for each $t \in [0, 1]$, (b) $h_t(x) = x$ if $t=0$ or $x \in \text{Bd}(M) - C$, and (c) $h_1(M) = \text{Cl}(M - C)$.

Proof for $p=3$. We follow Sanderson in Theorem 5 of [10]. Let n be the number of 2-cell faces of C lying in $\text{Bd}(M)$. There is exactly one $n-1$ simplex P of C not lying in $C \cap \text{Bd}(M)$, and a unique $3-n$ simplex Q which is not in $\text{Cl}(M - C)$, and is opposite P in the sense that it is the only $3-n$ simplex of C not containing P .

Let p_1 be the centroid of P and q_1 the centroid of Q . Let p_2 be a point on the extension of p_1q_1 through p_1 , close enough to p_1 so that $K_p = p_2(C \cap \text{Bd}(M)) \subset \text{Int}[\text{st}(P, T)] \cup (C \cap \text{Bd}(M))$. Let q_2 be a point on the extension of p_1q_1 through q_1 , close enough to q_1 such that $K_q = q_2(C \cap \text{Bd}(M)) \subset (E^3 - M) \cup (C \cap \text{Bd}(M))$. Still following Sanderson, we define $\bar{h}(q_1, t)$ to be the point of p_1q_1 a distance of $td(p_1q_1)$ from q_1 . For each t we let \bar{h} be fixed outside $K_p \cup K_q$ and let \bar{h} map every interval joining a point x of $M \cap \text{Bd}(K'_p \cup K'_q)$ to q_1 linearly onto the interval joining x to $\bar{h}(q_1, t)$. Define $h = \bar{h}|_M$.

Let C be denoted by $abcd$. We show how to obtain the triangulation T^1 only in the most difficult case, namely $Q = abc$ and $p_1 = P = d$. We subdivide C into solid tetrahedra abp_1q_1 , acp_1q_1 and $bc p_1q_1$, and do not subdivide any element of T which does not have p_1 as a vertex. If $D \in T_3$ has exactly a 2-simplex common with C , we subdivide D into two tetrahedra, $D \cap K_p$ and $\text{Cl}(D - K_p)$.

Now suppose $D \in T_3$ where $D \cap C \in T_0 \cup T_1$ and $p_1 \in D$. Then, there is a minimal triangulation T^D of $\text{Bd}(D) \cup (D \cap \text{Bd}(K_p))$ such that (1) if f is a 2-cell face of D having no interior point common with K_p then $f \in T^D$, (2) every 2-simplex in T^D lying in $\text{Bd}(D) \cap K_p$ has p_1 as a vertex. D is then subdivided into sets of the form xw where (a) $x = p_1$ and $w \in T_2^D$ lying in K_p but where $p_1 \notin w$, or (b) x is a fixed point of $\text{Int}(D - K_p)$ and $w \in T_2^D$ and $w \subset \text{Bd}(\text{Int}(D - K_p))$. (There is a fixed x in $\text{Int}(D - K_p)$ so that if y, z are two points of $\text{Bd}(\text{Int}(D - K_p))$ then xy and xz are different rays from x .) Remembering that in forming the T_2^D 's a face f must be subdivided the same way from both sides, T^1 is generated by the 3-simplexes described above.

In the worst possible case (which occurs when $p_2 \in \text{Int}(D)$) D is subdivided into ≤ 25 tetrahedra. In all other cases D is subdivided into no more than 15 tetrahedra and if $D \cap C$ is a 2-cell, D is subdivided into 2 tetrahedra. Thus, the number 15 in the conclusion of the lemma.

LEMMA 6. In E^p ($p=2$ or 3) suppose that M is a polyhedral p -cell with triangulation T and that $C \neq M$ is a p -simplex of T that is free in M . Let

$$S = \{t : t \in T \text{ and } t \subset \text{Cl}(M - C)\}.$$

Then, there is a subdivision S' of S such that (1) $n(S'_p) \leq (11p - 18)n(T_p)$ and (2) there is a simplicial isotopy $g: \text{Cl}(M - C) \times [0, 1] \rightarrow M$ such that (a) g_t is affine on each $s \in S'$ for each $t \in [0, 1]$, (b) $g_t(x) = x$ if $t=0$ or $x \in \text{Bd}(M) - C$, and (c) $g_1(\text{Cl}(M - C)) = M$.

