TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 160, October 1971

Γ-COMPACT MAPS ON AN INTERVAL AND FIXED POINTS

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Abstract. We characterize the Γ-compact continuous functions $f: X \to X$ where X is a possibly-noncompact interval. The map f is called Γ-compact if the closed topological semigroup $\Gamma(f)$ generated by f is compact, or equivalently, if every sequence of iterates of f under functional composition $(*)$ has a subsequence which converges uniformly on compact subsets of X. For compact X the characterization is that the set of fixed points of $f * f$ is connected. If X is noncompact an additional technical condition is necessary. We also characterize those maps f for which iterates of distinct orders agree ($\Gamma(f)$ finite) and state a result on common fixed points of commuting functions when one of the functions is Γ-compact.

Introduction. Let X be a connected subset of the real line and let g be a map (continuous function) from X into X^*, the closure of X. If we place on the set $C(X, X^*)$ of such maps the topology of uniform convergence on compact subsets and let $C(X, X)$ be the subspace of $C(X, X^*)$ consisting of those maps f for which $f(X) \subset X$, then $C(X, X)$ is a topological semigroup under functional composition: $(f * g)(x) = f(g(x))$; that is, composition is associative and jointly continuous in the given topology. ($C(X, X^*)$ is not a semigroup when X is not closed since $f * g$ is not defined at a point w for which $g(w) \in X^* \backslash X$.) For $f \in C(X, X)$ we set $f_1 = f$ and $f_{n+1} = f * f_n$ (the *iterates* of f), and we define [6, p. 15]

$$O(f) = \{f_n \mid n \geqq 1\} \subset C(X, X), \qquad \Gamma(f) = O(f)^*.$$

Since limit functions under the given topology are continuous, we have $\Gamma(f) \subset C(X, X^*)$, but $\Gamma(f)$ is not a semigroup unless $\Gamma(f) \subset C(X, X)$. We say that f is *Γ-compact* if $\Gamma(f)$ is a compact subset of $C(X, X^*)$. A map will be called *precompact* if it satisfies certain conditions necessary for Γ-compactness; when X is compact, for example, all maps are precompact. Our main result is:

THEOREM. *The map f is Γ-compact if and only if $f_2 = f * f$ is precompact and the fixed-point set J of f_2 is connected.*

Presented to the Society, January 25, 1970; received by the editors November 24, 1969 and, in revised form, September 18, 1970.

AMS 1969 *subject classifications*. Primary 2205, 2654, 5428, 5480; Secondary 4625, 4750, 4785, 5482, 5485.

Key words and phrases. Γ-compact, fixed point, functional composition, topological semigroup, convergence of iteration, commuting functions, common fixed point, precompact, equicontinuous, real functions.

From this and other theorems we obtain conclusions on common fixed points of commuting functions on an interval which parallel the results of Shields on commuting analytic functions on the unit disk [12]. There seems to be a common belief that Shields' work implies that commuting analytic functions on an interval must have a common fixed point, but our theorems show that Shields' methods applied to an interval give a different type of result. The question of whether commuting analytic functions on an interval must have a common fixed point is still undecided.

Some preliminaries. Throughout this paper X will denote a connected subset of the real line; that is, a bounded or unbounded, open, closed, or half-open interval. All maps will be from X into X or X^*, and f will always denote a map such that $f(X) \subset X$. A subset of X will be called closed or open if it is closed or open relative to X.

The following lemmas will be useful:

LEMMA 1. *These are equivalent*:

(a) *the map f is Γ-compact*;

(b) *every sequence of iterates of f has a convergent subsequence*; *that is, a subsequence which converges uniformly on each compact subset of X*;

(c) *the iterates of f are equicontinuous and uniformly bounded on each compact subset of X*;

(d) *the iterates of f are equicontinuous and bounded at each point of X.*

Proof. By [4, Exercise 8.5], $C(X, X^*)$ is metrizable. Thus by [9, p. 138, Theorem 5], (a) and (b) are equivalent. By [9, p. 233, Theorem 17], (a) and (d) are equivalent. Clearly (c) implies (d), while if the iterates of f were not bounded on a compact set C, we could find a sequence of integers $\{i_n\}$ and points $\{x_n\}$ in C such that $\{f_{i_n}(x_n)\}$ is unbounded; hence for a limit point x of $\{x_n\}$, $\{f_{i_n}(x)\}$ would be unbounded, contradicting (d). Thus (c) and (d) are equivalent.

LEMMA 2. *Let Y be a connected subset of X and let f be a map taking Y into itself and having no fixed point in Y. If $f(y) > y$ for each $y \in Y$, then the iterates of f diverge to infinity when Y is unbounded above and converge uniformly to the upper endpoint of Y on compact subsets of Y when Y is bounded above. The analogous conclusion holds when $f(y) < y$ for each $y \in Y$.*

Proof. If $f(y) > y$, then for each y, $\{f_n(y)\}$ is a monotone increasing sequence which diverges to infinity unless it is bounded above. Clearly the sequence cannot be bounded if Y is unbounded above since f has no fixed point in Y, so the iterates diverge at each y when Y is unbounded above. If Y is bounded above, then it has an upper endpoint z_0. To show that $\{f_n\}$ converges uniformly to z_0 on compact subsets of Y, let C be a compact subset of Y and let $\varepsilon > 0$. Let D be a continuum in Y containing C and the point $z_0 - \varepsilon/2$. Since D is compact and f has no fixed point in Y, there is a $\delta>0$ such that $f(y)\geqq y+\delta$ for $y\in D$. If $|D|$ is the length of D and $N>|D|/\delta$, then $|f_n(y)-z_0|<\varepsilon$ when $n>N$ and $y\in C$, so $\{f_n\}$ converges uniformly on C to the constant function with value z_0. The proofs for $f(y)<y$ are analogous.

Precompact maps. When X itself is compact, the criterion for Γ-compactness of a map f will be seen to depend only on the connectedness of the fixed-point set J of $f_2=f*f$. When X fails to be compact, however, f can jeopardize convergence by misbehaving below or above J. Maps that behave in a "compact" manner there will be called *precompact*. (The overlap of terminology with the "precompact" operators of functional analysis is coincidental.)

DEFINITION. (1) If f has a fixed point, then f is *precompact* if and only if (a) when x is less than every fixed point of f, then $f(x)>x$; and (b) when x is greater than every fixed point of f, then $f(x)<x$.

(2) If f has no fixed point, then f is *precompact* if and only if either (a) X is bounded above and $f(x)>x$ for all x; or (b) X is bounded below and $f(x)<x$ for all x.

Figure 1 illustrates the concept of precompactness for maps having fixed points. For this figure $X=(0,1)$, p and q are the least and greatest fixed points of f respectively, and the behavior of f on $(0,p)$ and $(q,1)$ determines whether or not f is precompact. Requirement (1.a) is satisfied for $x\in(0,p)$, but for $x\in(q,1)$ we have $f(x)>x$, so f is not precompact. Note that if $X=(0,1]$, then 1 would be the greatest fixed point of f, so f would be precompact.

