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WHEN IS $\mu * L_1$ CLOSED?

BY
I. GLICKSBERG[1]

Abstract. For a finite measure μ on a locally compact abelian group, we partially answer the question of when $\mu * L_1$ is closed in L_1.

1. Let $G^\wedge$ be a locally compact abelian group with dual $G^\wedge = \Gamma$. Clearly a measure[2] $\mu \in M = M(G)$ has $\mu * L_1(G)$ closed in $L_1(G)$ if it is invertible in M, or an idempotent, or if

(1.1) *μ is the convolve of an idempotent and an invertible in $M(G)$.*

Our question, for which we are indebted to Edwin Hewitt, might well be answered by the converse, but so far we can only give a complete answer when Γ is connected (so no nontrivial idempotents appear) or under some special hypothesis.

THEOREM 1.1. *If Γ is connected, $\mu * L_1(G)$ is closed if and only if μ is invertible in $M(G)$, or $\mu = 0$.*

THEOREM 1.2. *If $\mu \in L_1(G)$, $\mu * L_1(G)$ is closed iff* (1.1) *holds.*

Evidently (1.1) implies that $\mu * \mu * L_1(G)$ is also closed.

THEOREM 1.3. *$\mu * \mu * L_1(G)$ is closed iff* (1.1) *holds, and then the space coincides with $\mu * L_1(G)$. In particular if $\mu * L_1(G)$ is closed and μ has a square root in M (which thus must satisfy* (1.1)*), then* (1.1) *holds for μ.*

So what we shall leave open can be viewed as the question of whether $\mu * L_1(G)$ closed implies $\mu * \mu * L_1(G)$ is closed.

The same argument used to prove Theorem 1.1 can be applied to yield a bit of information on the analogous question in which L_1 is replaced by a closed ideal.

THEOREM 1.4. *Suppose I is a proper closed ideal in L_1. Then $\mu * I$ closed implies $\hat{\mu}^{-1}(0)$ is interior to* hull $(\mu * I)$ *and*

$$\mu * I = \{f \in I : \hat{f} = 0 \text{ on } \hat{\mu}^{-1}(0)\}. \tag{1.2}$$

In the special case that Γ is connected, $\mu \in L_1$ (or just if $\hat{\mu} \in C_0$) and hull I *is nowhere dense, $\mu * I$ is closed only if $\mu = 0$, or if Γ is compact and μ is invertible.*

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[2] We shall use M, rather than $M(G)$, for the algebra of measures on G, and similarly L_1, L_∞, C for the usual spaces *on G*.

Various questions related to the original one are in fact equivalent. Indeed, we shall see $\mu * L_1$ is closed iff $\mu * M$ is closed, and since an operator from one Banach space to another has closed range iff its adjoint has [1, p. 488] this implies

COROLLARY 1.5. *If any of $\mu * L_1$, $\mu * M$, $\mu * C_0$, $\mu * L_\infty$ are closed (in L_1, M, C_0, L_∞, resp.), then all are.*

A part of our argument yields a simple characterization of inclusions between ranges of convolution operators, which seems new, and is quite independent of our original problem.

THEOREM 1.6. *For $\mu, \nu \in M$, $\mu * L_1 \subset \nu * L_1$ iff $\mu \in \nu * M$ (i.e., iff $\mu * M \subset \nu * M$); more generally $\mu * L_1 \subset \sum_1^n \nu_i * L_1$ iff $\mu \in \sum_1^n \nu_i * M$.*

Finally, I am indebted to M. M. Hackman and I. Wik for several helpful conversations, and to Frank Pollard for pointing out a grievous error in an earlier attempt on the problem.

2. One easy special case, which we shall use later on, is that in which $\mu * L_1$ is closed and $\hat{\mu}$ never vanishes on Γ: then $\mu * L_1$ is a closed hull-less ideal and so all of L_1 by Wiener's Tauberian theorem, while $f \to \mu * f$ is 1-1. So the open mapping theorem says this map is topological and $\mu * f \to f$ is a bounded operator on $\mu * L_1 = L_1$ which evidently commutes with translations. As is well known [3] it is convolution with a λ in M, whence $\lambda * \mu * f = f$, $f \in L_1$ and $\lambda * \mu = \delta_0$ (the point mass at the identity of G), so μ is invertible.