Proof for $p=3$. Let h and T' be as in Lemma 5 and define $S' = \{t : t \in T' \text{ and } t \subset \text{Cl}(M - C)\}$. Define g by $g(x, t) = h(x, 1 - t)$ for $x \in \text{Cl}(M - C)$ and $t \in [0, 1]$.

DEFINITION. Given two points x and y of the interior of the compact convex set C , we define the S -isotopy h determined by (x, y, C) so that (1) h is first defined by $h_t(x) = (1 - t)x + ty$ for $t \in [0, 1]$, (2) if $P \in \text{Bd}(C)$ then Px is mapped linearly onto $Ph_t(x)$ for $t \in [0, 1]$ and (3) $h_t(P) = P$ for any element of domain h outside C .

LEMMA 7. Suppose P is a compact polyhedral 3-manifold with boundary in E^3 and S is a rectilinear triangulation of $\text{Bd}(P)$. Then, there is a triangulation T of P such that $n(T_3) \leq (j - 3)j2^j$, where $j = n(S_2)$.

Proof. Let W denote the set of all planes which contain an $s \in S_2$ and let g denote a component of $P - (UW)$. Then \bar{g} is a convex 3-cell having $\leq j$ flat faces, each of which is bounded by a simple closed curve which is the union of $\leq j - 1$ straight line intervals. Each flat face is then triangulated (using Lemma 1) into no more than $(j - 1) - 2$ 2-cells, and then \bar{g} is radially subdivided from some $x \in g$ into no more than $j(j - 3)$ solid tetrahedra. Since there are at most 2^j g 's the bound above is evident.

LEMMA 8. Suppose (1) P , S and j are as in Lemma 7, (2) the triangular disk $abc \subset \text{Int}(P)$, (3) K is a subset of P so that \bar{K} is a polyhedron and $\bar{K} \cap abc \subset \{a, c\}$, and (4) y is the midpoint of ac . Then, there is a triangulation W of P and a simplicial isotopy $g: P \times [0, 1] \rightarrow P$ such that (1) $g_t(x) = x$ if $t=0$ or $x \in \{a, b\} \cup K \cup \text{Bd}(P)$, (2) $g_1(b) = y$ (or $g_1(y) = b$), (3) $n(W_3) \leq (j + 8)(j + 5)2^{j+8} + 12 + 8(3 + (3j)2^{3j})$, and (4) g_t is affine on each $w \in W$ for each $t \in [0, 1]$.

Proof. The method of proof is a combination of ideas in Lemma 2 of [1] and Lemma 5. Let axc and azc be triangular disks such that x, z lie on line yb in the order $xybz$, and $axc \cup acz \subset (\text{Int}(P) - \bar{K}) \cup \{a, c\}$. Let $q_1p_1yp_2q_2$ be a short straight line interval which pierces disk $axc \cup acz$ at y , and has the property that the union Q of the tetrahedra q_1acx, q_1acz ($i=1, 2$) is a subset of $(\text{Int}(P) - \bar{K}) \cup \{a, c\}$.

The map g is defined as in Lemma 5 on y or b , depending on which is to be moved to which, and then extended linearly on intervals joining y (or b) to $\text{Bd}(R)$, where R is the union of $axcp_i, aczp_i$ ($i=1, 2$). Elsewhere, let $g(p, t) = p$. The elements of W_3 are now described.

- (1) Let $p_iaxy, p icy, p_iayb, p icyb, p_iabz, p icbz$ ($i = 1, 2$) be elements of W_3 .
- (2) The methods of Lemma 7 are used to triangulate $Cl(P - Q)$, where the 2-simplexes in the triangulation of the boundary either belong to S or are a face of one of q_iacx, q_iacz ($i = 1, 2$).
- (3) The tetrahedra $p_iq_iax, p_iq_ixc, p_iq_iaz, p_iq_icz$ ($i = 1, 2$) are triangulated the same way. For, consider p_1q_1ax . In the triangulation of $Cl(P - Q)$, the face q_1ax has been subdivided into $\leq (3j)2^j$ triangular disks, so p_1q_1ax is radially subdivided from some interior point t into $\leq 3 + (3j)2^j$ tetrahedra, including tp_1ax, tp_1qx and tp_1qa .