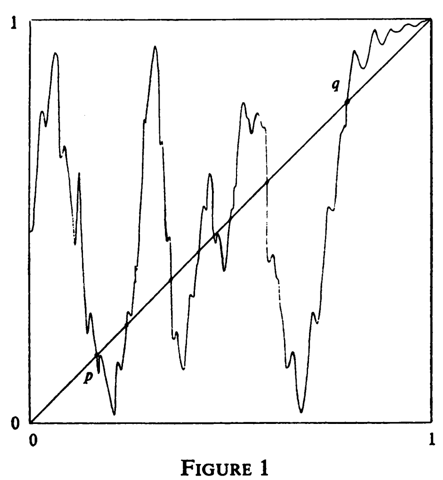

FIGURE 1

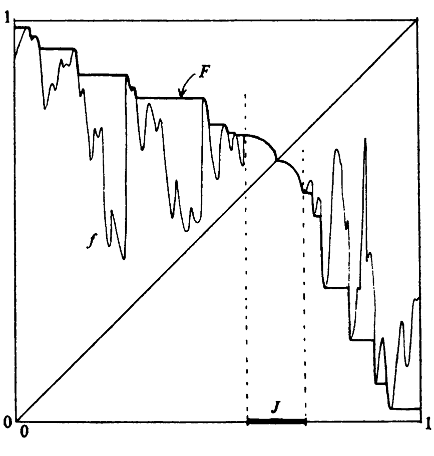

FIGURE 2

LEMMA 3. *For each $n\geqq 1$, with f_n the nth iterate of f,*

(a) *the map f has a fixed point if and only if f_n has;*

(b) *if f has no fixed point, then f is precompact if and only if f_n is.*

Proof. Any fixed point of f is a fixed point of f_n, while if f has no fixed point, then the sequence $\{f_n(x)\}$ is monotone away from x. Thus $f_n(x)$ can never equal x

(proving part (a)), and the graphs of f and f_n must be on the same side of the diagonal (above or below). Thus from the definition of *precompact*, the precompactness of both f and f_n depend on whether X is bounded on that side.

THEOREM 1. *If f has no fixed point, then f is Γ-compact if and only if f is precompact. If f is Γ-compact and has no fixed point, then $\Gamma(f)=O(f)\cup g$, where $g(x)=z_0\in X^*\backslash X$ for all $x\in X$, and the iterates of f are distinct.*

Proof. If f has no fixed point, then either $f(x)>x$ for all x or $f(x)<x$ for all x. Suppose that $f(x)>x$. If f is precompact, then X is bounded above, with upper endpoint z_0; define g by $g(x)=z_0$ for all x. By Lemma 2 the iterates of f converge uniformly to g, so every sequence of iterates of f for which n tends to infinity converges to g. Thus $\Gamma(f)$ is compact and $\Gamma(f)=O(f)\cup g$. If f is not precompact, then X is unbounded above, and by Lemma 2, the iterates of f diverge. Thus $\Gamma(f)$ is not compact. A similar argument holds if $f(x)<x$ for all x. We see that $z_0\notin X$, otherwise it would be a fixed point of f. The iterates of f are distinct since $\{f_n(x)\}$ is monotone for each x.

We note that $g\notin C(X, X)$, so $\Gamma(f)\not\subset C(X, X)$ when f is Γ-compact and has no fixed point. However, if we let $Y=X\cup z_0$ and define a function $f'\colon Y\to Y$ by $f'=f$ on X and $f'(z_0)=z_0$, then f' is continuous at z_0, $\Gamma(f')$ is compact, and $\Gamma(f')\subset C(Y, Y)$, so $\Gamma(f')$ is a compact semigroup.

THEOREM 2. (a) *If X is compact, then f is precompact*;

(b) *if f_2 is precompact, so is f*;

(c) *if f is Γ-compact, then f and f_2 are precompact.*

Proof. (a) If X is compact, then $X=[a, b]$, and f must have a fixed point. If $f(a)=a$, then there is no x less than each fixed point of f, while if $f(a)>a$, then for all x less than the least fixed point of f we have $f(x)>x$. The argument for points x greater than the fixed points of f is similar, so f is precompact.

(b) Lemma 3 applies if f has no fixed point. We thus assume that f has a fixed point and is not precompact, and we wish to show that f_2 is not precompact. Either there is an x less than all the fixed points of f for which $f(x)<x$, or there is an x greater than all the fixed points of f for which $f(x)>x$. The two cases are similar so we consider only the latter. Then f must have a greatest fixed point q, and $f(x)>x$ for all $x>q$. But then $f_2(x)>f(x)>x$ for all $x>q$. Thus no $x>q$ is a fixed point of f_2, so each $x>q$ is greater than every fixed point of f_2 and satisfies $f_2(x)>x$. Thus f_2 is not precompact.

(c) Theorem 1 and Lemma 3 apply when f has no fixed point, so by part (b) it suffices to show that $\Gamma(f)$ is not compact if f_2 has fixed points but is not precompact. If f_2 is not precompact, then either $f_2(x)<x$ for some x less than every fixed point of f_2, or $f_2(x)>x$ for some x greater than every fixed point of f_2. We assume the latter case and denote by q the greatest fixed point of f_2. Then $f_2(x)>x$ for all $x>q$.

Let $Y=\{x \in X \mid x>q\}$; then f_2 maps Y into Y with no fixed point, so Lemma 2 applies. If X is unbounded above, then the iterates of f_2 tend to infinity on Y, so $\Gamma(f)$ is not compact. If X has an upper endpoint z_0, then the iterates of f_2 converge to z_0 on Y, while $f_{2n}(q)=q$ for all n. Thus any limit function would be discontinuous at q, and hence would not be a limit in the given topology. Thus in this case also, $\Gamma(f)$ is not compact. By symmetry, the same conclusion holds for the former case, when $f_2(x)<x$ for some x less than every fixed point of f_2.

When $\Gamma(f)$ is a semigroup. If $\Gamma(f)$ is a compact semigroup, that is, if f is Γ-compact and $\Gamma(f) \subset C(X, X)$, then from [6, pp. 14–15] $\Gamma(f)$ has a unique closed minimal ideal $M(\Gamma(f))$ which is a group. We shall use the simpler notation $K(f)=M(\Gamma(f))$ of [13] and call $K(f)$ the *kernel.* From [13], $K(f)$ is the maximal group contained in $\Gamma(f)$, and $\Gamma(f)=O(f) \cup K(f)$. Since $\Gamma(f)=O(f)^*$ and $O(f)$ is abelian, $\Gamma(f)$ is abelian and $K(f)$ is an abelian group.

The next theorem, Theorem 3, describes the structure of $\Gamma(f)$ for the most interesting set of circumstances. In fact, Theorem 4 will show that Theorems 1 and 3 exhaust the possible structures of $\Gamma(f)$ when $\Gamma(f)$ is compact and f is a map on an interval. Much of the next theorem is not new; for instance, the fact that each $k \in K(f)$ is a homeomorphism of $e(X)$ onto itself is often stated as part of A. D. Wallace's "swelling lemma," one version of which is quoted in [6, p. 15].

