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WHITEHEAD PRODUCTS AS IMAGES OF PONTRJAGIN PRODUCTS

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Abstract. A method is given for computing higher order Whitehead products in the homotopy groups of a space X. If X can be embedded in an H-space E such that the pair (E, X) has sufficiently high connectivity, then we prove that a higher order Whitehead product element in the homotopy of X is the homomorphic image of a Pontrjagin product in the homology of E. The two main applications determine a higher order Whitehead product element in (1) $\pi_*(BU_t)$, the homotopy groups of the classifying space of the unitary group U_t, (2) the homotopy groups of a space with two nonvanishing homotopy groups.

1. **Introduction.** Suppose that one is interested in determining the Whitehead product $[\alpha, \beta] \in \pi_{p+q-1}(X)$ of $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$. If the space X can be embedded in an H-space E such that the pair (E, X) is $(p+q-1)$-connected, then we prove that the Whitehead product is the homomorphic image of a Pontrjagin product in $H_*(E)$. More precisely, there is a homomorphism

$$\theta\colon H_{p+q}(E) \to \pi_{p+q-1}(X)$$

such that $[\alpha, \beta] = \theta(h\alpha * h\beta)$, where $h\colon \pi_*(E) \to H_*(E)$ is the Hurewicz homomorphism, "$*$" denotes Pontrjagin product in $H_*(E)$ and $\alpha \in \pi_p(X) \equiv \pi_p(E)$, $\beta \in \pi_q(X) \equiv \pi_q(E)$. Thus, under the hypotheses, there is a homological method for computing the Whitehead product $[\alpha, \beta]$ which depends on a knowledge of (1) the effect of h on α and β, (2) the Pontrjagin product of $h\alpha$ and $h\beta$, (3) the homomorphism θ.

The method described in the preceding paragraph is actually generalized in two ways. First, we consider kth order Whitehead products instead of ordinary Whitehead products ($k=2$); the homological approach works equally well for these higher order homotopy operations. Secondly, we require that there exists a pair (E, A) with A operating on E instead of an H-space E. Our main theorem then yields for ordinary Whitehead products both the result of the first paragraph and a theorem of Meyer on Whitehead products and Postnikov systems. We next use the theorem to calculate some higher order Whitehead products. First we determine certain kth order Whitehead product elements in BU_t, the classifying space of the

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unitary group U_t, by embedding BU_t in the H-space BU_∞ and utilizing known results on $H_*(BU_\infty)$. For $k=2$ this gives a new proof of Bott's theorem on the Samelson product $\pi_{2r+1}(U_t) \otimes \pi_{2s+1}(U_t) \to \pi_{2t}(U_t)$, $t=r+s+1$. Secondly, we compute kth order Whitehead products in a space X with two nonvanishing homotopy groups, $\pi_n(X)=\pi$ and $\pi_{kn-1}(X)=G$. This is achieved by embedding X in the Eilenberg-Mac Lane H-space $K(\pi, n)$ and using known information on $H_*(\pi, n)$. As a corollary of this we retrieve some of Porter's results on kth order Whitehead products and, for $k=2$, some results due to Meyer and Stein. As a final application we embed in $(n-1)$-connected space X in $\Omega\Sigma X$. We thus obtain a new proof of the classical assertion that the Hopf invariant

$$H: \pi_{2n+1}(\Sigma X) \to \pi_{2n}(\Omega\Sigma X, X)$$

is an epimorphism if and only if the Whitehead product $\pi_n(X) \otimes \pi_n(X) \to \pi_{2n-1}(X)$ is trivial.

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Many of the results of this paper were announced without proof in [1].

2. **Preliminaries.** We shall assume that all spaces under consideration are 1-connected pointed spaces having the homotopy type of CW-complexes. Maps and homotopies are to preserve base points. The following notation shall be used throughout this paper: h for the Hurewicz homomorphism, "$\simeq$" for the relation of homotopy between maps, $f_\#$ for the homomorphism of homotopy groups and f_* for the homomorphism of homology groups induced by a map f, and ∂ for the boundary homomorphism in the exact homotopy sequence of a pair.