DEFINITION. Let the bound given in part (3) of the conclusion of Lemma 8 be denoted by $G(j)$.

4. The proofs of theorems.

THEOREM 1. Suppose Δ is a solid tetrahedron in E^3 , D_i ($i = 1, 2$) is a polyhedral disk in $Int(\Delta)$ with triangulation T^i , and $h: \Delta \rightarrow \Delta$ is an onto p.l. homeomorphism such that (a) $h(x) = x$ if $x \in Bd(\Delta)$, and (b) $h(D_1) = D_2$. Then, there is a simplicial isotopy $\phi: \Delta \times [0, 1] \rightarrow \Delta$ such that (1) $\phi_t(x) = x$ if $t = 0$ or $x \in Bd(\Delta)$, (2) $\phi_1(D_1) = D_2$, and (3) there is a triangulation T of Δ such that (a) ϕ_t is linear on each $s \in T$ for each $t \in [0, 1]$ and (b) $n(T_3) \leq (32G(4))^{8 + 2N_1 + 2N_2}$, where $N_i = 28(n(T_2^i) + 1)(16n(T_2^i) + 12)$, $i = 1, 2$.

Proof. Let w_1, \dots, w_{n_1} denote a shelling order [10] of D_1 , where each $w_p \in T_2^1$. Let $w_1 = acb$. We consider two cases.

Case 1. $abc \cap Cl(D_1 - abc) = bc$. Let $dbc \in T_2^1 - \{w_1\}$. (i) If a, b, c and d are coplanar, let $g: \Delta \times [0, 1] \rightarrow \Delta$ denote an S -isotopy such that (1) $g(x, t) = x$ if $t = 0$ or $x \in Bd(\Delta) \cup Cl(D_1 - (abc \cup bcd)) \cup \{b, c\}$, (2) $g_1(D_1) = Cl(D_1 - abc)$, and (3) there is a triangulation T^g of Δ so that (a) g_t is affine on each $s \in T^g$ for each $t \in [0, 1]$, and (b) $n(T_3^g) \leq G(4)$. (ii) Suppose a, b, c and d are not coplanar. Let $e = \frac{1}{2}(b + c)$ and let $x \in line\ de$ such that (a) $x \in Int(\Delta) - D_1$ and (b) $bxc - acb \subset Int(\Delta) - D_1$. We use the "composition" of two S -isotopies g, g' as above, where g moves acb onto acx and is fixed on $D_1 \cup Bd(\Delta)$ and where g' moves xbc into bcd analogous to g in (i) above. We also assume the bound $G(4)$ for each of g, g' .

Case 2. $abc \cap Cl(D_1 - abc) = ac \cup bc$. Utilizing the proof of Lemma 2 of [1], let J denote the simple closed curve $Bd|st(c, T_1)|$ and let J' denote the simple closed curve which is the projection of J on Δ from c . Since J is the union of no more than $n_1 = n(T_2^1)$ straight line intervals, then J' is the union of no more than $3n_1$ intervals each of whose endpoints are (a) the projection on Δ of a point x where $cx \in T_1^1$ or (b) on a 1-simplex of Δ . The simple closed curve $W = (plane\ abc) \cap \Delta$ is the union of no more than 4 intervals having their endpoints on the 1-skeleton of Δ . By Lemma 1 there is a triangulation R of $Bd(\Delta)$ such that (1) $W \cup J' \subset |R_1|$, (2) $n(R_1) \leq 28(n_1 + 1)$ and (3) $n(R_2) \leq 16n_1 + 12$.