THEOREM 3. *If f is Γ-compact and $\Gamma(f) \subset C(X, X)$, then the fixed-point set J of f_2 is nonempty and connected. Either $K(f)=\{e\}$, in which case $e(X)=J$ and e and f are the identity on J, or $K(f)=\{k, e\}$. If $K(f)=\{k, e\}$, then $e(X)=k(X)=J$, e is the identity on J and k and f agree on J, and k and f are sense-reversing self-inverse homeomorphisms of J onto J.*

Proof. Since $\Gamma(f)$ is compact and $\Gamma(f) \subset C(X, X)$, $\Gamma(f)$ is a compact semigroup and has a kernel $K(f)$. The kernel $K(f)$ is a group and hence has a unit e; letting $X'=e(X)$ (eventually we shall show that $X'=J$), then e is the identity on X', since e is idempotent. The kernel $K(f)$ is a group of maps into X, so each $k \in K(f)$ is a homeomorphism from X' onto X'. The set X' is an interval since it is the continuous image of X under e, so each k is either sense-reversing or sense-preserving on X'. If k is sense-preserving, then either k is the identity on X', in which case k is a unit for the group and so $k=e$, or there is a nonempty interval $Y \subset X'$ on which k is a fixed-point free homeomorphism. We will prove that the latter case cannot occur, as follows: since $K(f)$ is a closed group, $O(k) \subset K(f)$ and $\Gamma(k)=O(k)^* \subset K(f) \subset \Gamma(f)$, and since $\Gamma(f)$ is compact, $\Gamma(k)$ is compact. If such a Y exists, then by Lemma 2 the iterates of k must either diverge, in which case $\Gamma(k)$ would not be compact, or converge to function $g \in \Gamma(k) \subset K(f)$ which is constant on $Y \subset X'$. But each element of $K(f)$ is one-to-one on X', so no such g or Y exists. Thus the identity e is the only element of $K(f)$ which is sense-preserving on X'. If k and k' are sense-reversing, then $k * k'$ is sense-preserving and is a member of $K(f)$, so $k * k'=e$.

Thus the product of any two nonunit elements of $K(f)$ is e, so there can be at most one such element k. Thus either $K(f)=\{e\}$ or $K(f)=\{e, k\}$ where k is a self-inverse sense-reversing homeomorphism on X'.

Since $K(f)$ is an ideal of $\Gamma(f)$, $f * e \in K(f)$ so either $f * e=e$ or $K(f)=\{e, k\}$ and $f * e=k$. The maps f and $f * e$ agree on X', so if $f * e=e$ then $f(x)=e(x)=x$ and $f_n(x)=x$ for all $x \in X'$ and all n. Thus if $f * e=e$, then $K(f)=\{e\}$, and if $K(f)=\{e, k\}$, then $f * e=k$. In the latter case f is a self-inverse sense-reversing homeomorphism on X'. Now $f_2 * e$ is also a member of $K(f)$, and since $\Gamma(f)$ is abelian $f_2 * e= (f * f) * (e * e)=(f * e) * (f * e)$. Since $f * e \in K(f)$ and $g * g=e$ for all $g \in K(f)$, $f_2 * e=e$ and f_2 agrees with e on X'. Thus f_2 is the identity on X' and so $X' \subset J$. If w is a fixed point of f_2, then it is also a fixed point of f_{2n} for all n and a fixed point of any limit function g of $\{f_{2n}\}$, which would be in $K(f)$, so $w \in g(X)=e(X)=X'$. Thus $J \subset X'$ and so $J=X'$, an interval, and hence J is connected.

We note that if $K(f)=\{e\}$, then the sequence of iterates of f converges to e; while if $K(f)=\{e, k\}$, the odd iterates of f converge to k and the even iterates converge to e.

In Theorem 1 we considered maps with no fixed point, and in Theorem 3 we treated maps for which $\Gamma(f) \subset C(X, X)$. Our next theorem, as promised, shows that these two cases contain all the Γ-compact maps.

THEOREM 4. *If f is Γ-compact and has a fixed point, then* $\Gamma(f) \subset C(X, X)$.

Proof. In this proof we shall assume the falsity of the theorem, that is, that there is a $g \in \Gamma(f)$ which is not in $C(X, X)$, and then show that if f is Γ-compact and has a fixed point such a g cannot exist. Since by assumption $\Gamma(f) \not\subset C(X, X)$, $\Gamma(f)$ is not a semigroup and the kernel used in Theorem 3 is not available. Thus we will construct certain kernel-like elements from scratch; e will appear, along with the "inverse" h of $f_2 * e$, which, if $\Gamma(f)$ were a semigroup, would be e again.

Therefore we assume that there is a $g \in \Gamma(f) \backslash C(X, X)$. Then for some $z \in X$ and $z_0 \in X^* \backslash X$ we have $g(z)=z_0$. Since $C(X, X^*)$ is metrizable (see Lemma 1) there is a sequence $\{i_n\}$ of integers such that the sequence $\{f_{i_n}\}$ converges to g. Since $O(f) \subset C(X, X)$, g is not an iterate of f, so the sequence $\{i_n\}$ tends to infinity. Since the set of differences of elements of $\{i_n\}$ must be unbounded, and since by Lemma 1 any sequence of iterates of f has a convergent subsequence, it is notationally tedious but straightforward to produce sequences $\{j_n\}$ and $\{k_n\}$ such that $\{f_{j_n}\}$ converges to an element e of $\Gamma(f)$, $\{f_{j_n-2}\}$ converges to an element h of $\Gamma(f)$, $\{f_{k_n}\}$ converges to g, and $\{j_n+k_n\}$ is a subsequence of $\{i_n\}$. Using the fact that all functions are in $C(X, X^*)$ and are therefore continuous, and the fact that $f_{i+j}=f_i * f_j$, one is able to show that when $g(x) \in X$, $e(g(x))=g(x)$; and that $e(x)=h(f_2(x))$.

Since f has a fixed point, g must have the same one; it cannot be z, so g cannot be constant. Let $U=g(X) \cap X$. Since $g(X)$ is an interval containing z_0 and $g(X) \backslash U \subset X^* \backslash X$, U is an interval and U contains a nonempty neighborhood having z_0 as an endpoint. If $x \in U$, then $x=g(y)$ for some $y \in X$, so $x=g(y)=e(g(y))=e(x)$; thus e is the identity on U. For $x \in U$, we have $x=e(x)=h(f_2(x))$, so f_2 is one-to-one on U.