We now recall the definition of a kth order Whitehead product (see [10]). Let $\alpha_r \in \pi_{n_r}(X)$, $r=1, \ldots, k$, be k elements in the homotopy groups of X. We denote the cartesian product $S^{n_1} \times \cdots \times S^{n_k}$ by P and consider the following two subspaces of P: (1) the *lean wedge* $V=S^{n_1} \vee \cdots \vee S^{n_k}$ consisting of k-tuples with at least $k-1$ coordinates at the base point, (2) the *fat wedge* $T=T(S^{n_1}, \ldots, S^{n_k})$ consisting of k-tuples with at least one coordinate at the base point. Representatives of the elements α_r determine a map $g: V \to X$. We let $N=n_1+n_2+\cdots+n_k$ and choose a generator λ in $H_N(P, T) \approx \mathbf{Z}$ (see the proof of Theorem 3.3). If $\hat{g}: T \to X$ is an extension of g, then $\hat{g}_\# \partial h^{-1}(\lambda)$ is in $\pi_{N-1}(X)$:

$$H_N(P, T) \xleftarrow[\approx]{h} \pi_N(P, T) \xrightarrow{\partial} \pi_{N-1}(T) \xrightarrow{\hat{g}_\#} \pi_{N-1}(X).$$

DEFINITION 2.1. The kth order Whitehead product $[\alpha_1, \ldots, \alpha_k]$ of the elements $\alpha_r \in \pi_{n_r}(X)$ is the (possibly empty) subset $\{\hat{g}_\# \partial h^{-1}(\lambda) \mid \text{for every extension } \hat{g} \text{ of } g\}$ of $\pi_{N-1}(X)$.

When $k=2$ the subset $[\alpha_1, \alpha_2]$ consists of a single element, the ordinary Whitehead product of α_1 and α_2.

Next we consider the operation of a subspace A on a space E.

DEFINITION 2.2. *A operates* on E if there exists a map $\mu: E\times A \to E$ such that $\mu|E\simeq \mathrm{id}$ and $\mu|A\simeq i$, the inclusion map $A \hookrightarrow E$.

When A operates on E one inductively defines maps $\mu_{n+1}: E\times A^n \to E$ as follows: Let $\mu_2=\mu$; assume μ_n already defined and let μ_{n+1} be the composition

$$E\times A^n = (E\times A^{n-1})\times A \xrightarrow{\mu_n\times \mathrm{id}} E\times A \xrightarrow{\mu} E.$$

Clearly $\mu_{n+1}|E\simeq \mathrm{id}$ and $\mu_{n+1}|\text{each } A\simeq i$.

DEFINITION 2.3. If A operates on E, then $\mu_k: E\times A^{k-1}\to E$ induces a *k-fold generalized Pontrjagin product*

$$H_{n_1}(E)\otimes H_{n_2}(A)\otimes\cdots\otimes H_{n_k}(A) \hookrightarrow H_{n_1+\cdots+n_k}(E\times A^{k-1}) \xrightarrow{\mu_{k*}} H_{n_1+\cdots+n_k}(E).$$

If $a_1\in H_{n_1}(E)$ and $a_r\in H_{n_r}(A)$, $r=2,\ldots,k$, then μ_{k*} $(a_1\otimes a_2\otimes\cdots\otimes a_k)$ is denoted by $a_1 * a_2 *\cdots * a_k$.

If E is an H-space then E operates on E with $\mu: E\times E\to E$ the multiplication. In this case the k-fold generalized Pontrjagin product is just the k-fold ordinary Pontrjagin product.

3. **The main theorem.** We state the hypotheses of our main theorem which enables us to compute a kth order Whitehead product element of elements in $\pi_{n_r}(X)$, $r=1,\ldots,k$.

HYPOTHESIS OF THEOREM 3.3. (1) There exists a pair of spaces (E, A) such that A operates on E and the inclusion map $i: A\to E$ induces an isomorphism $i_\#: \pi_s(A)\to \pi_s(E)$ for $s=n_2,\ldots, n_k$.

(2) There exists a map $f: X\to E$ such that $f_\#: \pi_s(X)\to\pi_s(E)$ is an isomorphism for $s<N-1$ and an epimorphism for $s=N-1$, where $N=n_1+n_2+\cdots+n_k$.

Without loss of generality we will assume that the map $f: X\to E$ is an inclusion, for we can always replace E by the mapping cylinder of f. Then by Hypothesis (2), the pair (E, X) is $(N-1)$-connected and so the Hurewicz homomorphism $h: \pi_N(E, X)\to H_N(E, X)$ is an isomorphism.