Let $a'(b')$ denote the projection of $a(b)$ from c on Δ , let $A_1 = a'x_1b'$ denote the arc $\text{Cl}(J' - a'b')$, and let $a'x_1b' = B_1 = \text{Cl}(W - a'b')$, where we assume $a'b'$ is the projection on Δ from c of interval ab . There is a sequence of arcs $A_1, \dots, A_{N_1} = B_1$ such that (1) $A_p \subset |R_1| - \text{seg } a'b', p = 1, \dots, N_1$, (2) if $1 \leq p < N_1$, then there is an element s_p of R_2 so that $A_p - s_p = A_{p+1} - s_p$ and $\text{Bd}(s_p) \subset A_p \cup A_{p+1}$, and (3) $N_1 \leq 28(n_1 + 1)(16n_1 + 12)$.

Now suppose, for example, that $s_1 = xyz$ where $A_1 \cap s_1 = xy \cup yz$ and $A_2 \cap s_1 = xz$. Let $s = \frac{1}{2}(x + z)$ and let x', y', z', r', s', t' denote points so that (1) x', y', z' are elements of J whose projections from c are x, y, z , respectively, (2) s' is on $x'z'$ and the projection of s' from c on Δ is s , (3) r' and t' lie on line $y's'$ in the order $t's'y'r'$, and (4) the union K of the tetrahedra $cx'r'z', cx'z't', sx'r'z',$ and $sx'z't'$ is a subset of $\{s\} \cup \text{Int}(\Delta)$ and intersects $|\text{st}(a, T_1)|$ in the union of two triangular disks $cx'y'$ and $cy'z'$. Let g^1 be the S -isotopy determined by (y', s', K) .

Analogously, S -isotopies g^2, \dots, g^N are defined so that their "composition" together with g^0 moves $|\text{st}(c, T_1)|$ to a planar set while keeping abc fixed. Then an S -isotopy h is defined so that (1) $h_t(x) = x$ if $t = 0$ or $x \in g_1^N \cdots g_1^1[\text{Cl}(D_1 - |\text{st}(c, T_1)|)] \cup \text{Bd}(\Delta)$, (2) $h_1 g_1^N \cdots g_1^1(D_1) = \text{Cl}(g_1^N \cdots g_1^1(D_1 - abc))$. Finally, we let $v^1 = (g^1)^{-1} \cdots (g^N)^{-1} h g^N \cdots g^1$, which concludes Case 2.

Using Case 1 or 2 as needed for w_2, \dots, w_{n_1-1} we construct simplicial isotopies V^2, \dots, V^{n_1-1} such that (a) $V_1^{n_1-1} \cdots V_1^1(D_1) = w_{n_1}$ and (2) each V^p is a composition of no more than $2N_1 + 1$ S -isotopies, where the bound $G(4)$ applies to each factor.

Let u_1, \dots, u_{n_2} ($n_2 = n(T_2^2)$) denote a shelling order for D_2 , where each $w_p \in T_2^2$. As above there is a sequence of simplicial isotopies m^1, \dots, m^{n_2-1} whose composition takes D_2 onto u_{n_2} and each m^p is a composition of no more than $2N_2 + 1$ S -isotopies. Also, w_{n_1} can be moved onto u_{n_2} by an isotopy q which is the composition of no more than 6 S -isotopies, where the bound $G(4)$ applies to each factor.

The required map ϕ is defined by $(m^1)^{-1} \cdots (m^{n_2-1})^{-1} q V^{n_1-1} \cdots V^1$ and is the composition of no more than $6 + 2N_1 + 1 + 2N_2 + 1$ S -isotopies. Applications of Lemmas 4 and 8 finish the proof.

THEOREM 2. *Suppose that (1) K and L are polygonal knots in regular position in E^3 such that the xy projections πK and πL lie on different sides of some line in the xy plane, and have M and N crossing points, respectively, and (2) G_K and G_L are collections of straight line intervals such that (a) $\pi K = \bigcup G_K$ and $\pi L = \bigcup G_L$, (b) no two intervals of $G_K \cup G_L$ have more than an endpoint common, and (c) $n(G_K) \leq 2^{M+2} - 2$ and $n(G_L) \leq 2^{N+2} - 2$. Then, there is a solid tetrahedron $ABCD$ with triangulation T such that (1) $K \cup L \subset |T_1| \cap \text{Int}(ABCD)$, (2) $n(T_3) \leq 64(2^{M+2} + 2^{N+2}) - 8(M + N) + 96$, and (3) if C is a 3-cell which is the union of a nondegenerate subcollection G of T_3 , then there exist two elements of G that are free in C .*