Now $f(X)\subset X$ so $f_n(X)\subset f_2(X)\subset f(X)\subset X$ for all $n\geqq 2$, and $f_n(X)^*\subset f_2(X)^*\subset f(X)^*$. Since z_0 is a limit point of $\{f_{i_n}(z)\}$ and $\{i_n\}$ is unbounded, $z_0\in f_n(X)^*$ for all n; in particular, $z_0\in f_2(X)^*\subset f(X)^*$. Now X is an interval and z_0 is one of its endpoints, while f is bounded away from z_0 on any compact subset of X. Thus for z_0 to be in $f(X)^*$, either the point (z_0, z_0) is a limit point of the graph of f, or else X has another "endpoint" z_1, which may be $\pm\infty$, with (z_1, z_0) a limit point of the graph of f. Similarly, either (z_0, z_0) or (z_1, z_0) must be a limit point of the graph of f_2. If (z_0, z_0) is a limit point of the graph of f, then it will be a limit point of the graph of f_2 as well; but if (z_0, z_0) is not a limit point of the graph of f, (z_1, z_0) cannot be a limit point of the graphs of both f and f_2. Thus in either case (z_0, z_0) must be a limit point of the graph of f_2. Now f_2 is one-to-one on U, z_0 is an endpoint of U, and (z_0, z_0) is a limit point of the graph of f_2, therefore f_2 is monotone on U and $\lim_{x\to z_0} f_2(x)=z_0$. Since U must contain all fixed points of f, f_2 has fixed points in U. If z_0 is a limit of fixed points of f_2, then we claim that f_2 must be the identity in a neighborhood V of z_0. For if f_2 were not the identity, we could find w_1 and w_2 arbitrarily close to z_0 which are fixed points of f_2 and are such that (w_1, w_2) contains no fixed point of f_2. But then by Lemma 2 all the iterates of f_2 would converge to either w_1 or w_2, and any limit function would not be continuous. Thus if z_0 is a limit of fixed points of f_2, then $x=f_2(x)$ when x is a member of some nonempty neighborhood V having z_0 as an endpoint, and $f_2(V)=V$. If z_0 is not a limit of fixed points, then there is a fixed point w of f_2 such that $|w-z_0|$ is minimized. By Theorem 2 we know that f_2 is precompact, so if z_0 is an upper endpoint of X, then $w<f(x)<x$ on $(w, z_0)=V$, while if z_0 is a lower endpoint of X, then $w>f_2(x)>x$ on $(z_0, w)=V$. In either of these cases, $f_2(V)=V$. Thus whether z_0 is a limit of fixed points or not, we have the general conclusion that

$$|f_2(x)-z_0| \geqq |x-z_0|$$

in a nonempty neighborhood V having z_0 as an endpoint and for which $f_2(V)=V$. Therefore $|f_{2n}(x)-z_0|\geqq|x-z_0|$ for all n and $x\in V$.

Now consider the sequence $\{f_{i_n}(z)\}$ which by assumption converges to $g(z)=z_0$. Since each $f_{i_n}(z)\in X$, we can choose i, j, k from $\{i_n\}$ so that $f_i(z), f_j(z), f_k(z)$ are members of V, $i<j<k$, and $|f_i(z)-z_0|>|f_j(z)-z_0|>|f_k(z)-z_0|$. But of the three numbers i, j, k, there must be a pair of them whose difference is even: say $j-i=2n$. Then $|f_i(z)-z_0|>|f_{i+2n}(z)-z_0|$, and letting $x=f_i(z)\in V$ we have

$$|f_{2n}(x)-z_0| < |x-z_0|.$$

But this directly contradicts our earlier finding; thus no sequence $\{f_{i_n}(z)\}$ can converge to $z_0\in X^*\backslash X$ when f has a fixed point and is Γ-compact; thus for these hypotheses we must have $\Gamma(f)\subset C(X, X)$.

COROLLARY 1. *If f is Γ-compact and $\Gamma(f)\not\subset C(X, X)$, then f has no fixed point and the conclusions of Theorem 1 apply.*

The following summarizes Theorems 1, 3, and 4 giving a result analogous to the results of Shields [12]:

COROLLARY 2. *If f is Γ-compact, $\Gamma(f)$ consists of $\Gamma(f) \cap C(X, X)$ plus possibly a constant in $X^*\backslash X$.*

Corollary 3 summarizes the first four theorems in terms of precompactness and fixed points of f_2. Theorem 5 in the next section is its converse.

COROLLARY 3. *If f is Γ-compact, then f_2 is precompact and the fixed-point set J of f_2 is connected.*

Proof. The map f_2 is precompact by Theorem 2. If f has no fixed point, then by Lemma 3 J is empty and hence connected. If f has a fixed point, then by Theorem 4 $\Gamma(f) \subset C(X, X)$, so by Theorem 3 J is connected.

A sufficient condition.

THEOREM 5. *If f_2 is precompact and the fixed-point set J of f_2 is connected, then f is Γ-compact.*

Proof. If J is empty, then Theorem 1 and Lemma 3 imply that f is Γ-compact if and only if f_2 is precompact. Thus we assume for the rest of the proof that J is nonempty.

We shall prove that $\Gamma(f)$ is compact by showing that any sequence of all odd or all even iterates of f converges uniformly on each compact subset C of X. Thus $K(f)$ will have at most two elements, in accordance with Theorem 3.

We first prove that f is either the identity on J or a self-inverse sense-reversing homeomorphism of J, again conforming to Theorem 3. For if $x \in f(J)$, then $x = f(y)$ for some $y \in J$, so $f_2(x) = f_2(f(y)) = f(f_2(y)) = f(y) = x$, so $x \in J$ and $f(J) \subset J$. Then $J = f_2(J) = f(f(J)) \subset f(J)$, so $f(J) = J$. Since $f(f(x)) = x$ for each $x \in J$, we can conclude that f is a self-inverse homeomorphism. If f were sense-preserving and $f(x) \neq x$ for some $x \in J$, then x would be contained in an open interval $(a, b) \subset J$ such that f was a fixed-point-free homeomorphism on (a, b). But then f_2 would be fixed-point free on (a, b) also, which contradicts the definition of J.

We define an auxiliary function F on X (see Figure 2) to be the minimum monotone function majorizing f to the left of J, equal to f on J, and the maximum monotone function majorized by f to the right of J. More formally, if x is below J, let p be the least element of J (which is closed), and define

$$F(x) = \max\{f(y) \mid x \leqq y \leqq p\};$$

if $x \in J$, let $F(x) = f(x)$; and if x is above J, let q be the greatest element of J, and define $F(x) = \min\{f(y) \mid q \leqq y \leqq x\}$. Then F is nonincreasing on each component of $X \backslash J$ and continuous on X. The graph of F is indicated by the heavy line in Figure 2.

We denote by $I(x)$ whichever of the sets $[x, F(x)]$ and $[F(x), x]$ is nonempty. In the remainder of the paragraph we show that for each x, the sets $f(I(x))$ and $I(f(x))$ are subsets of $I(x)$. When $x \in J$ it is clear that the three sets are equal. If $x \notin J$, then either $x<p$, where p is the least element of J, or $x>q$, where q is the greatest element of J. We will prove the inclusions when $x<p$; the arguments for $x>q$ are analogous. First we show that $f(I(x)) \subset I(x)$. Since $x<p$ and $f(J)=J$, it follows that $p \leqq f(p) \leqq F(x)$. By Theorem 2 f is precompact, so $f(x)>x$. Thus if $y \in I(x)$ there are three cases to consider: $x \leqq y<p$, $p \leqq y \leqq f(p)$, and $f(p)<y \leqq F(x)$. If $x \leqq y<p$, then $f(y)>y \geqq x$ and $f(y) \leqq F(y) \leqq F(x)$, so $f(y) \in I(x)$. If $p \leqq y \leqq f(p)$ and $f(p)=p$, then $f(y)=y \in I(x)$. But if $p \leqq y \leqq f(p)$ and $f(p) \neq p$, then f must be sense-reversing on J, so it follows that J has a greatest element q with $f(p)=q$ and $J=[p, f(p)] \subset I(x)$. Thus y is a member of J, so $f(y) \in f(J)=J \subset I(x)$. The third case is more complicated. If $f(p)<y \leqq F(x)$ and $f(p)=q$, then since f is precompact, $f(y)<y$; while if $f(p)=p$, then $f(y)=y$ for $y \in J$ and $f(y)<y$ when $y \notin J$. Thus in the third case we have $f(y) \leqq y \leqq F(x)$. Since F is continuous and $f(p)<y \leqq F(x)$, there is a z for which $x \leqq z \leqq p$ and $f(z)=F(z)=y$, so $f(y)=f(f(z))=f_2(z)$. Since f_2 is precompact, $f_2(z) \geqq z$, so $f(y) \geqq z \geqq x$ and $f(y) \in I(x)$. Thus $f(I(x)) \subset I(x)$. To show that $I(f(x)) \subset I(x)$, we note that using F instead of f as the basic function will leave the sets $I(x)$ unchanged, so we have shown that $F(I(x)) \subset I(x)$. But then $f(x)$ and $F(f(x))$ are both in $I(x)$, so $I(f(x)) \subset I(x)$.