DEFINITION 3.1. Under Hypothesis (2) of Theorem 3.3 a homomorphism $\theta: H_N(E)\to\pi_{N-1}(X)$ is defined by $\theta=\partial h^{-1}j$, where j is induced by inclusion:

$$H_N(E) \xrightarrow{j} H_N(E, X) \underset{\approx}{\xrightarrow{h^{-1}}} \pi_N(E, X) \xrightarrow{\partial} \pi_{N-1}(X).$$

REMARK 3.2. We observe (but do not prove) that θ has an alternate description. Replace $f: X\to E$ by a fibre map and let F denote the fibre. Thus $F\to X\xrightarrow{f} E$ is essentially a fibre sequence and F is clearly $(N-2)$-connected. Then θ can be shown to be the composition

$$H_N(E) \xrightarrow{\tau} H_{N-1}(F) \underset{\approx}{\xrightarrow{h^{-1}}} \pi_{N-1}(F) \longrightarrow \pi_{N-1}(X)$$

where τ is the transgression homomorphism of the fibration and the homomorphism on the right is induced by inclusion.

Next we state and prove Theorem 3.3. By Hypothesis (2) $f_\#: \pi_{n_r}(X) \to \pi_{n_r}(E)$ is an isomorphism for $r=1, \ldots, k$. We will identify these groups and write ambiguously $\alpha_r \in \pi_{n_r}(X) \equiv \pi_{n_r}(E)$.

THEOREM 3.3. *Under the Hypotheses* (1) *and* (2) *above, the kth order Whitehead product set* $[\alpha_1, \ldots, \alpha_k] \subset \pi_{N-1}(X)$ *of elements* $\alpha_r \in \pi_{n_r}(X)$, $r=1, \ldots, k$, *is nonempty and one of its elements is*

$$\theta(h(\alpha_1) * h(i_\#^{-1}\alpha_2) * \cdots * h(i_\#^{-1}\alpha_k))$$

where $*$ *denotes the generalized Pontrjagin product.*

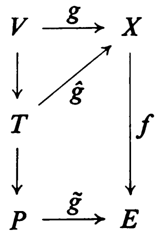
Proof. Let the α_r be represented by maps $a_r: S^{n_r} \to X$. By Hypothesis (1) there exist maps $a'_r: S^{n_r} \to A$ such that $ia'_r \simeq fa_r$ for $r=2, \ldots, k$. The maps $a_1, a_2, \ldots, a_k$ determine a map $g: V = S^{n_1} \vee \cdots \vee S^{n_k} \to X$ and a map $\tilde{g}: P = S^{n_1} \times \cdots \times S^{n_k} \to E$ defined by $\tilde{g} = \mu_k(fa_1 \times a'_2 \times \cdots \times a'_k)$:

$$S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k} \xrightarrow{fa_1 \times a'_2 \times \cdots \times a'_k} E \times A^{k-1} \xrightarrow{\mu_k} E.$$

Then there is homotopy-commutativity in the diagram

$$\begin{array}{ccc} V & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ P & \xrightarrow[\tilde{g}]{} & E \end{array}$$

and by the homotopy extension property of (P, V) we replace $\tilde{g}$ by a homotopic map (also called $\tilde{g}$) such that the above square commutes. Restrict $\tilde{g}$ to the fat wedge T to obtain a map of pairs $\tilde{g}: T, V \to E, X$. The obstructions to deforming this map into X lie in the group $H^i(T, V; \pi_i(E, X))$ (see [7, p. 197]). But (E, X) has trivial homotopy in dimensions $< N$ and for dimension reasons (T, V) has trivial cohomology in dimensions $\geqq N$. Therefore there exists a map $\hat{g}: T \to X$ such that in the following diagram



the triangle commutes and the trapezoid homotopy commutes. Here again we use the homotopy extension property and replace $\tilde{g}$ by a homotopic map such that the trapezoid commutes. Then $\hat{g}: T \to X$ is an extension of $g: V \to X$ and hence a Whitehead product element

$$\hat{g}_{\#}\, \partial h^{-1}(\lambda) \in [\alpha_1, \ldots, \alpha_k] \subset \pi_{N-1}(X)$$

exists.