Proof. Suppose each point in $K \cup L$ has a positive z coordinate. Let A, B, C, F denote points in the xy plane such that (1) line CF separates πK from πL , (2)

$F \in \text{seg } AB$, and (3) $\pi K \subset \text{Int}(ACF)$ and $\pi L \subset \text{Int}(BCF)$. Let D denote a point so high above the barycenter of ABC that the projection $P(K \cup L)$ from D through $K \cup L$ into the xy plane has the properties (1) $P(K) \subset \text{Int}(ACF)$ and $P(L) \subset \text{Int}(BCF)$, and (2) there is a p.l. homeomorphism h from disk ABC onto ABC such that (a) $h(x) = x$ if $x \in CF \cup \text{Bd}(ABC)$, (b) $h(\pi K) = P(K)$ and $h(\pi L) = P(L)$ and (c) if $g \in G_K \cup G_L$ then $h(g)$ is a straight line interval. There exist points H and E and intervals $g_1 \in G_K, g_2 \in G_L$ such that H is an endpoint of $h(g_1)$, E is an endpoint of $h(g_2)$, and intervals CH, CE, HF and EF have the property that $P(K) - H$ is a subset of the interior of the simple closed curve $AFHCA$ and $P(L) - E \subset \text{Int}(FBCEF)$. Note also that a knot projection of j crossings has exactly $j + 2$ complementary domains, and that no arc lying in the projection and not containing a crossing point is a subset of the boundaries of more than two complementary domains. Let

$$S = \{h(g) : g \in G_K \cup G_L\} \cup \{AF, FB, BC, CA, HC, HF, FE, EC, CF\}.$$

Consider a typical complementary domain U of $ABC - |S|$. $\text{Bd}(U)$ may be expressed as the union of straight line intervals $a_1 a_2, \dots, a_n a_1$ where (1) each $a_p a_{p+1} \in S$ and (2) there is a polygonal simple closed curve $b_1 b_2 \dots b_1 b_n$ in U so that (a) each b_p is close to a_p , and (b) each $b_p b_{p+1}$ is a straight line interval. Each quadrilateral disk $a_p b_p a_{p+1} b_{p+1}$ is triangulated as follows: Choose $c_p, e_p, f_p \in \text{seg } a_p a_{p+1}$ in the order $a_p c_p e_p f_p a_{p+1}$, choose $d_p \in \text{seg } b_p b_{p+1}$, and subdivide the quadrilateral disk into $a_p c_p b_p, c_p b_p d_p, c_p e_p d_p, e_p f_p d_p, f_p d_p b_{p+1}$ and $f_p a_{p+1} b_{p+1}$.

In a given quadrilateral we have used at most six 2-simplexes. Also, remembering the addition of the d_p 's, the disk bounded by $b_1 b_2 \dots b_n b_1$ may be triangulated with no more than $2n - 2$ 2-simplexes. Thus, a typical \bar{U} with $\text{Bd}(U) = a_1 a_2 \dots a_n a_1$ as above is triangulated into $\leq 6n + 2n - 2$ 2-simplexes. Then, if W is the triangulation of ABC with these 2-simplexes, then $n(W_2) \leq 16(2^M + 2) + 2^{N+2} - 2(M + N) + 24$. Note that if an $a_p a_{p+1}$ in $\text{Int}(ABC)$ is triangulated one way from one side it must be triangulated the same way from the other side.

We now proceed to construct the collection T . Given $w \in W_2$, if Dw does not intersect $K \cup L$, let $Dw \in T_3$. If Dw intersects $K \cup L$ there are three cases to consider. Let $w = abc$.

Case 1. $(K \cup L) \cap Dw$ is a single point d in the interior of a 1-simplex, say aD . We subdivide Dw into solid tetrahedra $Ddbc$ and $dabc$.

Case 2. $(K \cup L) \cap Dw$ is a straight line interval de where $d \in \text{seg } aD, e \in \text{seg } Dc$. Subdivide Dw into $bdeD, bdec$, and $abcd$, provided c corresponds to the role of c_p or f_p above.

