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EXPANDABILITY AND COLLECTIONWISE NORMALITY

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Abstract. In 1958 M. Katětov proved that in a normal space X, X is expandable if and only if X is collectionwise normal and countably paracompact. This result has since been used to answer many questions in various areas of general topology. In this paper Katětov's theorem is generalized for nonnormal spaces and various characterizations of collectionwise normality are shown. Results concerning metrization, paracompactness, sum theorems, product theorems, mapping theorems and M-spaces are then obtained as applications of these new theorems.

Introduction. In [14] L. Krajewski investigated the property of expanding locally finite collections to open locally finite collections and obtained various results relating this property with certain topological covering properties, metrization theorems, sum theorems and product theorems. In §1 we summarize the known results concerning expandability, collectionwise normality and these topological covering properties; and then we show that every metacompact space is almost expandable. In §2 we introduce various generalizations of the notion of expandability and give characterizations for collectionwise normality. The theorem of Katětov [13] (Theorem 1.4 below) is then generalized. Characterizations of expandability properties in terms of open covers are given in §3; and in §4 we characterize the properties of paracompactness, subparacompactness, and metacompactness and establish equivalences in certain expandable spaces. Various mapping theorems are proved in §5, and product theorems are obtained in §6. We obtain results concerning subspaces and sum theorems in §7. As applications of the previous results, metrization theorems are observed in §8. Examples and unanswered questions are given in §9.

REMARK. A space X is *subparacompact* (F_σ-screenable) if every open cover has a σ-discrete closed refinement. The symbol $|A|$ will be used to denote the cardinality of the point set A. A space X will be a T_1 topological space unless otherwise stated.

1. **Preliminary results.**

The following definition is due to L. Krajewski [14].

DEFINITION 1.1. A space X is called *m-expandable*, where m is an infinite cardinal, if for every locally finite collection $\{F_\alpha : \alpha \in A\}$ of subsets of X with $|A| \leq m$, then

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there exists a locally finite open collection of subsets $\{G_\alpha : \alpha \in A\}$ such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in A$. X is called *expandable* if X is m-expandable for all cardinals m.

REMARK. We will refer to collections with the above property as being "expandable" or "expandable to locally finite open collections". We may also assume the collection $\{F_\alpha : \alpha \in A\}$ to be a collection of closed sets as noted by Remark 2.3 of [14].

DEFINITION 1.2. A locally finite collection $\mathscr{F}=\{F_\alpha : \alpha \in A\}$ is called *bounded locally finite* if there exists an integer $n>0$ such that each point $x \in X$ has an open neighborhood $N(x)$, such that $N(x)$ intersects at most n members of $\mathscr{F}$.

It is easy to show that $\mathscr{F}=\{F_\alpha : \alpha \in A\}$ is bounded locally finite if and only if $\bar{\mathscr{F}}=\{\bar{F}_\alpha : \alpha \in A\}$ is bounded locally finite.

DEFINITION 1.3. A space X is called *boundedly expandable* if every bounded locally finite collection of subsets of X is expandable. We note at this point that *m-boundedly expandable* (as well as all other cardinality-dependent definitions which naturally follow) are defined analogously as above and hence will be omitted.

The following theorem is due to Katětov [13].

THEOREM 1.4. *Let X be a normal space. Then*
(i) *X is collectionwise normal iff X is boundedly expandable,*
(ii) *X is expandable iff X is collectionwise normal and countably paracompact.*

DEFINITION 1.5. A space X is called *almost expandable* if every locally finite collection of subsets of X is expandable to a point finite open collection. X is *almost discretely expandable* if every discrete collection can be expanded to an open point finite collection.

THEOREM 1.6. (i) *Every paracompact space is expandable.*
(ii) *Every metacompact space is almost expandable.*
(iii) *Every subparacompact expandable space is paracompact.*
(iv) *Every subparacompact almost expandable space is metacompact.*
(v) *A space X is countably paracompact iff X is $\aleph_0$-expandable.*
(vi) *A space X is countably metacompact iff X is almost $\aleph_0$-expandable.*

Proof. Parts (i), (iii), (v) and (vi) are proved in [14].

(ii) Let $\{F_\alpha : \alpha \in A\}$ be a locally finite collection of subsets of X. For each $x \in X$, let $N(x)$ be an open neighborhood of x which intersects only finitely many F_α. Define $\mathscr{N}=\{N(x) : x \in X\}$. Since X is metacompact, $\mathscr{N}$ has a point finite open refinement $\mathscr{V}=\{V_\delta : \delta \in D\}$. Define $G_\alpha=\mathrm{St}\,(F_\alpha, \mathscr{V})=\bigcup\{V \in \mathscr{V} : V \cap F_\alpha \neq \varnothing\}$ for each $\alpha \in A$. Then $F_\alpha \subseteq G_\alpha$ for each $\alpha \in A$ and $\{G_\alpha : \alpha \in A\}$ is point finite. Hence X is almost expandable.

(iv) This is a corollary to Theorem 4.3 below.

EXAMPLE. Let Ω denote the first uncountable ordinal and $X=[0, \Omega)$ with the usual order topology. It is well known that X is countably compact and collectionwise normal and hence expandable and almost expandable. X is neither paracompact, subparacompact, nor metacompact however.

2. **Discrete expandability and H. C. expandability.**

DEFINITION 2.1. A space X is called *discretely expandable* if every discrete collection of subsets of X is expandable to a locally finite open collection.

REMARK. We note here that discrete expandability is a somewhat weaker condition than collectionwise normal. In fact any nonnormal finite space is expandable and hence discretely expandable. We also note that every normal space is discretely $\aleph_0$-expandable. The following gives a characterization for collectionwise normal spaces.