Now we prove that this element is $\theta(h\alpha_1 * hi_{\#}^{-1}\alpha_2 * \cdots * hi_{\#}^{-1}\alpha_k)$. Consider the commutative diagram

$$\begin{array}{ccc}
\pi_{N-1}(T) & \xrightarrow{\hat{g}_{\#}} & \pi_{N-1}(X) \\
\uparrow{\scriptstyle\partial} & & \uparrow{\scriptstyle\partial} \\
\pi_N(P, T) & \xrightarrow{\tilde{g}_{\#}} & \pi_N(E, X) \\
{\scriptstyle h}\downarrow{\scriptstyle\approx} & & {\scriptstyle\approx}\downarrow{\scriptstyle h} \\
H_N(P, T) & \xrightarrow{\tilde{g}_{*}} & H_N(E, X) \\
\uparrow{\scriptstyle j} & & \uparrow{\scriptstyle j} \\
H_N(P) & \xrightarrow{\tilde{g}_{*}} & H_N(E)
\end{array}
\qquad \theta: H_N(E) \to \pi_{N-1}(X)$$

For the generator $\lambda \in H_N(P, T)$ we choose $j(s_1 \otimes \cdots \otimes s_k)$ where $s_r \in H_{n_r}(S^{n_r})$ is the standard generator and $s_1 \otimes \cdots \otimes s_k \in H_{n_1}(S^{n_1}) \otimes \cdots \otimes H_{n_r}(S^{n_r}) \equiv H_N(P)$. Then

$$\hat{g}_{\#}\, \partial h^{-1}(\lambda) = \partial h^{-1}\tilde{g}_*(\lambda) = \partial h^{-1} j \tilde{g}_*(s_1 \otimes \cdots \otimes s_k) = \theta \tilde{g}_*(s_1 \otimes \cdots \otimes s_k).$$

But $\tilde{g} \simeq \mu_k(fa_1 \times a_2' \times \cdots \times a_k'): P \to E$ and so

$$\begin{aligned}
\theta\tilde{g}_*(s_1 \otimes \cdots \otimes s_k) &= \theta\mu_{k*}((fa_1)_*(s_1) \otimes a_{2*}'(s_2) \otimes \cdots \otimes a_{k*}'(s_k)) \\
&= \theta(h(\alpha_1) * h(i_{\#}^{-1}\alpha_2) * \cdots * h(i_{\#}^{-1}\alpha_k)).
\end{aligned}$$

For the first corollary of the theorem assume that Y is a $(p-1)$-connected space and $2 \leqq p \leqq q$. Let Y_n denote the nth Postnikov section of Y and let $Y_{q,p+q-2}$ denote the fibre of $Y_{p+q-2} \to Y_{q-1}$. Since this fibration is principal there is an action of $Y_{q,p+q-2}$ on Y_{p+q-2}. We let $X = Y_{p+q-1}$, $A = Y_{q,p+q-2}$, $E = Y_{p+q-2}$ and $f: X \to E$ be the fibre map. The hypotheses of Theorem 3.3 are then satisfied for the ordinary Whitehead product of $\alpha \in \pi_p(Y) \equiv \pi_p(X)$ and $\beta \in \pi_q(Y) \equiv \pi_q(X)$. We obtain as a corollary Meyer's theorem [8].

COROLLARY 3.4. $[\alpha, \beta] = \theta(h(\alpha) * h(i_{\#}^{-1}\beta))$.

By Remark 3.2, $\theta: H_{p+q}(Y_{p+q-2}) \to \pi_{p+q-1}(Y_{p+q-1}) = \pi_{p+q-1}(Y)$ can be identified with the transgression homomorphism

$$H_{p+q}(Y_{p+q-2}) \to H_{p+q-1}(F_{p+q-1}) \equiv \pi_{p+q-1}(F_{p+q-1}) = \pi_{p+q-1}(Y)$$

of the fibration $F_{p+q-1} \to Y_{p+q-1} \to Y_{p+q-2}$.

For applications in the next section we will need the following corollary.

COROLLARY 3.5. *Suppose there exists a map of X into an H-space E such that $\pi_s(X) \to \pi_s(E)$ is an isomorphism for $s<N-1$ and an epimorphism for $s=N-1$, where $N=n_1+\cdots+n_k$. Then, for $\alpha_r \in \pi_{n_r}(X) \equiv \pi_{n_r}(E)$,*

$$\theta(h\alpha_1 * \cdots * h\alpha_k) \in [\alpha_1, \ldots, \alpha_k] \subset \pi_{N-1}(X)$$

where $*$ denotes the ordinary Pontrjagin product.

4. **Applications.** In this section we use Corollary 3.5 to compute certain higher order Whitehead product elements. Our first result concerns Whitehead products in $\pi_*(BU_t)$, where BU_t is the classifying space of U_t, the unitary group of $t \times t$ matrices. Recall that for $i<2t+1$, $\pi_i(BU_t)$ is either 0 or Z depending on whether i is odd or even, and that $\pi_{2t+1}(BU_t) \approx Z_{t!}$.