Case 3. $(K \cup L) \cap Dw$ is the union of a straight line interval de and a point g , where $g, d \in \text{seg } aD$ and $e \in \text{seg } Dc$.

- (a) If $g \in \text{seg } Dd$, subdivide Dw into $bgeD, bdeg, bdec$, and $abcd$.
- (b) If $g \in \text{seg } da$, subdivide Dw into $bdeD, decb, abcg$, and $dcgb$.

We now let T denote the triangulation of $ABCD$ whose 3-simplexes are of the form Dw where $Dw \subset ABCD - (K \cup L)$ or a subset of a Dw as given in Cases 1, 2, 3. Since no Dw is subdivided into more than 4 tetrahedra, then $n(T_3) \leq 4n(W_2)$, the stated bound.

Now let $C = \bigcup G$ where G is a nondegenerate subset of T_3 and C is a 3-cell, and let $R = \{w \mid w \in W_2 \text{ and } Dw \text{ contains an element of } G\}$. Let $V = |R|$, and for each $w \in R$ let $G_w = \{g : g \in G \text{ and } g \subset Dw\}$. If V_1 denotes the set of all 1-simplexes t of elements of R where $t \subset \text{Int}(V)$ and t does not contain the P projection (from D) of a point of $\text{Int}(C)$, then we find that $V - |V_1|$ is connected and simply connected. So, if we imagine that V is "split open" along the simplexes in V_1 , then the new R (say R_s) yields a cellular subdivision of the new V (say V_s). By Lemma 0 either R_s is degenerate or there are two 2-cells w, w' of R_s that are free in V_s and have the following additional property:

(P₁) If v is a 0-simplex of w (or w') and $v \in \text{Int}(V_s)$, then $|\text{st}(v, R_s)|$ is free in V , and if v, v' are 0-simplexes of w, w' , respectively, that lie in $\text{Int}(V_s)$ and $W_0 \cap \text{Int}(V_s)$ is nondegenerate, then $v \neq v'$.

If R_s is degenerate, then it is easy to see that the various cases yield two elements of F , the set of all elements of G free in C . We therefore suppose that w, w' are two elements of R_s free in V , where property (P₁) holds, and that we are also proceeding by induction on the number of elements in G , supposing the desired result to hold for all such G 's with fewer elements. The basic idea of the proof is to show outright there are two elements of F or to show there exists $f \in F$ having a certain property relative to w , i.e. either $f \subset w$, or if $v \neq v'$ as in (P₁), then f has a 0 or 1-simplex on vD and would not have the same property relative to $v'D$.

If w has no vertex in $\text{Int}(V_s)$, it is straightforward to show in all the cases that (1) if Dw contains at least two elements of G , then Dw contains two elements of F , and (2) if Dw contains only one element g of G then $g \in F$, and the induction hypothesis may be applied to $\text{Cl}(C - g)$ to produce another element of F . We thus now assume that w and w' have 0-simplexes v and v' , respectively, in $\text{Int}(V_s)$, and that property (P₁) holds.

Now suppose that $g \in G$ and g has two 2-cell faces f_1, f_2 such that $f_1 \cup f_2 \subset \text{Bd}(C)$. (a) If $g \cap \text{Bd}(C) = f_1 \cup f_2$ or $f_1 \cup f_2 \cup f_3$, where f_3 is a third 2-cell face of g , then $g \in F$. (b) The other possible case is that $g \cap \text{Bd}(C) = f_1 \cup f_2 \cup s$, where s is a 1-simplex of g , and $g \notin F$. There is then a third face f_3 of g such that $C - f_3$ is the union of two mutually separated sets C_1, C_2 , and the induction hypothesis may be applied to \bar{C}_i to produce a free cell g_i ($i = 1, 2$) such that g_i does not have f_3 as a face unless $g_i = \bar{C}_i$. Then $g_1, g_2 \in F$. Hereafter, we shall assume

(S₁) In the above circumstances case (a) always holds.