THEOREM 2.2. *Let X be a normal space. Then X is m-collectionwise normal iff X is discretely m-expandable.*

Proof. The sufficiency is clear. Let $\{F_\alpha : \alpha \in A\}$ be a discrete collection of closed subsets of X with $|A| \leqq m$. Since X is m-discretely expandable there exists a locally finite open collection $\{G_\alpha : \alpha \in A\}$ such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in A$. Define $K_\alpha = G_\alpha - \bigcup_{\beta \neq \alpha} F_\beta$ so that $F_\alpha \subseteq K_\alpha$ for each $\alpha \in A$. Since X is normal, there exists an open collection $\{V_\alpha : \alpha \in A\}$ such that $F_\alpha \subseteq V_\alpha \subseteq \overline{V}_\alpha \subseteq K_\alpha$ for each $\alpha \in A$. Define $U_\alpha = V_\alpha - \bigcup_{\beta \neq \alpha} \overline{V}_\beta$ for each $\alpha \in A$. Then $\{U_\alpha : \alpha \in A\}$ is a mutually disjoint open collection such that $F_\alpha \subseteq U_\alpha$ for each $\alpha \in A$. Hence X is m-collectionwise normal.

COROLLARY 2.3. *Let X be a normal space. Then X is collectionwise normal iff X is discretely expandable.*

The next definition is due to Lašnev [15].

DEFINITION 2.4. A collection of subsets $\{F_\alpha : \alpha \in A\}$ is called *hereditarily conservative* (H.C.) if every collection $\{V_\alpha : \alpha \in A\}$, such that $V_\alpha \subseteq F_\alpha$ for every $\alpha \in A$, is closure preserving.

DEFINITION 2.5. A space X is called *H.C. expandable* if every locally finite collection of subsets of X can be expanded to a hereditarily conservative open collection. X is called *discretely H.C. expandable* if every discrete collection of subsets of X can be expanded to a hereditarily conservative open collection.

Clearly expandable implies H.C. expandable and discretely expandable implies discretely H.C. expandable. We also note that the property of expanding collections to open closure preserving collections is useless, since each set can be expanded to the whole space itself in every case.

THEOREM 2.6. *In a normal space X the following are equivalent*:

(i) *X is collectionwise normal,*
(ii) *X is boundedly expandable,*
(iii) *X is discretely expandable,*
(iv) *X is discretely H.C. expandable.*

Proof. The proof that (iv) implies (i) follows the same argument as in Theorem 2.2 above.

A natural question now arises as to whether the last three properties in the preceding theorem are equivalent in a nonnormal space. The fact that (iii) and (iv) are not equivalent is given in §9.

THEOREM 2.7. *A space X is boundedly expandable iff X is discretely expandable.*

Proof. The sufficiency is clear. Suppose X is discretely expandable. The proof is by induction on n.

(i) For $n=1$, every 1-bounded locally finite closed collection is discrete and hence expandable.

(ii) Assume that for $i=1, 2, \ldots, n-1$, every locally finite closed collection $\mathscr{F}=\{F_\alpha : \alpha \in A\}$ with $\sup\{\text{ord}(x, \mathscr{F}) : x \in X\} \leqq i$ is expandable.

(iii) Suppose $\mathscr{F}=\{F_\alpha : \alpha \in A\}$ is a locally finite closed collection with

$$\sup\{\text{ord}(x, \mathscr{F}) : x \in X\} = n.$$

We note that for every $x \in X$, $N(x)=X-\bigcup\{F \in \mathscr{F} : x \notin F\}$ is an open neighborhood of x which intersects at most n members of $\mathscr{F}$. Now define

$$\mathscr{B}_n = \{B \subseteq A : |B| = n\},\ K_B=\bigcap_{\beta \in B} F_\beta \text{ and } \mathscr{K}_n=\{K_B : B \in \mathscr{B}_n\}.$$

Then $\mathscr{K}_n$ is a discrete closed collection in X; for if $N(x)$ intersects two distinct members of $\mathscr{K}_n$, then $N(x)$ must intersect at least $n+1$ distinct members of $\mathscr{F}$. Since X is discretely expandable there exists an open locally finite collection $\mathscr{G}_n=\{G_B : B \in \mathscr{B}_n\}$ such that $K_B \subseteq G_B$ for each $B \in \mathscr{B}_n$. Define $G=\bigcup_{B\in \mathscr{B}_n} G_B$ and $\mathscr{F}^*=\{F_\alpha - G : \alpha \in A\}$ so that $\sup\{\text{ord}(x, \mathscr{F}^*) : x \in X\} \leqq n-1$. By the induction assumption (ii) above, there exists a locally finite open collection $\mathscr{V}=\{V_\alpha : \alpha \in A\}$ such that $F_\alpha - G \subseteq V_\alpha$ for each $\alpha \in A$. Now define $G_\alpha = V_\alpha \cup \{\bigcup G_B : K_B \subseteq F_\alpha\}$ so that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in A$. Since $\mathscr{V}$ and $\mathscr{G}_n$ are locally finite, for each $x \in X$ there exists an open neighborhood $U(x)$ of x such that $U(x)$ intersects only finitely many members of $\mathscr{V}$ and $\mathscr{G}_n$. Also for fixed $B \in \mathscr{B}_n$, $K_B \subseteq F_\alpha$ if and only if $\alpha \in B$. Therefore $U(x)$ intersects only finitely many G_α, and hence $\{G_\alpha : \alpha \in A\}$ is locally finite. Thus $\mathscr{F}$ is expandable and X is boundedly expandable.