Both Theorems 1.1 and 1.2 follow from the following fact, which we put in a more general form for later use.

LEMMA 2.1. *If I is an ideal in L_1 and $\mu * I$ is closed then $\hat{\mu}$ is bounded away from zero on $\Gamma \backslash (\hat{\mu}^{-1}(0) \cup \text{hull } I) = \Gamma \backslash \text{hull } (\mu * I)$.*

Indeed, suppose not, so we can find distinct $\gamma_n \notin \text{hull } I$ with $0 < |\hat{\mu}(\gamma_n)| < 1/n!$. We can find symmetric compact Baire neighborhoods V_n of the identity in Γ for which $\{\gamma_n + 2V_n\}$ is a pairwise disjoint sequence of subsets of $\Gamma \backslash \text{hull } (\mu * I)$, so the functions

$$\psi_n = (mV_n)^{-1} R_{\gamma_n}(\chi_{V_n} * \chi_{V_n})$$

(where m is Haar measure in Γ and R_γ translation by γ) have pairwise disjoint supports contained in $\Gamma \backslash \text{hull } (\mu * I)$. Now ψ_n is the Fourier transform of an f_n in L_1 with $\bar{\gamma}_n f_n = (mV_n)^{-1} \hat{\chi}_{V_n}^2 \geqq 0$ (since V_n is symmetric) so $\|f_n\|_1 = \|\bar{\gamma}_n f_n\|_1 = (mV_n)^{-1} \chi_{V_n} * \chi_{V_n}(0) = 1$ by Plancherel say. Hence

$$f = \sum_{n=1}^{\infty} \frac{2^n}{n!} f_n$$

is in L_1 and $\hat{f}(\gamma_n) = (2^n/n!)\psi_n(\gamma_n) = 2^n/n!$. Since $0 < |\hat{\mu}(\gamma_n)| < 1/n!$ we have $|\hat{f}(\gamma_n)/\hat{\mu}(\gamma_n)| > 2^n$ and $\hat{f}/\hat{\mu}$ is not bounded on $\Gamma \backslash \hat{\mu}^{-1}(0)$. On the other hand, since $\hat{f}_n$

has compact support disjoint from hull $(\mu * I)$, f_n lies in the smallest ideal with that hull [3] and so in $\mu * I$. So $f = \lim_{k \to \infty} \sum_{n=1}^{k} 2^n f_n/n!$ is in the closed ideal $\mu * I$, whence $f = \mu * g$, $g \in L_1$, and, on $\Gamma \backslash \hat{\mu}^{-1}(0)$, $\hat{f}/\hat{\mu} = \hat{g}$ must be bounded, a contradiction completing our proof(3).

Again suppose $\mu * L_1$ is closed. By Lemma 2.1 with $I = L_1$, $\hat{\mu}$ is bounded away from zero on $\Gamma \backslash \hat{\mu}^{-1}(0)$, so the boundary of $\hat{\mu}^{-1}(0)$ must be void, and $\hat{\mu}^{-1}(0)$ is open. This, of course, yields Theorem 1.1, since there if $\mu \neq 0$ we must have $\hat{\mu}^{-1}(0)$ empty, whence μ is invertible by our easy special case.

Now to obtain Theorem 1.2, note that there $\hat{\mu}$ is in C_0 and is bounded away from zero on $\Gamma \backslash \hat{\mu}^{-1}(0)$, so this closed set is compact. By regularity there is an η in L_1 with $\hat{\eta} = 1$ on $\Gamma \backslash \hat{\mu}^{-1}(0)$ and 0 on $\hat{\mu}^{-1}(0)$. This η is the desired idempotent, and to complete our proof we note that $\mu = \eta * \mu = \eta * (\mu + \delta_0 - \eta)$, while $\mu + \delta_0 - \eta$ is an invertible in the subalgebra $C\delta_0 + L_1$ of M since its Gelfand representative (on the one point compactification of Γ) has the value 1 where $\hat{\mu}$ (and hence $\hat{\eta}$) vanish, and at infinity, while on $\Gamma \backslash \hat{\mu}^{-1}(0)$ where $\hat{\eta} = 1$ the values coincide with those of $\hat{\mu}$, hence are bounded away from 0.