Remembering that V has been split along certain 1-simplexes to form V_s , we show how to handle the argument in the more difficult cases. We suppose that $w = abc$ has only one 1-simplex lying in $\text{Bd}(V_s)$, and where Dw is subdivided as in Case 3(a) above. Using the notation of 3(a) let A_1, \dots, A_{10} denote the 2-simplexes $bgD, bgd, bda, Dge, gde, dec, adc, beD, bec$, and abc , respectively.

Case (F1). $ac = abc \cap \text{Bd}(V_s)$. In this case $|\text{st}(bd, T)| \subset C$ so $bgeD \in G_w$. Since every element of G_w has a 2-simplex on acD , then some element of G_w has two 2-simplexes lying in $\text{Bd}(C)$. By assumption (S_1) , G_w contains an element of F .

Case (F2). $ab = abc \cap \text{Bd}(V_s)$. Either (a) $|\text{st}(De, T)| \subset C$ or (b) $|\text{st}(ec, T)| \subset C$. In either case the only way for (S_1) to fail to yield an $f \in F \cap G_w$ is for

$$G_w = \{bgdD, bged, bdec\}$$

and for $\bigcup_{i \in J} \text{Int}(A_i) \subset \text{Int}(C)$, where $J = \{4, 5, 6, 8, 9\}$. There exist triangles abc , bcr , cfr , cfu , cut , and atc in W_2 such that in forming T (1) $Dabc$ is as above, (2) $Dbcr$ is subdivided into $bcre$ and $berD$, (3) $Drcf$ is subdivided into $Dres$, $refe$, and $rfes$, (4) $Ducf$ is subdivided into $Desu$, $ucfe$, and $uefs$, (5) $Dcut$ is subdivided into $teuD$ and $tuce$ and (6) $Dtac$ is subdivided into $tgeD$, $dtge$, $tdec$, and $tacd$.

Suppose (a) holds. Then $bgeD$, $berD$, $Ders$, $Desu$, $Detu$, and $Degt$ are also in G . Also, by the conditions on $\bigcup_{i \in J} A_i$ above, $becr$, $tged$ and $tdec$ are in G .

At this stage we consider the two cases:

(i) c is the only 0-simplex of W_0 in $\text{Int}(V_s)$ (and thus $\bigcup \{ab, at, tu, uf, fr, br\} = \text{Bd}(V_s)$).

(ii) There are at least two elements of W_0 in $\text{Int}(V_s)$.

Suppose (i) holds. If $recf \notin G$ then $becr$ and one of $resf$ and $Desr$ are in F . So suppose $recf \in G$. If $ucfe \notin G$, then $rcfe$ and one of $uesf$ and $uesD$ are in F . Suppose $ucfe \in G$. If $uesf \notin G$, then $ucfe$ and $uesD$ are in F , so suppose $uesf \in G$. If $tuce \notin G$, then $uefc$ and $teuD$ are in F , so suppose $tuce \in G$. At this stage we have $|\text{st}(e, T)| \subset C$, so $bgeD$ is in F and is a subset of Dw .

Now suppose (ii) holds. If $recf \in G$, then $cebr \in F$, where $cebr \cap cD = ce$, so suppose $recf \in G$. If $esfr \notin G$, then $ecfr \in F$, so suppose $esfr \in G$. If $ecfu \notin G$, then $ecfr \in F$, so suppose $ecfu \in G$. If $esfu \notin G$, then $ecfu \in F$, so suppose $esfu \in G$. If $tuce \notin G$, then $ecfu \in F$, so suppose $tuce \in G$. As above we have $|\text{st}(e, T)| \subset C$, so $bgeD \in F$.

We now omit Case (F2(b)) and proceed to

Case (F3). $abc \cap \text{Bd}(V_s) = bc$. In this case either (a) $|\text{st}(gD, T)| \subset C$, or (b) $|\text{st}(gd, T)| \subset C$, or (c) $|\text{st}(ad, T)| \subset C$.

Assuming, as explained above, that $a \in \text{Int}(V_s)$, we label the 3-simplexes of T that contain a point of aD . $Dabc$ is subdivided as above, and in each of the seven lines below the left-hand tetrahedron is the union of the four on the right.