REMARK. It should be noted at this point that expanding a bounded locally finite collection to an open H.C. collection is equivalent to discrete H.C. expandability by an argument similar to that for the previous theorem. It is also easy to show that expanding a bounded locally finite collection to a point finite open collection is equivalent to almost discrete expandability. We now have a generalization of Katětov's theorem.

THEOREM 2.8. (i) *A space X is expandable iff X is discretely expandable and countably paracompact.*

(ii) *A space X is almost expandable iff X is almost discretely expandable and countably metacompact.*

Proof. (i) By part (v) of Theorem 1.6 above the sufficiency is clear. Let $\mathscr{F}=\{F_\alpha : \alpha \in A\}$ be a locally finite closed collection of subsets of X. We show that $\mathscr{F}$ is expandable to an open locally finite collection. For each integer $n \geqq 0$, define $S_n=\{x \in X : \operatorname{ord}(x, \mathscr{F}) \leqq n\}$. It is easy to show that $\{S_n : n=0, 1, 2, \ldots\}$ is a countable open cover of X. Since X is countably paracompact and $S_n \subseteq S_{n+1}$ for each n, there exists by Theorem 5 of [16], a locally finite open cover of X, $\{U_n : n = 0, 1, 2, \ldots\}$, such that $U_n \subseteq \overline{U}_n \subseteq S_n$ for each n. Again since X is countably paracompact there exists a locally finite open collection $\{V_n : n=0, 1, 2, \ldots\}$ such that $U_n \subseteq \overline{U}_n \subseteq V_n \subseteq S_n$ for each n. Now define $\mathscr{F}_n=\{\overline{U}_n \cap F : F \in \mathscr{F}\}=\{F(n, \alpha) : \alpha \in A\}$ so that $\mathscr{F}_n$ is a bounded locally finite closed collection in X. Since X is discretely expandable and hence boundedly expandable, there exists a locally finite collection $\mathscr{G}_n=\{G(n, \alpha) : \alpha \in A\}$ such that $F(n, \alpha) \subseteq G(n, \alpha)$ for each $\alpha \in A$ and $G(n, \alpha) \subseteq V_n$ for each n. Now define $G_\alpha=\bigcup_{n=1}^{\infty} G(n, \alpha)$ for each $\alpha \in A$ so that $F_\alpha \subseteq G_\alpha$ and $\{G_\alpha : \alpha \in A\}$ is locally finite. Therefore X is expandable.

(ii) By part (vi) of Theorem 1.6 above the sufficiency is clear. Let $\mathscr{F} = \{F_\alpha : \alpha \in A\}$ be a locally finite closed collection in X and $\{S_n : n=0, 1, 2, \ldots\}$ be defined as in part (i) above. Since X is countably metacompact there exists a point finite open cover $\{V_n : n=0, 1, 2, \ldots\}$ of X such that $V_n \subseteq S_n$ for each n. Define $H_n=X-\bigcup_{k>n} V_k$ so that H_n is a closed set such that $H_n \subseteq \bigcup_{j=0}^{n} V_j \subseteq S_n$ for each n. We observe that $\{H_n^*=H_n \cap V_n : n=0, 1, 2, \ldots\}$ is a point finite cover of X such that $\overline{H}_n^* \subseteq H_n \subseteq S_n$ for each n. If $x \in X$, then choose the largest i such that $x \in V_i$. Then $x \notin \bigcup_{k>i} V_k$ and hence $x \in H_i \cap V_i=H_i^*$. As in part (i) above,

$$\mathscr{F}_n = \{F(n, \alpha) = \overline{H}_n^* \cap F_\alpha : \alpha \in A\}$$

is a bounded locally finite collection. Since X is almost discretely expandable there exists, for each n, a point finite open collection $\mathscr{G}_n=\{G(n, \alpha) : \alpha \in A\}$ such that $F(n, \alpha) \subseteq G(n, \alpha)$ for each $\alpha \in A$. Define $G_n=\bigcup_{\alpha \in A} G(n, \alpha)$ so that

$$\mathscr{G} = \{G_n : n = 1, 2, \ldots\} \cup \left\{X-\bigcup_{\alpha \in A} F_\alpha\right\}$$

is a countable open cover of X. Again since X is countably metacompact, $\mathscr{G}$ has a point finite open refinement $\{G_n^* : n=0, 1, 2, \ldots\}$. Finally

$$\left\{U_\alpha = \bigcup_{n=1}^{\infty} [G(n, \alpha) \cap G_n^*] : \alpha \in A\right\}$$

is a point finite open collection such that $F_\alpha \subseteq U_\alpha$ for each $\alpha \in A$. Therefore X is almost expandable.

3. **Characterizations in terms of open covers.**

DEFINITION 3.1. An open cover $\mathscr{U}$ is called an *A-cover* (A_σ-cover) if $\mathscr{U}$ has a locally finite (σ-locally finite), not necessarily open, refinement.

REMARK. Every countable open cover is an A-cover [12] and clearly every A-cover is an A_σ-cover. There are A_σ-covers which are not A-covers. Let X be any regular subparacompact space that is not paracompact (Example 4.4 of [5]). Then every open cover of X is an A_σ-cover but not every open cover is an A-cover, since then X would be paracompact by a theorem of E. Michael [17].

The following lemma is due to Katětov [13].

LEMMA 3.2. *A space X is expandable if and only if for every locally finite collection $\mathscr{F}=\{F_\alpha \mid \alpha\in A\}$ there is a locally finite open cover $\mathscr{U}$ of X such that each element of $\mathscr{U}$ intersects only finitely many elements of $\mathscr{F}$.*

THEOREM 3.3. *A space X is expandable if and only if every A_σ-cover has an open locally finite refinement.*

Proof. (i) Let X be expandable. If $\mathscr{U}$ is an A_σ-cover, then $\mathscr{U}$ has a σ-locally finite open refinement. Since X is countably paracompact, $\mathscr{U}$ has a locally finite open refinement by E. Michael [17].