We obtain Theorem 1.3 by first noting that since $(\mu * \mu)^{\wedge -1}(0) = \hat{\mu}^{-1}(0)$ is open, it is a set of synthesis so that

$$\mu * L_1 = \mu * \mu * L_1 = \{f \in L_1 : \hat{f} = 0 \text{ on } \hat{\mu}^{-1}(0)\}$$

since the latter two are closed ideals with hull $\hat{\mu}^{-1}(0)$, while the first lies between these two. Consequently $f \to \mu * f$ is a map of $\mu * L_1$ onto itself by the first equality, evidently 1-1 by the second and thus an invertible operator on the Banach space $\mu * L_1$. So nearby operators on $\mu * L_1$ are invertible, in particular $f \to (z\delta_0 - \mu) * f$, for $|z| < \delta$, some $\delta > 0$. Thus for $0 \neq |z| < \delta$ we have $(z\delta_0 - \mu) * \mu * L_1 = \mu * L_1$, and so $f \in L_1$ implies $\mu * f = (z\delta_0 - \mu) * \mu * g$, $g \in L_1$, whence $f - (z\delta_0 - \mu) * g \in N_\mu$, the nullity of $f \to \mu * f$ in L_1. But $h \in N_\mu$ implies $(z\delta_0 - \mu) * h = zh$ so that $N_\mu \subset (z\delta_0 - \mu) * L_1$, and we conclude that any f in L_1 lies in $(z\delta_0 - \mu) * L_1$ if $0 \neq |z| < \delta$, and $L_1 = (z\delta_0 - \mu) * L_1$. By our easy special case $z\delta_0 - \mu$ is an invertible in M, and thus 0 is an isolated point in the spectrum of the element μ of M. So the characteristic function χ of the complement of a small disc about $0 \in C$ is analytic near the spectrum of μ, and (via the Cauchy integral) [2] there is an η in M, necessarily idempotent, with $\hat{\eta} = \chi \circ \hat{\mu}$ the characteristic function of the spectrum of M less

(3) I am indebted to M. Hackman for suggesting this type of construction, and to I. Wik for pointing out how simple the final argument could be made. A shorter but slightly less elementary alternate proof of the lemma can be obtained from the open mapping theorem and the fact that for $\gamma \in \Gamma \backslash F$, F closed, we have $\|\gamma + (kF)^\perp\| = \inf\{\|\gamma + h\|_\infty : h \in (kF)^\perp\} = \|\gamma\|_\infty$ (which also follows from spectral synthesis for open sets). Indeed since the dual to $I \to \mu * I \subset L_1$ induces a topological map of $L_\infty/(\mu * I)^\perp$ into $L_\infty/I^\perp$, $\|\varphi + (\mu * I)^\perp\| \leqq k\|\mu * \varphi + I^\perp\| \leqq k\|\mu * \varphi\|_\infty$ for $\varphi \in L_\infty$. With $\varphi \doteq \gamma \in \Gamma \backslash \text{hull}(\mu * I)$ then $1 = \|\gamma\| = \|\gamma + (\mu * I)^\perp\| \leqq k\|\mu * \gamma\|_\infty = k\|\hat{\mu}(\gamma)\gamma\|_\infty = k|\hat{\mu}(\gamma)|$.

$\hat{\mu}^{-1}(0)$, where the hat now denotes the Gelfand representative. So $\mu=\eta * \mu =\eta * (\mu+\delta_0-\eta)$, and we see $\mu+\delta_0-\eta$ is invertible in M exactly as before([4]).

To obtain Theorem 1.4, suppose I is a closed ideal and $0\neq\mu * I$ is closed. By Lemma 2.1 no element of $\hat{\mu}^{-1}(0)$ can lie in the boundary of hull $(\mu * I)$, and thus the boundary of $\hat{\mu}^{-1}(0)$ is interior to hull I as asserted.