THEOREM 4.1. *If $\alpha_r \in \pi_{2m_r+2}(BU_t) \approx Z$ and $\gamma \in \pi_{2t+1}(BU_t) \approx Z_{t!}$ are suitable generators, $r=1, 2, \ldots, k$ and $t=m_1+\cdots+m_k+k-1$, then*

$$m_1! \cdots m_k! \gamma \in [\alpha_1, \ldots, \alpha_k] \subset \pi_{2t+1}(BU_t).$$

Proof. We write B_n for BU_n, $1 \leqq n \leqq \infty$, and let $f: B_t \to B_\infty$ be the inclusion map. It is well known that B_∞ is an H-space and that f is an isomorphism of homotopy groups in dimensions $<2t+1=N-1$ and an epimorphism in dimension $2t+1 = N-1$. By Corollary 3.5, the Whitehead product set $[\alpha_1, \ldots, \alpha_k] \subset \pi_{N-1}(B_t)$ is nonempty and one of its elements is $\theta(h\alpha_1 * \cdots * h\alpha_k)$, where $h\alpha_r \in H_{2m_r+2}(B_\infty)$ is the Hurewicz homomorphism image of $\alpha_r \in \pi_{2m_r+2}(B_\infty)$ and θ is the composition

$$H_N(B_\infty) \xrightarrow{j} H_N(B_\infty, B_t) \underset{\approx}{\xrightarrow{h^{-1}}} \pi_N(B_\infty, B_t) \xrightarrow{\partial} \pi_{N-1}(B_t).$$

It remains to show that $\theta(h\alpha_1 * \cdots * h\alpha_k) = m_1! \cdots m_k! \gamma$.

We recall some known facts on the homology and homotopy of B_n. The general reference is [5], especially 11-12–11-13 and 17-01–17-03. The multiplication in $H_*(B_\infty)$ (i.e., the Pontrjagin product) is denoted by $*$ or by juxtaposition or by a dot.

(1) The homomorphism $\partial: \pi_N(B_\infty, B_t) \to \pi_{N-1}(B_t)$ is the projection $Z \to Z_{t!}$, for $\pi_N(B_\infty) \to \pi_N(B_\infty, B_t)$ is the homomorphism $Z \to Z$ consisting of multiplication by $t!$.

(2) $h(\alpha_r) = m_r! p_{2m_r+2}$, where $p_{2i} \in H_{2i}(B_\infty)$ is the generator in dimension $2i$ of the primitive elements of the Hopf algebra $H_*(B_\infty)$.

(3) $H_*(B_\infty)$ is a polynomial algebra on generators $x_2, x_4, x_6, \ldots$, where degree $x_{2i} = 2i$. Furthermore $p_{2m_1+2} \cdot p_{2m_2+2} \cdot \cdots \cdot p_{2m_k+2} = x_2^{t+1} +$ linear combination of other monomials in the x's. This last assertion is an easy consequence of the following result [5, 17-03]:

$$p_2 = x_2, \ldots, p_{2i} = (-1)^{i+1} i x_{2i} + \sum_{j=1}^{i-1} (-1)^{j+1} x_{2j} p_{2i-2j}, \ldots.$$

(4) The homomorphism $j: H_N(B_\infty) \to H_N(B_\infty, B_t)$ maps all monomials $x_{2k_1} \cdot \cdots \cdot x_{2k_s}$ $(2k_1+\cdots+2k_s=N)$ except x_2^{t+1} to zero and maps x_2^{t+1} onto a generator of $H_N(B_\infty, B_t) = Z$.

We put these facts together and obtain

$$\begin{aligned}\theta(h\alpha_1 * \cdots * h\alpha_k) &= m_1!\cdots m_k!\,\theta(p_{2m_1+2}\cdot\cdots\cdot p_{2m_k+2})\\ &= m_1!\cdots m_k!\,\theta(x_2^{t+1}+\text{other monomials})\\ &= m_1!\cdots m_k!\,\partial h^{-1}j(x_2^{t+1}).\end{aligned}$$

But $j(x_2^{t+1})$ is a generator of $H_N(B_\infty, B_t)\approx \boldsymbol{Z}$ by (4) and so $\partial h^{-1}j(x_2^{t+1})$ is a generator γ of $\pi_{N-1}(B_t)\approx Z_{t!}$ by (1). Thus $m_1!\cdots m_k!\gamma\in[\alpha_1,\ldots,\alpha_k]$.