(ii) Let $\{F_\alpha : \alpha\in A\}$ be a locally finite collection of closed subsets of X. Define $\mathscr{B}_n=\{B\subseteq A : |B|=n\}$ for $n=1, 2, \ldots$, $\mathscr{U}_n=\{X-\bigcup_{\alpha\notin B} F_\alpha : B\in\mathscr{B}_n\}$ and $\mathscr{U}_0=\{X-\bigcup_{\alpha\in A} F_\alpha\}$. Then $\mathscr{U}=\bigcup_{i=0}^\infty \mathscr{U}_i$ is an open cover of X such that for $U\in\mathscr{U}$, U intersects only finitely many members of $\mathscr{F}$. Now define

$$H_n = \{x\in X : \operatorname{ord}(x, \mathscr{F}) = n\}$$

for $n=1, 2, \ldots$ and $\mathscr{V}_0=\mathscr{U}_0$. For each n, let $\mathscr{V}_n=\{[\bigcap_{\alpha\in B} F_\alpha]\cap H_n : B\in\mathscr{B}_n\}$. Since $\mathscr{F}$ is locally finite, $\mathscr{V}=\bigcup_{i=0}^\infty \mathscr{V}_i$ is a σ-locally finite refinement of $\mathscr{U}$; so that $\mathscr{U}$ has a locally finite open refinement. Therefore X is expandable by Lemma 3.2 above.

COROLLARY 3.4. *The following are equivalent:*

(i) *X is collectionwise normal and countably paracompact,*

(ii) *X is normal and every A_σ-cover has a locally finite open refinement,*

(iii) *X is normal and every A-cover has a locally finite open refinement.*

Proof. Katuta [12] showed the equivalence of (i) and (iii). The fact that (i) is equivalent to (ii) follows from Theorem 3.3 and Theorem 1.4 above.

DEFINITION 3.5. An open covering $\mathscr{U}$ is called a *B-covering* if $\mathscr{U}$ has a bounded locally finite refinement.

LEMMA 3.6. *A space is discretely expandable if and only if for every discrete collection $\mathscr{F}=\{F_\alpha : \alpha\in A\}$ there is a locally finite open cover $\mathscr{U}$ of X such that each member of $\mathscr{U}$ intersects at most one member of $\mathscr{F}$.*

Proof. The proof follows the same argument as that of Lemma 3.2 above.

THEOREM 3.7. *A space X is discretely expandable if and only if every B-cover has a locally finite open refinement.*

Proof. The proof follows from a simple modification of the proof of Theorem 3.3 above.

COROLLARY 3.8 (KATUTA [12]). *A space X is collectionwise normal if and only if X is normal and every B-cover has an open locally finite refinement.*

4. Characterizations of paracompact, subparacompact and metacompact spaces.

The following definition is due to J. Worrell and H. Wicke [27].

DEFINITION 4.1. A space X is *θ-refinable* if every open cover of X has a refinement $\mathscr{U}=\bigcup_{i=1}^{\infty}\mathscr{U}_i$ satisfying the following:

(i) $\mathscr{U}_i$ is an open cover of X for each i,

(ii) for each $x\in X$ there exists a positive integer $n(x)$ such that x belongs to only finitely many members of $\mathscr{U}_{n(x)}$.

Clearly every metacompact space is θ-refinable.

E. Michael [19] and D. Burke [5] have shown that in a collectionwise normal space the properties of paracompactness, subparacompactness, and metacompactness are equivalent. We generalize this to the following

THEOREM 4.2. *Let X be a regular space. If X is discretely H.C. expandable the following are equivalent*:

(i) *X is paracompact,*

(ii) *X is subparacompact,*

(iii) *X is metacompact,*

(iv) *X is θ-refinable.*

Proof. It is clear that (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv). Also (ii) $\Rightarrow$ (iv) by Theorem 1.6 of [6]. We show (iv) $\Rightarrow$ (i) by a method similar to that of E. Michael [19]. Let $\mathscr{G}=\{G_\alpha : \alpha\in A\}$ be an open cover of X and $\mathscr{U}=\bigcup_{i=1}^{\infty}\mathscr{U}_i$ be a refinement of $\mathscr{G}$ satisfying properties (i) and (ii) of Definition 4.1. Now for each i we construct a sequence $\{\mathscr{G}(i,j) : j=0,1,\ldots\}$ of open collections such that

(1) $\mathscr{G}(i,j)$ is hereditarily conservative for each j,

(2) each member of $\mathscr{G}(i,j)$ is a subset of some member of $\mathscr{U}_i$,

(3) if $x\in X$ and x belongs to at most m members of $\mathscr{U}_i$, then

$$x\in\bigcup\{G\in\mathscr{G}(i,j) : 0\leqq j\leqq m\},$$

(4) if $x\in\bigcup\{G : G\in\mathscr{G}(i,j)\}$ then x belongs to at least j members of $\mathscr{U}_i$.

The proof is by induction on j. Define $\mathscr{G}(i,0)=\varnothing$ and assume $\mathscr{G}(i,j)$ has been constructed satisfying (1)–(4) above for $0\leqq j\leqq n$. We now construct $\mathscr{G}(i,n+1)$.

Let $\mathscr{U}_i=\{U_\alpha : \alpha\in A_i\}$, $\mathscr{B}=\{B\subseteq A_i : |B|=n+1\}$ and $G(i,j)=\bigcup\{G : G\in\mathscr{G}(i,j)\}$. Define $F(B)=[X-\bigcup_{j=0}^{n}G(i,j)]\cap[X-\bigcup\{U_\alpha : \alpha\in A_i-B\}]$ for each $B\in\mathscr{B}$. Then $\mathscr{F}=\{F(B) : B\in\mathscr{B}\}$ is a closed collection. We assert that $\mathscr{F}$ is discrete. Let $x\in X$.