Now $\mu * I\subset\{f\in I : \hat{f}=0 \text{ on } \hat{\mu}^{-1}(0)\}$, while if f is in the larger set then $\hat{f}=0$ near $\hat{\mu}^{-1}(0)$ so that $f\in\mu * L_1$, and we have an $h\in L_1$ with $f=\mu * h\in L_1$. Near any point of $\Gamma\backslash\hat{\mu}^{-1}(0)$, $1/\hat{\mu}$ coincides with a Fourier transform, so h belongs locally to I at all γ in $\Gamma\backslash\hat{\mu}^{-1}(0)$. On the other hand $\hat{f}$ vanishes on an open set $U\supset\hat{\mu}^{-1}(0)$, so that $\hat{f}/\hat{\mu}=\hat{h}$ vanishes on $U\backslash\hat{\mu}^{-1}(0)$ and if we set $\varphi=\hat{h}$ off U and $\equiv 0$ on U then φ belongs locally to $I^\wedge$ at all points of Γ, and even at ∞ if $\hat{f}$ has compact support, so $\varphi=\hat{g}$, $g\in I$. For such an f we thus have $f\in\mu * I$, as desired; and for any $f\in I$ with $\hat{f}=0$ on $\hat{\mu}^{-1}(0)$ we have a net $\{f_\delta\}$ in L_1 with $f_\delta * f\to f$, $\hat{f}_\delta\equiv 0$ near ∞, so that $f_\delta * f\in\mu * I$, and therefore $f\in\mu * I$ since $\mu * I$ is closed, proving (1.2).

Suppose hull I is nowhere dense. Then since $\hat{\mu}^{-1}(0)$ is in the interior of $\hat{\mu}^{-1}(0)\cup$ hull I, one easily concludes $\hat{\mu}^{-1}(0)$ is open (if U is the interior, $U\backslash\hat{\mu}^{-1}(0)$ is an open set contained in hull I). So if $\mu\neq 0$, $\hat{\mu}^{-1}(0)=\varnothing$ if we assume Γ to be connected, and by Lemma 2.1 $\hat{\mu}$ is bounded away from zero on $\Gamma\backslash$hull $(\mu * I)=\Gamma\backslash$hull I. Since $\hat{\mu}$ lies in C_0 by hypothesis, we conclude then that $\Gamma\backslash$hull I must have compact closure, which easily implies that Γ is compact since hull I is nowhere dense. But now $\hat{\mu}^{-1}(0)=\varnothing$ means μ is invertible in $M=L_1$ and Theorem 1.4 is proved.

3. **Proof of Corollary 1.5 and Theorem 1.6.** We shall first obtain 1.5 as a corollary to 1.6. If $\mu * L_1$ is closed then, as we have noted, since $\hat{\mu}^{-1}(0)$ is open

$$\mu * L_1 = \{f\in L_1 : \hat{f} = 0 \text{ on } \hat{\mu}^{-1}(0)\},$$

and this implies any $\nu\in M$ with $\hat{\nu}=0$ on $\hat{\mu}^{-1}(0)$ has $\nu * L_1\subset\mu * L_1$, so $\nu\in\mu * M$ by 1.6. Conversely $\nu\in\mu * M$ implies $\hat{\nu}=0$ on $\hat{\mu}^{-1}(0)$, so

$$\mu * M = \{\nu\in M : \hat{\nu} = 0 \text{ on } \hat{\mu}^{-1}(0)\}$$

which shows $\mu * M$ is closed.