Now we can consider the convergence of iterates on compact subsets of X. If such a compact subset C of X is given, let C_1 be a continuum containing C. The set $F(C_1)$ is also a continuum, so let C_2 be the minimum continuum containing C_1 and $F(C_1)$. Then $C_2=\bigcup \{I(x) \mid x \in C_1\}$, so since $f(I(x)) \subset I(x)$ we have $f(C_2) \subset C_2$. Thus without loss of generality we may assume that C_2 is the whole space; that is, that X is compact. Then J coincides with $J \cap C_2$ and is thus a closed finite interval, so we may write $J=[p, q]$.

Let an $\varepsilon>0$ be given; we wish to determine an N such that if $n, m>N$ then $|f_{2n}(x)-f_{2m}(x)|<\varepsilon$ and $|f_{2n+1}(x)-f_{2m+1}(x)|<\varepsilon$ for $x \in C \subset C_2$. We will define a set J_ε, containing J in its interior, such that if $f_{2N}(x) \in J_\varepsilon$ for all $x \in C_2$, then all successive even iterates are within ε of one another; the analogous condition will hold for $f_{2N+1}(x)$ and the odd iterates. The definition of J_ε will differ for the cases (i) $p<q$, and (ii) $p=q$. (i) When $p<q$, let $\varepsilon_1=\min(\varepsilon/2, (q-p)/2)$. Then since $f_2(p)=p$ and $f_2(q)=q$, we can choose a $\delta>0$ such that $|x-p| \leqq \delta$ implies $|f_2(x)-p|<\varepsilon_1$ and $|x-q| \leqq \delta$ implies $|f_2(x)-q|<\varepsilon_1$. Since $p+\delta \in J$, $f_2(p+\delta)=p+\delta$ so $\delta<\varepsilon_1$. Let $J_p=[p-\delta, p+\varepsilon_1]$, $J_q=[q-\varepsilon_1, q+\delta]$, and $J_\varepsilon = J_p \cup J \cup J_q$. Since $f_2(x)>x$ for $x<p$, $f_2(J_p \backslash J) \subset f_2([p-\delta, p+\delta]) \subset [p-\delta, p+\varepsilon_1]=J_p$. This inclusion plus the fact that $f_2(x)=x$ on J gives $f_2(J_p)=f_2(J_p \backslash J) \cup f_2(J_p \cap J) \subset J_p \cup (J_p \cap J)=J_p$. In a similar manner $f_2(J_q) \subset J_q$, so $f_2(J_\varepsilon) \subset J_\varepsilon$. (ii) When $p=q$, we can choose a $\delta>0$ such that $\delta \leqq \varepsilon/2$ and $|x-p| \leqq \delta$ implies $|F(x)-p|<\varepsilon/2$. Then $|x-p| \leqq \delta$ implies $I(x) \subset (p-\varepsilon/2, p+\varepsilon/2)$. Let $J_\varepsilon=I(p-\delta) \cup I(p+\delta)$. Then $f(J_\varepsilon)=f(I(p-\delta)) \cup f(I(p+\delta)) \subset I(p-\delta) \cup I(p+\delta)=J_\varepsilon$, so $f_2(J_\varepsilon) \subset f(J_\varepsilon) \subset J_\varepsilon$. Also if $x, y \in J_\varepsilon$ then $|x-y|<\varepsilon$.

For either case, $p<q$ or $p=q$, let $U_\varepsilon=(p-\delta, q+\delta)$. Since U_ε is an open set containing J and $f(J)=J$, $V_\varepsilon=f^{-1}(U_\varepsilon)\cap U_\varepsilon$ is an open set containing J also, and $J\subset V_\varepsilon\subset U_\varepsilon\subset J_\varepsilon$. Then $C_2\backslash V_\varepsilon$ is a compact set containing no fixed points of f_2, hence no fixed points of f, so we can choose a $\Delta>0$ such that $|f(x)-x|\geqq\Delta$ and $|f_2(x)-x|\geqq\Delta$ when $x\in C_2\backslash V_\varepsilon$. Let L_2 be the length of C_2 and let $L(x)$ be the length of $I(x)$, $L(x)=|F(x)-x|$. Then for $x\in C_2$ we have $I(x)\subset C_2$ so $L_2\geqq L(x)\geqq 0$. Since $I(f(x))\subset I(x)$, we know that $L(f(x))\leqq L(x)$. In the following paragraphs we prove that for all $x\in C_2$, either $f_2(x)\in J_\varepsilon$ or else $L(f_2(x))\leqq L(x)-\Delta$. Therefore if $x\in C_2$ and $N>L_2/\Delta$, then $f_{2N}(x)$ must be in J_ε, for otherwise $0\leqq L(f_{2N}(x))\leqq L(x)-N\Delta<L(x)-L_2\leqq 0$, a contradiction.

So let $x\in C_2$. If $x\in V_\varepsilon$, then $f(x)\in f(V_\varepsilon)\subset U_\varepsilon\subset J_\varepsilon$, and $f_2(x)\subset f_2(J_\varepsilon)\subset J_\varepsilon$. Thus if $f_2(x)$ is not in J_ε, then $x\in C_2\backslash V_\varepsilon$. There are two possibilities, either $x<p$ or $x>q$. We shall prove that either $f_2(x)\in J_\varepsilon$ or $L(f_2(x))\leqq L(x)-\Delta$ when $x\in C_2\backslash V_\varepsilon$ and $x<p$; the proof for $x>q$ is analogous. Since neither $f(x)$ nor $f_2(x)$ is in J, either may be either below p or above q. If $f(x)<p$, then $F(f(x))\leqq F(x)$ and $f(x)\geqq x+\Delta$, so $L(f(x))=F(f(x))-f(x)\leqq F(x)-(x+\Delta)=L(x)-\Delta$, and $L(f_2(x))\leqq L(f(x))\leqq L(x)-\Delta$. If $f_2(x)<p$, then $F(f_2(x))\leqq F(x)$ and $f_2(x)\geqq x+\Delta$, so $L(f_2(x))=F(f_2(x))-f_2(x)\leqq F(x)-(x+\Delta)=L(x)-\Delta$. The only remaining case is when $f(x)>q$ and $f_2(x)>q$. Then $f_2(x)=f(f(x))\leqq f(x)-\Delta$ and $F(f_2(x))\geqq F(f(x))$, so $L(f_2(x))=f_2(x)-F(f_2(x))\leqq (f(x)-\Delta)-F(f(x))=L(f(x))-\Delta\leqq L(x)-\Delta$. Thus if $n\geqq N>L_2/\Delta$ and $x\in C_2$, then $f_{2n}(x)$ and $f_{2n+1}(x)$ are members of J_ε.