Case 1. x belongs to more than $n+1$ members of $\mathscr{U}_i$, say $U(i,1), U(i,2),\ldots, U(i,k)$. Then $U(x)=\bigcap_{j=1}^{k}U(i,j)$ is an open neighborhood of x which intersects no member of $\mathscr{F}$.

Case 2. x belongs to less than $n+1$ members of $\mathscr{U}_i$. From (3) above

$$x\in\bigcup\{G\in\mathscr{G}(i,j) : 0\leqq j\leqq n\}$$

which intersects no member of $\mathscr{F}$.

Case 3. x belongs to exactly $n+1$ members of $\mathscr{U}_i$; say $\{U_{\alpha_k} : k=1, 2, \ldots, n+1\}$. Then $U(x)=\bigcap_{k=1}^{n+1} U_{\alpha_k}$ intersects no member of $\mathscr{F}$ other than $F(B)$ where

$$B = \{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}.$$

Since X is discretely H.C. expandable, there exists an open H.C. collection $\mathscr{K}=\{K(B) : B \in \mathscr{B}\}$ such that $F(B) \subseteq K(B)$ for each $B \in \mathscr{B}$, and each $K(B)$ is contained in some member of $\mathscr{U}_i$. Now define $L(B)=K(B) \cap [\bigcap_{\alpha \in B} U_\alpha]$ and $\mathscr{G}(i, n+1) =\{L(B) : B \in \mathscr{B}\}$. It is clear that (1), (2), and (4) above are satisfied by $\mathscr{G}(i, n+1)$. To show (3), let $x \in X$ such that x belongs to at most $n+1$ members of $\mathscr{U}_i$. If $x \notin \bigcup_{j=0}^{n} G(i, j)$, then $x \in F(B)$ for some $B \in \mathscr{B}$ and hence belongs to some member of $\mathscr{G}(i, n+1)$.

Define $\mathscr{G}^*=\bigcup_{i=1}^{\infty} \{\bigcup_{j=0}^{\infty} \mathscr{G}(i, j)\}$. Then $\mathscr{G}^*$ is a σ-H.C. open refinement of $\mathscr{G}$. Therefore X is paracompact by Theorem 2 of [18].

THEOREM 4.3. (i) *Let X be an almost discretely expandable space. If X is subparacompact then X is metacompact.*

(ii) *Let X be an almost expandable space. Then X is θ-refinable iff X is metacompact.*

Proof. (i) Let $\mathscr{G}$ be an open cover of X. Then X subparacompact implies that $\mathscr{G}$ has a σ-discrete closed refinement $\mathscr{V}=\bigcup_{i=1}^{\infty} \mathscr{V}_i$ where $\mathscr{V}_i=\{V(i, \alpha) : \alpha \in A_i\}$. Since X is almost discretely expandable there exists for each i, a point finite open collection $\mathscr{K}_i=\{K(i, \alpha) : \alpha \in A_i\}$, such that $V(i, \alpha) \subseteq K(i, \alpha)$ for all $\alpha \in A_i$. Again we may assume without loss of generality that $\mathscr{K}=\bigcup_{i=1}^{\infty} \mathscr{K}_i$ refines $\mathscr{G}$. Hence X is σ-metacompact. X is countably metacompact by Theorem 3 of [10]; and hence X is metacompact.

(ii) Repeating the proof of Theorem 4.2 above and expanding the resulting discrete collections at each stage to point finite open collections, it is easy to show that X is σ-metacompact and hence metacompact by Theorem 1.6(vi).

COROLLARY 4.4. *Let X be a regular discretely H.C. expandable space and $X=\bigcup_{i=1}^{\infty} F_i$, where each F_i is a closed subparacompact subspace of X. Then X is paracompact.*

Proof. D. Burke [5] has shown that X is subparacompact. Hence X is paracompact by Theorem 4.2 above.

COROLLARY 4.5. *Let X be a regular discretely H.C. expandable space. If X has a σ-locally finite closed cover, each element of which is subparacompact, then X is paracompact.*

Proof. Burke [5] has shown that subparacompactness is preserved under closed maps and is hereditary for closed subsets. By a similar technique as that used in the proof of Theorem 7.4 below, it then follows that locally finite unions of closed subparacompact subspaces is subparacompact. Therefore X is paracompact by Corollary 4.4 above.

5. Mapping theorems.

DEFINITION 5.1. A *quasi-perfect* (perfect) mapping $f: X \to Y$ is a closed onto map such that $f^{-1}(y)$ is countably compact (compact) for each $y \in Y$. The following lemmas are due to Lašnev [15] and Okuyama [24] respectively.

LEMMA 5.2. *The image of a hereditarily conservative collection under a closed map is hereditarily conservative.*

LEMMA 5.3. *Let $f: X \to Y$ be quasi-perfect. Then $\{F_\alpha : \alpha \in A\}$ is locally finite in X iff $\{f(F_\alpha) : \alpha \in A\}$ is locally finite in Y.*

The following theorem generalizes a result of J. Mack [16].