On the other hand that $\mu * M$ is closed is equivalent to continuity of the inverse of the induced map of M/N_μ^0 into the subspace $\mu * M$ of M (where N_μ^0 is the

([4]) At this point it should be apparent that the gap between Theorem 1.1 and the result in full generality lies in showing that 0 isolated in the range of $\hat{\mu}$ means it must be isolated in the spectrum of μ in M. Actually, any root of μ will do in the final assertion, and more generally, if $\mu=\lambda * \nu$, where $\hat{\lambda}^{-1}(0)=\hat{\nu}^{-1}(0)=\hat{\mu}^{-1}(0)$, the conclusion follows, by just the same argument: λ is invertible on $\nu * L_1=\lambda * \nu * L_1$, whence 0 is isolated in the spectrum of λ in M, so the Gelfand representative $\hat{\lambda}$ of λ is bounded away from 0 where it is not 0. By symmetry the same is true of $\hat{\nu}$, hence $\hat{\lambda}\hat{\nu}=\hat{\mu}$ is bounded away from 0 where it is not $=0$.

nullity of $\nu \to \mu * \nu$), i.e. to the existence of a k with

$$\|\nu + N_\mu^0\| \leqq k\|\mu * \nu\|, \qquad \nu \in M. \tag{3.1}$$

With N_μ the nullity in L_1, for $\nu \in L_1$ one easily sees $\|\nu + N_\mu\| \leqq \|\nu + N_\mu^0\|$ using an approximate identity in L_1 and the fact that L_1 is an ideal in M, so (3.1) implies our map of L_1/N_μ into the subspace $\mu * L_1$ of L_1 is topological, whence $\mu * L_1$ is closed. Because of the theorem on adjoints cited earlier the proof of 1.5 is complete, and we turn to 1.6.

First we need the simple

LEMMA 3.1. *For λ, μ in M with*

$$\|\lambda * \varphi\| \leqq k\|\mu * \varphi\|, \qquad \varphi \in C_0,$$

we have $\lambda \in \mu * M$.

We have only to note that $\lambda * \varphi \in C_0$ so $\mu * \varphi \to \lambda * \varphi(0)$ is a bounded linear functional on a subspace of C_0. By Hahn-Banach it is given by a $\nu \in M$, $\lambda * \varphi(0) = \nu * \mu * \varphi(0)$, and replacing φ by $R_{-x}\varphi$, $\lambda * \varphi(x) = \nu * \mu * \varphi(x)$, hence $\lambda = \mu * \nu$ as desired.

We can now obtain 1.6.

Suppose that $\mu * L_1 \subset \nu * L_1$. Then we have a linear map of L_1 into L_1/N_ν sending f into $g + N_\nu$ if $\mu * f = \nu * g$, which has a closed graph: if $f_n \to f$ in L_1 and $g_n + N_\nu \to g + N_\nu$ in L_1/N_ν, $\mu * f_n = \nu * g_n$, then $\mu * f = \lim \mu * f_n = \lim \nu * g_n = \nu * g$. So our map is continuous, and has a continuous adjoint sending $(L_1/N_\nu)^*$ into L_∞. Of course $(L_1/N_\nu)^*$ is the subspace $N_\nu^\perp$ of L_∞, which clearly contains[5] $\nu * L_\infty$. With $\varphi \in L_\infty$, $f, g \in L_1$ and $\mu * f = \nu * g$,

$$\langle g + N_\nu, \nu * \varphi \rangle = \langle \nu * g, \varphi \rangle = \langle \mu * f, \varphi \rangle = \langle f, \mu * \varphi \rangle$$

so the adjoint maps $\nu * \varphi$ into $\mu * \varphi$, and we have $k > 0$ for which

$$\|\mu * \varphi\|_\infty \leqq k\|\nu * \varphi\|_\infty, \qquad \varphi \in L_\infty,$$

so $\mu \in \nu * M$ follows from Lemma 3.1. Conversely $\mu \in \nu * M$ implies $\mu * L_1 \subset \nu * L_1$ trivially.

The more general assertion in Corollary 1.5 follows in the same way, noting that $\mu * L_1 \subset \sum_1^n \nu_i * L_1$ yields a continuous linear map of L_1 into $(L_1 \oplus \cdots \oplus L_1)/N$, where $N = \{(f_1, \ldots, f_n) : f_i \in L_1, \sum \nu_i * f_i = 0\}$, with an adjoint taking $(\nu_1 * \varphi, \ldots, \nu_n * \varphi)$ in $N^\perp$ into $\mu * \varphi$. So for $\varphi \in C_0$, $(\nu_1 * \varphi, \ldots, \nu_n * \varphi) \to \mu * \varphi(0)$ defines a continuous linear functional on a subspace of $C_0 \oplus \cdots \oplus C_0$, whence we obtain $\lambda_1, \ldots, \lambda_n \in M$ with $\mu * \varphi(0) = \sum \lambda_i * \nu_i * \varphi(0)$, yielding the result.