REMARK 4.2. For $k=2$, Theorem 4.1 provides a new proof of Bott's theorem [2] on the Whitehead product $\pi_{2r+2}(BU_t)\otimes\pi_{2s+2}(BU_t)\to\pi_{2t+1}(BU_t)$ (or what is the same thing, the Samelson product $\pi_{2r+1}(U_t)\otimes\pi_{2s+1}(U_t)\to\pi_{2t}(U_t)$), where $t=r+s+1$. In addition, we can similarly prove a result which is completely analogous to Theorem 4.1 for the symplectic group Sp_t. In this way we compute a higher order Whitehead product element in $\pi_*(BSp_t)$ and obtain another proof of Bott's theorem on a Samelson product in $\pi_*(Sp_t)$.

Our next application of Corollary 3.5 concerns k-fold Whitehead products in a space X such that $\pi_i(X)=0$ for $i<n$ and $n<i<kn-1$, where $n>1$. Let $\pi_n(X)=\pi$ and $\pi_{kn-1}(X)=G$ and let $l\in H^{kn}(\pi, n; G)$ denote the first Postnikov invariant of X and $\gamma_k\colon H_n(\pi, n)=\pi\to H_{kn}(\pi, n)$ be the kth divided power in the ring $H_*(\pi, n)$ [4].

THEOREM 4.3. *Let X satisfy the conditions of the preceding paragraph. If $\alpha\in\pi_n(X)=\pi$ and $s_1,\ldots,s_k$ are any k integers, then the kth order Whitehead product $[s_1\alpha,\ldots,s_k\alpha]$ is a unique element of $\pi_{kn-1}(X)=G$.*

(a) *If n is odd, $[s_1\alpha,\ldots,s_k\alpha]=0$.*

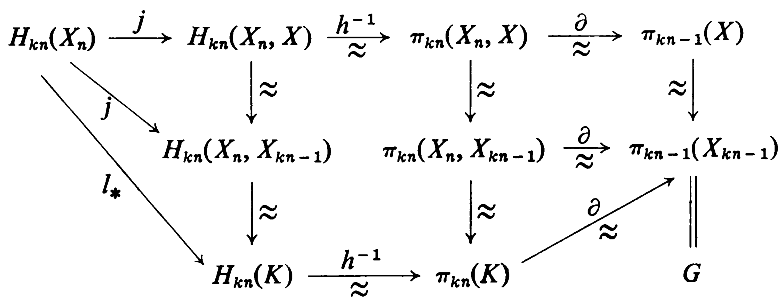
(b) *If n is even, $[s_1\alpha,\ldots,s_k\alpha]=s_1\cdots s_k k!\,l_*\gamma_k(\alpha)$, where $l_*\colon H_{kn}(\pi, n)\to H_{kn}(G, kn)$ $=G$ is induced by l.*

Proof. We embed X in an Eilenberg-Mac Lane complex $K(\pi, n)$ in the usual way. Clearly $K(\pi, n)$ is an H-space and the map $X\to K(\pi, n)$ satisfies the hypotheses of Corollary 3.5. Then the kth order Whitehead product $[s_1\alpha,\ldots,s_k\alpha]$ exists and one of its elements is $s_1\cdots s_k\theta(h\alpha*\cdots*h\alpha)$. We show first that this Whitehead product element is unique. If there are two extensions of a map $V=S^n\vee\cdots\vee S^n\to X$ to the fat wedge $T=T(S^n,\ldots,S^n)$ then the obstructions to their being homotopic lie in the groups $H^i(T, V; \pi_i(X))$ which are always zero. Thus there is up to homotopy exactly one extension of a map $V\to X$ to T. Therefore $[s_1\alpha,\ldots,s_k\alpha]$ consists of the element $s_1\cdots s_k\theta(h\alpha*\cdots*h\alpha)$. Here $\alpha\in\pi=\pi_n(X)$ $\equiv\pi_n(K(\pi, n))\xrightarrow[\approx]{h} H_n(\pi, n)=\pi$ and we shall write α for the element in each of these groups (identifying α with $h\alpha$). Thus we must determine $\theta(\alpha*\cdots*\alpha)=\theta(\alpha^k)$ for $\alpha\in H_n(\pi, n)$. We recall the following known facts about $H_*(\pi, n)$ (the general reference is [4]). Let $U(\pi, n)$ be the universal graded strictly anticommutative algebra with divided powers generated by the graded abelian group which is trivial in dimensions $\neq n$ and is π in dimension n. Then the subalgebra with divided powers generated by $H_n(\pi, n)=\pi$ is isomorphic to $U(\pi, n)$. If n is odd, $U(\pi, n)$ is the exterior algebra of π, suitably graded. Hence $\alpha^k=\alpha*\cdots*\alpha=0$ for