THEOREM 5.4. *Let $f: X \to Y$ be quasi-perfect. If X is m-paracompact then Y is m-paracompact.*

Proof. By Theorems 2.4 and 3.4 of [14] Y is m-expandable. Hence it is sufficient to show Y is m-metacompact by Corollary 2.11.2 of [14]. Let $\mathscr{U}=\{U_\alpha : \alpha \in A\}$ be an open cover of Y with $|A| \leqq m$. Then $\{f^{-1}(U_\alpha) : \alpha \in A\}$ has a locally finite open refinement $\mathscr{V}=\{V_\alpha : \alpha \in A\}$ such that $V_\alpha \subseteq f^{-1}(U_\alpha)$ for each $\alpha \in A$. Let $\mathscr{B}$ be the family of all finite subsets of A so that $|\mathscr{B}|=|A| \leqq m$. Define $W(B)=\bigcap_{\alpha \in B} V_\alpha$ and let $G(B)=Y-f(X-W(B))$ for each $B \in \mathscr{B}$. Since f is a closed map $G(B)$ is open for each $B \in \mathscr{B}$. Also $y \in G(B)$ implies that $f^{-1}(y) \subseteq W(B)$ and hence $y \in U_\alpha$ for each $\alpha \in B$. Therefore $\mathscr{G}=\{G(B) : B \in \mathscr{B}\}$ is a point finite open refinement of $\mathscr{U}$.

THEOREM 5.5. *Let $f: X \to Y$ be a closed map. If X is almost expandable then Y is almost expandable.*

Proof. Let $\mathscr{F}=\{F_\alpha \mid \alpha \in A\}$ be a locally finite collection of subsets of Y. Then $\{f^{-1}(F_\alpha) \mid \alpha \in A\}$ is locally finite in X, and hence there exists a point finite open collection $\{G_\alpha \mid \alpha \in A\}$ such that $f^{-1}(F_\alpha) \subseteq G_\alpha$ for each $\alpha \in A$. If $V_\alpha = Y-f(X-G_\alpha)$, then V_α is open and $F_\alpha \subseteq V_\alpha$ for each $\alpha \in A$. Also $\{V_\alpha \mid \alpha \in A\}$ is point finite since $y \in V_\alpha$ if and only if $f^{-1}(y) \subseteq G_\alpha$.

THEOREM 5.6. *Let $f: X \to Y$ be quasi-perfect. Then*

(i) *X is m-expandable iff Y is m-expandable,*
(ii) *if X is discretely expandable then Y is discretely expandable,*
(iii) *X is H.C. expandable iff Y is H.C. expandable,*
(iv) *if X is discretely H.C. expandable then Y is discretely H.C. expandable,*
(v) *X is almost expandable iff Y is almost expandable.*

Proof. Part (i) is proved by Theorem 3.4 of [14]. Using Lemmas 5.2 and 5.3 above and applying the technique used in the proof of Theorem 5.5 above, the remaining parts follow directly.

THEOREM 5.7. *Let $f: X \to Y$ be a closed onto map. If X is discretely H.C. $\aleph_0$-expandable and Y satisfies the first axiom of countability then $\partial f^{-1}(y)$ is countably compact for each $y \in Y$.*

Proof. The proof is similar to that of Lemma 1 of [22].

COROLLARY 5.8 (MORITA-HANAI [22]). *Let f: $X \to Y$ be a closed onto map. If X is normal and Y satisfies the first axiom of countability, then $\partial f^{-1}(y)$ is countably compact for each $y \in Y$.*

THEOREM 5.9. *Let f: $X \to Y$ be a closed onto map such that $\partial f^{-1}(y)$ is countably compact for each $y \in Y$.*

(i) *If X is expandable then Y is expandable.*

(ii) *If X is discretely expandable then Y is discretely expandable.*

(iii) *If X is H.C. expandable then Y is H.C. expandable.*

(iv) *If X is discretely H.C. expandable then Y is discretely H.C. expandable.*

(v) *If X is m-paracompact then Y is m-paracompact.*

Proof. By noting that each of the properties satisfies the closed subset theorem (Theorem 7.2 below), all parts of the theorem are proved by a similar technique to that used in Theorem 3.7 of [14] using Theorem 5.6 above.

THEOREM 5.10. *Let f: $X \to Y$ be a closed onto map, where Y satisfies the first axiom of countability.*

(i) *If X is m-expandable, then Y is m-expandable.*

(ii) *If X is discretely expandable then Y is discretely expandable.*

(iii) *If X is m-paracompact then Y is m-paracompact.*

(iv) *If X is discretely H.C. expandable then Y is discretely H.C. expandable.*

Proof. The proofs follow from Theorems 5.7 and 5.9 above.

REMARK. Theorem 5.10(iii) extends a result of P. Zenor [28].

THEOREM 5.11. *Let f: $X \to Y$ be a closed onto map, where Y is normal and satisfies the first axiom of countability. If X is an M-space then Y is an M-space.*

Proof. Morita [21] has shown that if f is quasi-perfect, Y is normal and X is an M-space, then Y is an M-space. From part (i) of Theorem 5.10 above Y is expandable. By Theorem 5.7 above, $\partial f^{-1}(y)$ is countably compact for each $y \in Y$. Now for each $y \in Y$, let $p_y \in f^{-1}(y)$ and define

$$L(y) = \begin{Bmatrix} \operatorname{Int}(f^{-1}(y)) \text{ if } \partial f^{-1}(y) \neq \varnothing \\ f^{-1}(y) - \{p_y\} \text{ if } \partial f^{-1}(y) = \varnothing \end{Bmatrix} \quad \text{and} \quad X^* = X - \bigcup_{y \in Y} L(y).$$

Then X^* is a closed subset of X and hence is an M-space. Also $f^* = f|X^*$ is a closed continuous mapping onto Y such that $(f^*)^{-1}(y)$ is countably compact for each $y \in Y$. Therefore Y is an M-space by Morita's theorem.

6. **Product spaces.** The following lemma is essentially due to O. T. Alas [1]. Since a proof is not readily available we supply one here.