In the case $p=q$, since $f_2(J_\varepsilon)\subset f(J_\varepsilon)\subset J_\varepsilon$ and $|x-y|<\varepsilon$ when $x, y\in J_\varepsilon$, it is clear that once $f_{2n}(x)$ or $f_{2n+1}(x)$ is in J_ε, then all successive iterates are also, hence all are within ε of one another. Thus the sequence $\{f_n\}$ converges uniformly to p when $J=\{p\}$, and $\Gamma(f)$ is compact.

If $p<q$, then $f_{2N}(x)\in J_\varepsilon$ and $f_{2N+1}(x)\in J_\varepsilon$ for all $x\in C_2$. Let M denote either $2N$ or $2N+1$. We have $f_M(x)\in J_\varepsilon=J_p\cup J\cup J_q$. If $f_M(x)\in J$, then since J is the fixed-point set of f_2, $f_2*f_M(x)=f_M(x)$ and $f_{M+2i}(x)=f_M(x)$ for all i. Thus for $i,j>0$, $|f_{M+2i}(x)-f_{M+2j}(x)|=0<\varepsilon$. If $f_M(x)\in J_p$, then since $f_2(J_p)\subset J_p$ we have $f_{2i}(J_p)\subset J_p$ for all $i>0$, $f_{M+2i}(x)\in J_p$. But if $y, z\in J_p$, then $|y-z|<2\varepsilon_1\leqq\varepsilon$, so $i,j>0$ implies $|f_{M+2i}(x)-f_{M+2i}(x)|<\varepsilon$. The same conditions hold if $f_M(x)\in J_q$, since $f_2(J_q)\subset J_q$. Thus if $n, m>N$ and $x\in C_2$, $|f_{2n}(x)-f_{2m}(x)|<\varepsilon$ and

$$|f_{2n+1}(x)-f_{2m+1}(x)| < \varepsilon,$$

so the sequences of odd and even iterates of f converge uniformly on each compact subset of X. Thus by Lemma 1 $\Gamma(f)$ is compact and f is Γ-compact, so the proof is complete.

Theorem 5 together with Corollary 3 yield the theorem quoted in the introduction:

THEOREM. *The map f is Γ-compact if and only if $f_2=f*f$ is precompact and the fixed-point set J of f_2 is connected.*

An alternative but somewhat less useful version of Theorem 5 is

COROLLARY 4. *If f is precompact and has a connected set of fixed points, and if there is a "square root" map g such that $g * g = f$, then f is Γ-compact.*

Proof. By the theorem $\Gamma(g)$ is compact, and $\Gamma(f)$ is a closed subset of $\Gamma(g)$, so $\Gamma(f)$ is compact.

When $\Gamma(f)$ is finite. If $\Gamma(f)$ is finite, then $O(f)$ must be finite, so $f_m = f_n$ for some m and n with $n \neq m$. Then $\Gamma(f) = O(f)$ and so $\Gamma(f)$ is automatically a compact semigroup. Since the iterates of f are not distinct, by Theorem 1 the map f must have a fixed point, so the conclusions of Theorem 3 apply, with $e = f_{2N}$ and $k = f_{2N+1}$ for some N. The method of proof of Theorem 5 can be adapted to the following theorem.

THEOREM 6. *If X is bounded and f is Γ-compact, then $\Gamma(f)$ is finite if and only if there is a neighborhood V of J and a $\delta > 0$ such that $f_2(V) \subset J$ and $|f_2(x) - x| \geqq \delta$ for $x \in X \setminus V$.*

Proof. Assume that f_2 satisfies the conditions of the theorem. If J were empty, then V would have to be empty also, and the bound δ would force X to be unbounded; thus J must be nonempty. If there is no $\Delta > 0$ such that $\Delta \leqq \delta$ and $|f(x) - x| \geqq \Delta$ when $x \in X \setminus V$, then since V is open and f has no fixed points outside of V, X must have an endpoint z_0 not in X such that (z_0, z_0) is a limit point of the graph of f. But then (z_0, z_0) would be a limit point of the graph of f_2 as well, contradicting the lower bound δ for $|f_2(x) - x|$ when $x \in X \setminus V$. Thus we may take a $\Delta > 0$ as a lower bound on $|f(x) - x|$ and $|f_2(x) - x|$ outside of V, let L_2 be the length of X, define $J_\varepsilon = V_\varepsilon = V$, and apply the proof of Theorem 5 to obtain an N such that $f_{2N}(X) \subset V$ and $f_{2N+1}(x) \subset V$. Then since $f_2(V) \subset J$, it follows that $f_{2N+2}(X) \subset J$ and $f_{2N+3}(X) \subset J$. Since f_2 is the identity on J, for $i \geqq N$ we have $f_{2i+2} = f_{2N+2}$ and $f_{2i+3} = f_{2N+3}$. Thus $\Gamma(f)$ is finite.

Conversely, if $\Gamma(f)$ is finite then $e = f_N$ for some N, so $f_N(X) = J$. Let n be the least integer such that $f_n(X) \subset J$, and let $Y = f_{n-1}(X)$; then $J \subset Y$, $f(Y) \subset J$, Y is an interval, and $Y \setminus J$ is nonempty. If $X \setminus J$ has only one component, or if both $X \setminus J$ and $Y \setminus J$ have two components, then since J is closed we can pick an open V such that $J \subset V \subset Y$, so $f(V) \subset J$ and $f_2(V) \subset J$. If $Y \setminus J$ has only one component and $X \setminus J$ has two, then $J = [p, q]$ and either p or q is an endpoint of $Y \setminus J$; without loss of generality we may assume it is q. Then there is an $\varepsilon > 0$ such that $[q, q+\varepsilon] \subset Y$, and we can choose a $\delta > 0$ such that $|x - p| \leqq \delta$ implies $x \in X$ and $f(x) \in [p-\varepsilon, q+\varepsilon]$. If $x \in [p-\delta, p]$, then since f is precompact, $f(x) \geqq x$; thus $f([p-\delta, p]) \subset [p-\delta, q+\varepsilon]$. Let $Y_0 = [p-\delta, q+\varepsilon]$; then $f(Y_0) \subset f([p-\delta, p] \cup Y) \subset Y_0 \cup J = Y_0$, so $f_n(Y_0) \subset Y_0$ for all n. Since $\Gamma(f)$ is finite and $f_N(X) = J$, there is a least n such that $f_n(Y_0) \subset J$; if $n = 1$ or 2, we can pick $V = (p-\delta, q+\varepsilon)$ and we are done. If not, let $Y_1 = f_{n-2}(Y_0)$, which is a compact interval containing J, so $Y_1 = [p', q']$ with $p' \leqq p$, $q' \geqq q$. The set $Y_1 \setminus Y$ cannot be empty since $f(Y) \subset J$, so $p' < p$; then we may pick $V = (p', q+\varepsilon) \subset Y \cup Y_1$, and $f_2(V) \subset f_2(Y) \cup f_2(Y_1) \subset J$.