$abpD$; $abpd, pdgb, ygbp, ygDb$

$qapD$; $apqd, qpdg, qpgy, qgyD$

$arqD$; $arqd, rdqs, sdgq, sDgq$

$tarD$; $tard, rsdt, sdgt, sgDt$

$tavD$; $tavd, tgdv, tguw, tguD$

$vamD$; $vamd, vdmg, uvgm, ugmD$

$macD$; $macd, mdec, mdeg, mgeD$

In Case (F3) we show how to handle only case (b) above, all critical ideas of the others being represented here. Recalling assumption (S₁) above, we find that Dw contains an element of G obviously in F or else (i) $G_w = \{bged, bgeD\}$ or (ii) $G_w = \{t \in T_3 : t \subset Dw\}$ or (iii) $G_w = \{bged, bdec, abcd\}$. In each of these three cases, if $t \in T_3$ and $t \cap t' = s$ for $t' \in G_w$ and s a 2-simplex subset of $Dab \cup Dac$, then t is assumed to be in G . Otherwise, there is an element of F in Dw .

Suppose case (i) of case (b) now holds. Suppose also $\{a\} = W_0 \cap \text{Int}(V_s)$. If $ygpb \notin G$, then $ygDb$ and one of $pgdb$ and $abpd$ are in F , so suppose $ygpb \in G$. Also, since at least two elements of G are in $\text{st}(gD, T)$ we assume $|\text{st}(gD, T)| \subset C$; otherwise we can find two elements of F . Now, we also assume that every element T of T_3 lying directly below $t' \in \text{st}(gD, T) \cap G$ and having a 2-simplex common with t' is in G . For suppose, for example, that $gypq \notin G$. Then, $gyqD \in F$ and an element of $\{qdgp\} \cup \text{st}(ad, T)$ is in F . We now have the case $|\text{st}(g, T)| \subset C$, so $gebD \in F$.

Still supposing that $\{a\} = W_0 \cap \text{Int}(V_s)$, we handle cases (ii) and (iii) above. Since there are at least two 3-simplexes of $\text{st}(ad, T)$ in G we suppose they all are, for otherwise we obtain two elements of F . If $rsdq \notin G$ then $raqd$ and one of $rdat$ and $rdst$ are in F . So suppose that $rsdq \in G$ and, analogously, that $rsdt \in G$. We now have $|\text{st}(d, T)| \subset C$, so $abcd \in F$.

We now handle case (b)(i) in the case that $W_0 \cap \text{Int}(V_s)$ is nondegenerate. Since $abcd \notin G$, we may as well assume no element of $\text{st}(ad, T)$ is in G , for if one is, then one is in F and has a 1-simplex common with aD . If $pgyb \notin G$, then $pdbg \in F$ and has a 1-simplex common with aD . So suppose $pgyb \in G$, and, analogously, that $qgyp, tguw, uvgm \in G$. If $qgyD \notin G$, then $q \in \text{Bd}(V_s)$ so (1) $\bigcup \{bc, bp, qp\} \subset \text{Bd}(V_s)$ or (2) $\bigcup \{rq, rt, tv, vm, mc, bc\} \subset \text{Bd}(V_s)$. But case (1) implies $gyDb \in F$ and (2) implies one of $sdgq, rsdq$, and $sgDq \in F$. Thus we assume $qgyD \in G$ and, likewise, that $ugmD, tguD \in G$. If $sgDq \notin G$ then $q \in \text{Bd}(V_s)$, and as above we find an element of F . So suppose $sgDq$ and likewise $sgDt \in G$. We now have the case $|\text{st}(g, T)| \subset C$, so $bgde \in F$.

Consider now cases (b)(ii),(iii) where $W_0 \cap \text{Int}(V_s)$ is nondegenerate. Since $abcd \in G$, we may as well assume $|\text{st}(ad, T)| \subset C$. If $rsdq \notin G$, then $raqd \in F$. So suppose that $rsdq$ and, likewise, $rsdt \in G$. We now have $|\text{st}(d, T)| \subset C$, so $abcd \in F$. This completes the proof for Case (F3(b)). Since the arguments given are typical of, and as difficult as, the other cases in Theorem 2, this completes the proof given for Theorem 2.

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