LEMMA 6.1. *Assume Y has two locally finite covers $\{C_\alpha : \alpha \in A\}$ and $\{U_\alpha \mid \alpha \in A\}$ where, for each $\alpha \in A$, $C_\alpha \subseteq U_\alpha$, C_α is compact and U_α is open. If X is m-expandable, then $X \times Y$ is m-expandable.*

Proof. Let $\mathscr{F}=\{F_\delta : \delta \in D\}$ be a locally finite collection of closed subsets of $X\times Y$ (with $|D|\leqq m$). For each $\alpha\in A$, let $H(\alpha, \delta)=(X\times C_\alpha)\cap F_\delta$ for each $\delta\in D$. Then $\{\Pi_x[H(\alpha, \delta)] : \delta\in D\}$ is a locally finite collection of subsets of X (with cardinality $\leqq m$), where $\Pi_x: X\times Y\to X$ denotes the projection map. Since X is m-expandable there exists a locally finite collection $\{W(\alpha, \delta) : \delta\in D\}$ of open sets such that $\Pi_x[H(\alpha, \delta)]\subseteq W(\alpha, \delta)$ for every $\delta\in D$. Define $G_\delta=\bigcup_{\alpha\in A}[W(\alpha, \delta)\times U_\alpha]$. Then G_δ is open and $F_\delta\subseteq G_\delta$ for all $\delta\in D$. It is easy to verify that $\{G_\delta : \delta\in D\}$ is locally finite. Hence $X\times Y$ is m-expandable.

COROLLARY 6.2. *If X is expandable and Y is locally compact and paracompact then $X\times Y$ is expandable.*

COROLLARY 6.3. *If X is countably paracompact and Y is locally compact and paracompact then $X\times Y$ is countably paracompact.*

THEOREM 6.4. *If X is almost expandable and Y is locally compact and metacompact then $X\times Y$ is almost expandable.*

Proof. The proof is similar to that of Lemma 6.1 with the obvious modifications.

COROLLARY 6.5. *If X is countably metacompact and Y is locally compact and metacompact then $X\times Y$ is countably metacompact.*

S. Hanai [7] has shown the following

THEOREM 6.6. *If Y is compact and X is any topological space, then $\Pi_x: X\times Y\to X$ is a closed map.*

THEOREM 6.7. *Let X be a space and Y be compact. Then*
(i) *X is expandable iff $X\times Y$ is expandable,*
(ii) *X is H.C. expandable iff $X\times Y$ is H.C. expandable,*
(iii) *X is almost expandable iff $X\times Y$ is almost expandable.*

Proof. The proofs follow directly from Theorems 5.6 and 6.6 above.

7. **Subspaces and sum theorems.** Any countably compact nonnormal space is an example of a discretely expandable and discretely H.C. expandable space which is not collectionwise normal.

To see that the various properties are not hereditary, consider the Sorgenfrey plane $S\times S$ in [26]. It can be shown that $S\times S$ is not discretely expandable and not almost expandable. However $S\times S$ is a subset of its Stone-Čech compactification.

LEMMA 7.1. *Let $\mathscr{V}=\{V_\alpha : \alpha\in A\}$ be a hereditarily conservative collection in S and $S\subseteq X$. Then $\mathscr{V}\cap S=\{V_\alpha\cap S : \alpha\in A\}$ is a hereditarily conservative collection in S.*

Proof. Let $\{U_\alpha : \alpha\in A\}$ be a collection of subsets of S such that $U_\alpha\subseteq V_\alpha\cap S$ for each $\alpha\in A$. Note that $\overline{U}{}_\alpha^S=\overline{U}_\alpha\cap S\subseteq\overline{V}_\alpha$ for each $\alpha\in A$. Therefore for $B\subseteq A$,

$\bigcup_{\beta \in B} \bar{U}_\beta^S = \bigcup_{\beta \in B} [\bar{U}_\beta \cap S] = S \cap [\bigcup_{\beta \in B} \bar{U}_\beta]$ which is closed in S since $\bigcup_{\beta \in B} \bar{U}_\beta$ is closed in X. Hence $\mathscr{V} \cap S$ is hereditarily conservative.

The proofs of part (i) in Theorems 7.2 and 7.3 below are found in [14].

THEOREM 7.2. *The Closed Subset Theorem holds for the following properties*:

(i) *expandable*,
(ii) *almost expandable*,
(iii) *discretely expandable*,
(iv) *H.C. expandable*,
(v) *discretely H.C. expandable.*

Proof. We prove (iv). The other parts follow similarly.

Let K be a closed subset of X and $\{F_\alpha : \alpha \in A\}$ be a locally finite collection of subsets of K. Then $\{F_\alpha : \alpha \in A\}$ is locally finite in X so that there exists a H.C. collection of open subsets of X, $\{G_\alpha : \alpha \in A\}$ such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in A$. By Lemma 7.1 above $\{G_\alpha \cap K : \alpha \in A\}$ is a H.C. (in K) collection of open (in K) subsets of K satisfying $F_\alpha \subseteq G_\alpha \cap K$ for each $\alpha \in A$.

THEOREM 7.3. *For the properties* P *listed in Theorem* 7.2 *above the following statement is true*: *If every open subset of X satisfies property* P, *then every subset of X satisfies property* P.