If C is a compact subset of X, then f_2 is bounded away from the diagonal on $C\setminus V$, so the only problem arises if there is an endpoint z_0 of X not in X such that (z_0, z_0) is a limit point of the graph of f_2. But in that case it is easy to construct a sequence $\{x_i\}$ tending to z_0 such that $x_i \notin J$ and $f_2(x_i)=x_{i-1}$, so that some $f_{2n}(x_i)$ would not be in J for each n; then $\Gamma(f)$ would not be finite. Thus if $\Gamma(f)$ is finite, then f_2 is bounded away from the diagonal on $X\setminus V$.

An example. We illustrate Theorems 3, 5, and 6 using the function f shown in Figure 3. The map f is defined on $(0, 5)$, $J=[2, 3]$, and on J, f is symmetric with respect to the diagonal. Thus f is a self-inverse homeomorphism of J. The exact definition is $f(x)=(1-(x-2)^2)^{1/2}+2$ for $2\leqq x\leqq 3$. On $(0, 2]$ f is interpolated between the bounding functions α and β by a function τ with range $[0, 1]$:

$$f(x) = \alpha(x)\tau(x)+\beta(x)[1-\tau(x)], \qquad \tau(x) = \tfrac{1}{2}(1+\sin 5\pi/x).$$

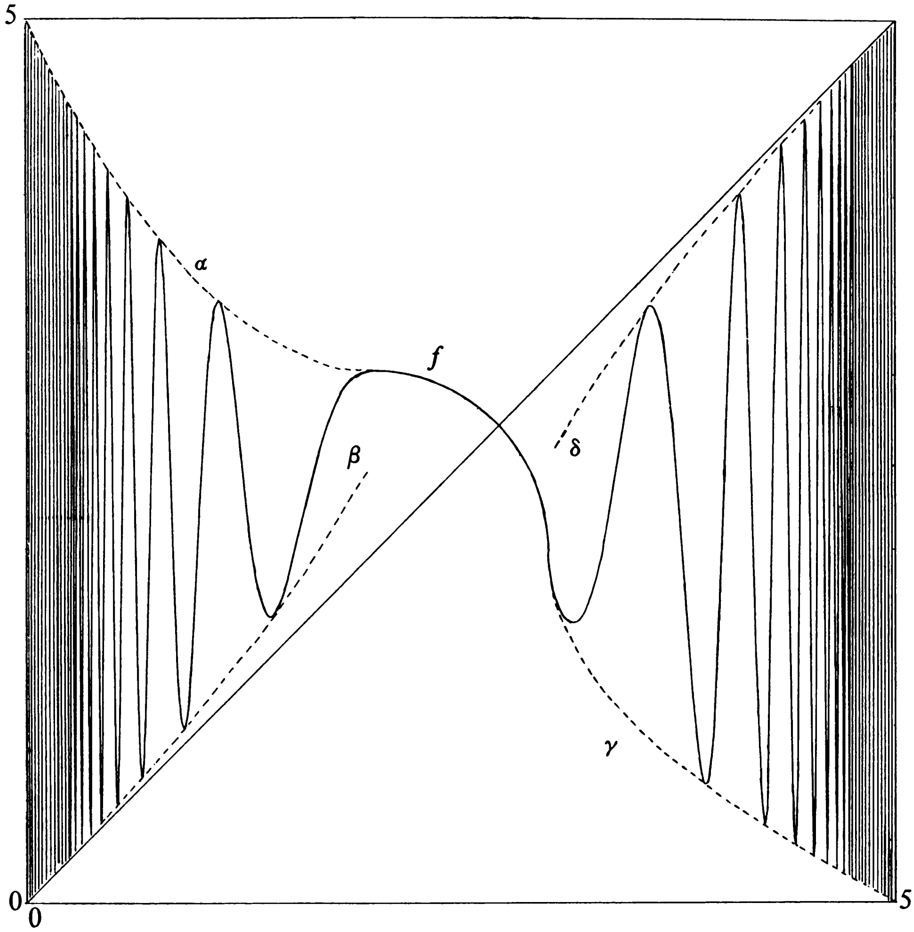

FIGURE 3

The function $\tau(x)$ equals 0 and 1 infinitely often for x near 0; it is difficult to graph f there, so we have shaded the general area instead. The function α passes through (0, 5) and β through (0, 0), so the whole interval $\{(0, y) \mid 0 \leqq y \leqq 5\}$ is in the closure of the graph of f. On [3, 5) f is similarly interpolated between γ and δ:

$$f(x) = \gamma(x)\tau(5-x) + \delta(x)[1-\tau(5-x)]$$

and the interval $\{(5, y) \mid 0 \leqq y \leqq 5\}$ consists of limit points of the graph.

$$\alpha(x) = 3+\tfrac{1}{2}(x-2)^2, \qquad \beta(x) = x^2/10+x,$$
$$\gamma(x) = 2-(2x-6)^{1/2}, \qquad \delta(x) = (5-x)^2/10+10-x.$$

Figure 4 shows the function $f_2 = f * f$. We claim that f_2 can have no fixed points outside of J. If $f_2(x) = x$, then letting $y = f(x)$, since $f(y) = x$ both the points (x, y) and (y, x) must lie on the graph G of f. Since (x, y) and (y, x) are symmetric with respect to the diagonal, if we reflect G through the diagonal to obtain a set G', the point (x, y) must lie in the intersection of G and G'. Figure 5 shows G as a solid line,

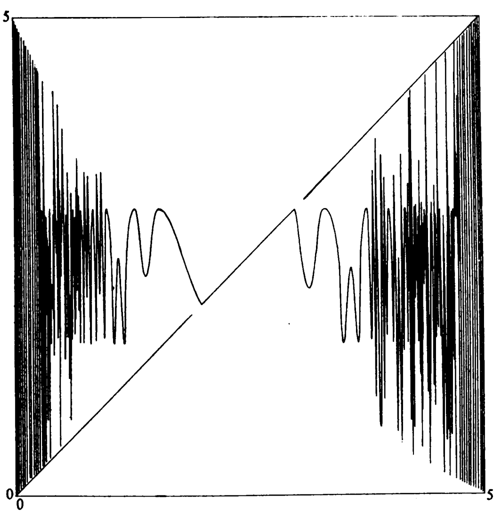

FIGURE 4

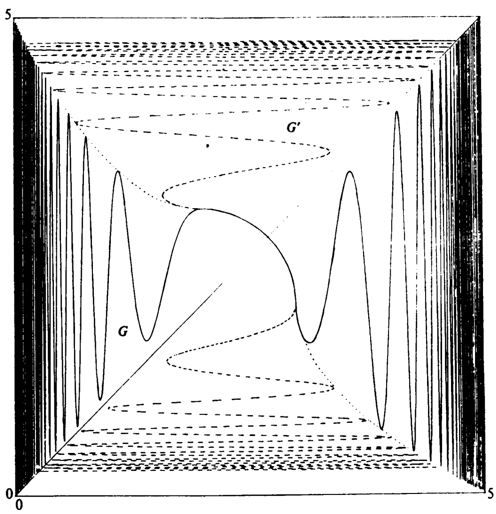

FIGURE 5

G' as a dotted line. We see that the bounding functions α and γ are symmetric with respect to the diagonal, and that if (x, y) and (y, x) are both on G, one of them, say (x, y), must lie on the graph of α, and the other on the graph of γ; thus

$$y = f(x) = \alpha(x) = 3+\tfrac{1}{2}(x-2)^2, \qquad x = f(y) = \gamma(y).$$

On the other hand, if $f(x) = \alpha(x)$, then $\tau(x) = 1$ and $\sin 5\pi/x = 1$; similarly, if $f(y) = \gamma(y)$, then $\tau(5-y) = 1$ and $\sin 5\pi/(5-y) = 1$. Thus x and y must satisfy

$$\sin 5\pi/x = 1, \qquad y = 3+\tfrac{1}{2}(x-2)^2, \qquad \sin 5\pi/(5-y) = 1.$$