n odd, and so (a) is proved. To prove (b), we note that $\alpha^k=(\gamma_1(\alpha))^k=k!\gamma_k(\alpha)$ and so $\theta(\alpha^k)=k!\theta(\gamma_k(\alpha))$. We must show that under appropriate identification the homomorphism $\theta\colon H_{kn}(\pi, n)\to\pi_{kn-1}(X)=G$ is just the homomorphism $l_*\colon H_{kn}(\pi, n)\to H_{kn}(G, kn)=G$. This follows easily from Remark 3.2, but we will establish it directly. If X_i denotes the ith Postnikov section of X then $X_n=K(\pi, n)$. We denote $K(G, kn)$ by K and regard maps between spaces as inclusions whenever convenient. First we identify $j\colon H_{kn}(X_n)\to H_{kn}(X_n, X)$ with

$$j\colon H_{kn}(X_n)\to H_{kn}(X_n, X_{kn-1}),$$

for the inclusion map $(X_n, X)\to(X_n, X_{kn-1})$ induces an isomorphism $H_{kn}(X_n, X)\xrightarrow{\approx} H_{kn}(X_n, X_{kn-1})$ (e.g., by the five lemma). But $X_{kn-1}\to X_n\xrightarrow{l} K$ is essentially a fibre sequence and so $H_{kn}(X_n, X_{kn-1})\to H_{kn}(K)$ is an isomorphism (either by the Serre theorem or by arguing on the homotopy groups and then passing to homology). Thus we have the following commutative diagram:



Since the top line $\partial h^{-1}j$ is just θ, under the identification of $\pi_{kn-1}(X)$ with G and of $H_{kn}(K)$ with G via ∂h^{-1}, $\theta\colon H_{kn}(X_n)\to G$ is identified with $l_*\colon H_{kn}(X_n) =H_{kn}(\pi, n)\to H_{kn}(K)=G$. This completes the proof.

In the case when n is even, more complete information can be obtained when π and G are the group of integers.

COROLLARY 4.4. *Let X satisfy the hypothesis of Theorem 4.3 with $G=\pi=\mathbf{Z}$. Let the first Postnikov invariant $l=mb^k$, a multiple of the kth cup product of the basic class $b\in H^n(\mathbf{Z}, n)$. Let $\alpha\in\pi$ and $\gamma\in G$ be generators. Then if n is even and $s_1,\dots,s_k$ are any integers,*

$$[s_1\alpha,\dots,s_k\alpha] = mk!s_1\cdot\cdots\cdot s_k\gamma.$$

The corollary follows immediately from the fact that $(b^k)_*(\gamma_k(\alpha))\in H_{kn}(\mathbf{Z}, kn) =\mathbf{Z}$ is a generator.

REMARKS 4.5. (a) Porter's result [10] on the kth order Whitehead product in complex projective $(k-1)$-space follows from Corollary 4.4 by setting $n=2$ and $m=1$.

(b) The last two results provide another way to obtain some of Porter's examples for certain phenomena regarding higher order Whitehead products [9]. For example, by Corollary 4.4 there is for every k a space in which all ith order Whitehead products are trivial, $i<k$, and in which there exists a nontrivial kth order Whitehead product. Also, by Theorem 4.3, one can easily find spaces in which all kth order Whitehead products vanish, but which are not H-spaces.

(c) For $k=2$, Theorem 4.3 yields a result on ordinary Whitehead products which is similar to but somewhat weaker than a result of Stein and Meyer [11]. Our proof for $k=2$ is not substantially different from that of [11]. We note that one direction of Stein's theorem can be generalized to higher order Whitehead products as follows: Let X satisfy the hypothesis of Theorem 4.3 with n even. Then there exists a function $\mathscr{K}: \pi \to G$ such that $\mathscr{K}(s\alpha)=s^k\mathscr{K}(\alpha)$ and

$$[\alpha_1, \ldots, \alpha_k] = \mathscr{K}(\alpha_1+\cdots+\alpha_k)-\sum_{i=0}^{k} \mathscr{K}(\alpha_1+\cdots+\hat{\alpha}_i+\cdots+\alpha_k) \\ +\cdots+(-1)^{k-1}(\mathscr{K}(\alpha_1)+\cdots+\mathscr{K}(\alpha_k))$$

for any $\alpha_i \in \pi$. The proof proceeds just like the proof of Theorem 4.3 with $\mathscr{K}=l_*\gamma_k$.