Proof. Again we prove only (iv). Let S be a subset of X and $\mathscr{F} = \{F_\alpha : \alpha \in A\}$ a locally finite collection of subsets of S. For each $x \in S$, let $N(x)$ be an open set containing x such that $N(x)$ intersects only finitely many members of $\mathscr{F}$. Define $U = \bigcup_{x \in S} N(x)$ so that $S \subseteq U$ and U is open in X. Since $\mathscr{F}$ is locally finite in U, there exists a collection $\{G_\alpha : \alpha \in A\}$ of open sets in X which is hereditarily conservative and satisfies $F_\alpha \subseteq G_\alpha$ for each $\alpha \in A$. By Lemma 7.1 above, $\{G_\alpha \cap S : \alpha \in A\}$ is hereditarily conservative in S and $F_\alpha \subseteq G_\alpha \cap S$ for each $\alpha \in A$. Hence S is H.C. expandable.

THEOREM 7.4. *The Locally Finite Sum Theorem is satisfied by all the properties listed in Theorem* 7.2 *above.*

Proof. We prove (iv). Let $X = \bigcup_{\alpha \in A} F_\alpha$ where $\mathscr{F} = \{F_\alpha : \alpha \in A\}$ is a locally finite closed collection. Let S be the disjoint topological sum of the members of $\mathscr{F}$. Then it is easy to show that $f: S \to X$, where $f(x) = x$ for all x, is a quasi-perfect map. Clearly S is H.C. expandable since each F_α is open and closed in S. Hence X is H.C. expandable by Theorem 5.6.

Let Q be a class of topological spaces satisfying the following properties:

(I) If X is a topological space such that every open subset of X belongs to Q, then every subset of X belongs to Q.

(II) If $X = \bigcup_{\alpha \in A} F_\alpha$, where $\{F_\alpha : \alpha \in A\}$ is a locally finite closed collection such that each F_α belongs to Q, then X belongs to Q.

R. Hodel [9] has proved the following

THEOREM 7.5. *Let X be a totally normal space such that X belongs to Q. Then every subset of X belongs to Q.*

In view of Theorems 7.3 and 7.4 above we therefore obtain the next two results.

THEOREM 7.6. *If X is totally normal with property* P, *then every subset of X has property* P *for the following properties*:
(i) *expandable,*
(ii) *almost expandable,*
(iii) *discretely expandable,*
(iv) *H.C. expandable,*
(v) *discretely H.C. expandable.*

THEOREM 7.7. *Let $\mathscr{G}$ be a σ-locally finite cover of X such that the closure of each element of $\mathscr{G}$ satisfies property* P. *Then X satisfies property* P, *where* P *is any of the properties listed in Theorem* 7.6 *above.*

8. **Metrization theorems.** The authors have noted that the previous results can be used to generalize a number of metrization theorems. These observations are summarized in Theorems 8.3 and 8.4 below with proofs left for the interested reader. Two important definitions are included. The remaining definitions and results are found in K. Morita [20], A. Okuyama [24], F. Siwiec and J. Nagata [25] and K. Nagami [23].

DEFINITION 8.1. A space X is called *developable* if there exists a sequence $\mathscr{G}_1, \mathscr{G}_2, \ldots$ of open covers of X such that, for each $x \in X$ and each open set U containing x, there exists a positive integer $n(x, U)$ such that $\mathrm{St}\,(x, \mathscr{G}_{n(x,U)}) \subseteq U$. A regular developable T_2 space is called a *Moore space.*

DEFINITION 8.2. A base $\mathscr{B}$ for a space X is called a *uniform* base if every infinite subfamily of $\mathscr{B}$ having a common element $x \in X$ is a neighborhood base for x.

THEOREM 8.3. *The following are equivalent for a* T_2 *space X*:
(i) *X is metrizable,*
(ii) *X is a discretely expandable Moore space,*
(iii) *X is an expandable space with a uniform base,*
(iv) *X is an M-space and a σ-space.*

THEOREM 8.4. *The following are equivalent for a regular* T_2 *space X*:
(i) *X is metrizable,*
(ii) *X is an $M\#$-space, a $\sigma\#$-space and discretely H.C. expandable,*
(iii) *X is a $W\Delta$-space, a σ-space and discretely H.C. expandable,*
(iv) *X is a σ-space, discretely H.C. expandable and has a point countable base,*
(v) *X is discretely H.C. expandable and has a uniform base.*

9. **Examples and questions.** Professor O. T. Alas [1] has pointed out to the authors the first two examples.

EXAMPLE 1. Let R be the set of real numbers with the usual topology and $p \notin R$. Let $X = R \cup \{p\}$ and define the topology τ on X by the following:

(1) For $t \in R$, $V \subseteq X$ is an open neighborhood of t in X iff $V \cap R$ is an open neighborhood of t in R.

(2) For the point $p \notin R$, $U \subseteq X$ is an open neighborhood of p iff $p \in U$ and $(X-U) \cap R$ is a countable closed set in R.

It then follows that (X, τ) is a Lindelöf T_1-space which is not Hausdorff. The space (X, τ) can easily be shown to be discretely H.C. expandable but not discretely expandable.

EXAMPLE 2. In Example 1 above, if the "ideal" point p is replaced by countably many such points, then the resulting space can be shown to be a space which is θ-refinable but not countably metacompact.

EXAMPLE 3. An example of an almost expandable space which is not discretely H.C. expandable, and hence not expandable is Michael's modification of Bing's example in [19].

A number of open questions still remain:

(1) Are almost discretely expandable spaces countably metacompact?

(2) Are expandable screenable spaces paracompact?

(3) Which of the expandable properties are hereditary for F_σ-subsets? (The referee has noted that almost expandability is inherited by F_σ-subsets.)

(4) Is the necessary condition true for parts (ii) and (iv) of Theorem 5.6?

(5) Consider any cardinal $m > \aleph_0$. Is there a space X which is m-expandable but not $(m+1)$-expandable?

(6) Are the properties of discretely expandable and discretely H.C. expandable equivalent in Hausdorff spaces?

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