We show that this cannot occur outside of [2, 3], as follows: if $\sin 5\pi/x=1$, then $5\pi/x=2\pi n+\pi/2$, and $x=10/(4n+1)$. Similarly, if $\sin 5\pi/(5-y)=1$, then $y=5-10/(4m+1)=5-z$. Then $3+\frac{1}{2}(x-2)^2=y=5-z$ so $4x-x^2=2z$. Substituting $x=10/(4n+1)$ and $z=10/(4m+1)$ gives the expression

$$2m = n+(3n+2)/(8n-3).$$

The right-hand side is integral only when $n=1$, which gives $x=2\in[2,3]$. Thus f_2 has no fixed points outside of J. However, since $f(x)$ equals $\alpha(x)$ infinitely often near 0, and $f(x)$ equals $\gamma(x)$ infinitely often near 5, G and G' get arbitrarily close near (0, 5) and (5, 0) and share those limit points. Thus f_2 would get arbitrarily close to the diagonal near 0 and 5 even if the graph of f was bounded away from the diagonal outside of J.

Since f_2 has the fixed-point set [2, 3] and is thus precompact, we can apply Theorems 3 and 5, which tell us that the odd iterates of f will converge to k and the even iterates to e. Figures 6 and 7 show these functions, or at least a computer-plotted approximation of them.

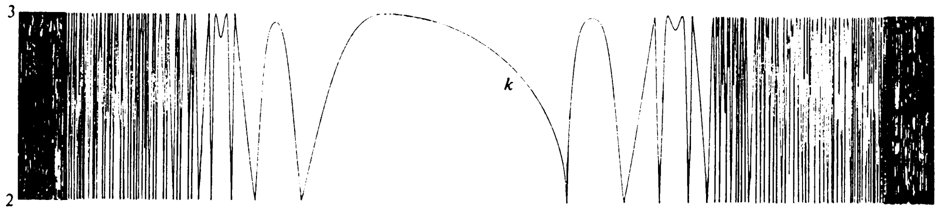

FIGURE 6

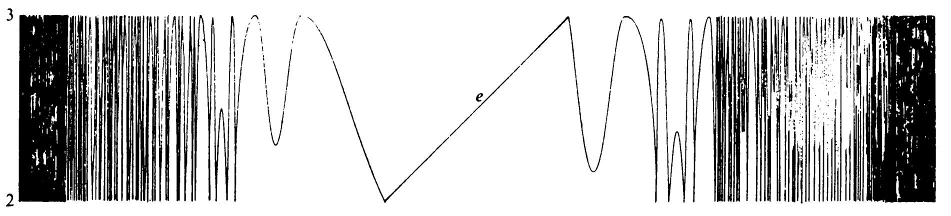

FIGURE 7

To apply Theorem 6 we would like to have X bounded, $f_2(V)\subset J$, and f_2 bounded away from the diagonal outside V. Of course X is bounded, and Figure 4 shows that there is a V for which $f_2(V)\subset J$, but f_2 approaches the diagonal near 0 and 5, so by Theorem 6, $\Gamma(f)$ will not be finite. But if we pick an $x\in(0,2)$ such that $\tau(x)=1$, that is, $x=10/(4n+1)$ for some n, then $I(x)=[x,\alpha(x)]$ and f maps $I(x)$ into itself. If we take $X=I(x)$, then f_2 is bounded away from the diagonal on X, $\Gamma(f)$ is finite by Theorem 6, and $e=f_{2N}$, $k=f_{2N+1}$ for some N. For $X=(0,5)$, $\Gamma(f)$ is infinite only because for large i, the maps f_{2i} and e fail to agree for values very close to 0 and 5.

Applications to commuting functions. Two maps $f, g \in C(X, X)$ are said to *commute* if $f * g = g * f$; that is, if $f(g(x)) = g(f(x))$ for all $x \in X$. A long-standing conjecture was that commuting maps on a compact interval must have a fixed point in common, but recent examples [2], [7] have shown that there need not be a common fixed point unless additional restrictions are placed on f or g. Since the examples are rather complicated, they give few clues as to how mild the restrictions might be. Previous work has shown that if the maps are polynomials [11], [1], or if one of them is full ($f(U)$ is open when U is) [5], [8] or contracting [3], there must be a common fixed point. If the functions are differentiable or have a finite number of extrema, however, the problem is still open.

In [12] Shields considered the case where the space is the unit disc D and f and g are commuting maps from D into D. He was able to show that if f is analytic on the interior of D, then f and g have a common fixed point. A key point in his proof was that under such circumstances f must be Γ-compact. There seems to be a common belief that Shields' results imply that commuting analytic functions on an interval must have a common fixed point, but his methods are not applicable to maps on the unit interval analytic on the interior since such maps need not be Γ-compact. The question of whether such maps must have a common fixed point is still undecided.

The present paper originated in an effort to apply Shields' topological semigroup method to maps on the unit interval. The following conclusions are obtained; as previously noted, they have little or no relevance to analyticity of the maps. For completeness we first prove a lemma which is a special case of the results of Folkman [5] and Joichi [8] on full maps.

LEMMA 4. *If Y is a compact interval and f and g commute and map Y into Y, and if f is a homeomorphism, then f and g have a common fixed point.*

Proof. The map f is sense-reversing or sense-preserving; if the former f has exactly one fixed point p. Since $f(g(p)) = g(f(p)) = g(p)$, $g(p)$ is also a fixed point of p, so $g(p) = p = f(p)$. If f is sense-preserving, let q be the greatest fixed point of the map $h = f * g = g * f$. Then if $h(x) = x$, $f(x) = f(h(x)) = f(g(f(x))) = h(f(x))$, so $f(x)$ is a fixed point of h, and so $f(q) = q$. Then $g(q) = g(f(q)) = h(q) = q$.

THEOREM 7. *If f and g are commuting members of $C(X, X)$, if X is compact, and if the fixed-point set J of f_2 is connected, then f and g have a common fixed point.*

Proof. If $x \in J$, then $g(x) = g(f_2(x)) = f_2(g(x))$, so $g(J) \subset J$. We have previously seen that f is a homeomorphism on J. Thus by Lemma 4, f and g have a common fixed point in J, hence in X.

The generalization of Shields' theorem, Corollary 6, is equivalent by Lemma 1 to Corollary 5, which is similar to an announcement of Mitchell [10].

COROLLARY 5. *If f and g are commuting members of $C(X, X)$, X is compact, and the iterates of f are equicontinuous, then f and g have a common fixed point.*

COROLLARY 6. *If f and g are commuting members of $C(X, X)$, X is compact, and f is Γ-compact, then f and g have a common fixed point.*

Proof. Apply Theorems 3 and 7.

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