Despite the fact that proof of Corollary 3.5 is not difficult, the result appears to be new even for ordinary Whitehead products (see the first paragraph of §1 for an explicit statement in that case). We next use Corollary 3.5 to derive a few classical results on ordinary Whitehead products.

PROPOSITION 4.6. *If X is an H-space, then all ordinary Whitehead products in X vanish.*

For we take $X=E$ and note that θ factors through $H_*(X, X)$.

Our final application is the theorem relating the vanishing of Whitehead products to the existence of elements of Hopf invariant one.

Let X be an $(n-1)$-connected space, $n>1$. Let $e: X \to \Omega\Sigma X$ denote the canonical embedding of X into the loop space of the suspension of X and let $j: \Omega\Sigma X \to \Omega\Sigma X, X$ be the inclusion map. Then the generalized Hopf invariant homomorphism $H: \pi_{2n+1}(\Sigma X) \to \pi_{2n}(\Omega\Sigma X, X)$ has been defined (see e.g. [4, Exposé 22]) as the composition

$$\pi_{2n+1}(\Sigma X) \approx \pi_{2n}(\Omega\Sigma X) \xrightarrow{j_\#} \pi_{2n}(\Omega\Sigma X, X).$$

In the case $X=S^n$ this is just a homomorphism $H: \pi_{2n+1}(S^{n+1}) \to \pi_{2n}(\Omega S^{n+1}, S^n) = \mathbf{Z}$. The following theorem is known.

THEOREM 4.7. *The generalized Hopf invariant $H: \pi_{2n+1}(\Sigma X) \to \pi_{2n}(\Omega\Sigma X, X)$ of an $(n-1)$-connected space X is an epimorphism if and only if $[\alpha, \beta]=0$ for every $\alpha, \beta \in \pi_n(X)$.*

Proof. $\Omega\Sigma X$ is an H-space and $e\colon X \to \Omega\Sigma X$ satisfies the hypothesis of Corollary 3.5 with $k=2$ and $n_1=n_2=n$. Therefore $[\alpha, \beta]=\theta(h\alpha * h\beta)=\theta(e_*\alpha * e_*\beta)$ where $\alpha, \beta \in \pi_n(X)\equiv\pi_n(\Omega\Sigma X)$ are identified with their Hurewicz images in $H_n(X)$. Consider the diagram

$$\begin{array}{ccccc}
H_{2n}(\Omega\Sigma X) & \xrightarrow{\ j\ } & H_{2n}(\Omega\Sigma X, X) & & \\
 & & \approx\big\downarrow h^{-1} & & \\
\pi_{2n}(\Omega\Sigma X) & \xrightarrow{\ j_\#\ } & \pi_{2n}(\Omega\Sigma X, X) & \xrightarrow{\ \partial\ } & \pi_{2n-1}(X) \\
\approx\big\uparrow & \nearrow_{H} & & & \\
\pi_{2n+1}(\Sigma X) & & & &
\end{array}$$

It is not difficult to show that $H_{2n}(\Omega\Sigma X, X)$ consists of elements of the form $j(e_*\alpha * e_*\beta)$ for every $\alpha, \beta \in H_n(X)$. When $H_*(X)$ is torsion-free this follows from the Bott-Samelson theorem [3] which expresses the Pontrjagin algebra $H_*(\Omega\Sigma X)$ as a tensor algebra on $e_*H_+(X)$. In general a theorem of Copeland enables one to express $H_*(\Omega\Sigma X)$ as a torsion-tensor algebra on $e_*H_+(X)$. In any case $H_{2n}(\Omega\Sigma X) \approx H_{2n}(X) \oplus H_n(X) \otimes H_n(X)$. Thus $\pi_{2n}(\Omega\Sigma X, X)$ consists of all elements of the form $h^{-1}j(e_*\alpha * e_*\beta)$ and $[\alpha, \beta]=\partial h^{-1}j(e_*\alpha * e_*\beta)$. Hence $[\alpha, \beta]=0$ for every α and β if and only if $\partial=0$. By exactness this is so if and only if $j_\#$ and consequently H is an epimorphism.

COROLLARY 4.8 [12, 3.49]. *Let $\iota_n \in \pi_n(S^n)$ be the class of the identity map. Then $[\iota_n, \iota_n]=0 \in \pi_{2n-1}(S^n)$ if and only if there is an element in $\pi_{2n+1}(S^{n+1})$ of Hopf invariant one.*

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