Finally, the argument applies to a more general setting: suppose E is a space of Radon measures on G which is a Banach space under some norm, closed under

(5) We pair L_1 and L_∞ via $\langle f, \varphi \rangle = f * \varphi(0)$.

convolution with finite measures, with the property

(3.2) $\mu \in M$ and $e_n \to 0$ in E imply $|\mu * e_n|(K) \to 0$ for all compact K in G.

(For example E could be $C(G)m$, where m is Haar measure.) Defining $\mu*$ on the dual E^* as the adjoint operator, so $\langle e, \mu * e^* \rangle = \langle \mu * e, e^* \rangle$, $e^* \in E^*$, suppose D is a closed subspace of E^* which is invariant and continuously translating (so $\|\delta_x * d - d\| \to 0$ as $x \to 0$ in G). Then if $\mu * L_1 \subset \nu * E$, where $\mu, \nu \in M$, (3.2) shows the nullity N_ν of $e \to \nu * e$ is closed, and then that the map of L_1 into E/N_ν sending f into $e + N_\nu$ if $\mu * f = \nu * e$ has a closed graph, hence is continuous, as before. The adjoint, mapping $N_\nu^\perp$ in E^* into L_∞ in particular takes $\nu * D \subset \nu * E^* \subset N_\nu^\perp$ into L_∞, sending $\nu * d$ into $\mu * d$ since $\langle e + N_\nu, \nu * d \rangle = \langle \nu * e, d \rangle = \langle \mu * f, d \rangle = \langle f, \mu * d \rangle$ when $\mu * f = \nu * e$. Moreover, since $\delta_x * \mu * d = \mu * \delta_x * d$, $\mu * d$ is a continuously translating element of L_∞, hence can be taken as a continuous function, so that $\nu * d \to \mu * d(0)$ is a well-defined continuous linear functional on D. Thus there is a $d^* \in D^*$ with

$$(3.3) \qquad \mu * d(0) = \langle \nu * d, d^* \rangle = \langle d, \nu * d^* \rangle, \qquad d \in D.$$

For example, if $E = C(G)m$ and we take $D = L_1(G)$ (with $\langle e \cdot m, f \rangle = e * f(0)$ of course) then (3.3) says there is an $h \in L_\infty$ for which

$$\mu * d(0) = d * \nu * h(0) \quad \text{all } d \in L_1,$$

whence we can identify μ with the measure $(\nu * h) \cdot m$ in $L_\infty \cdot m$. So $\mu * L_1 \subset (\nu * C)m$ $(= \nu * (C \cdot m))$ implies $\mu \in (\nu * L_\infty)m$; conversely that implies $\mu * L_1 \subset (\nu * L_\infty * L_1)m$ $\subset (\nu * C)m$ of course.

4. We should note that if $\sum_{i=1}^n \mu_i * L_1$ is closed the argument of Lemma 2.1 can be used to show $\sum |\hat{\mu}_i|$ is bounded away from zero on $\Gamma \setminus \bigcap \hat{\mu}_i^{-1}(0)$, and thus that $\bigcap \hat{\mu}_i^{-1}(0)$ is open. So *if* Γ *is connected* L_1 *has no proper nonzero closed subspaces of the form* $\sum_1^n \mu_i * L_1$.

Finally, we note that the arguments of §2 apply in a more general setting: in place of L_1 we can take any regular commutative Banach algebra A which is tauberian (so the analogue of Wiener's theorem holds) and in place of $M(G)$ the (Banach) algebra of multipliers of A (the algebra M of all operators T on A satisfying $T(ab) = a \cdot Tb$). With just the special hypothesis that for each neighborhood V of any element γ of the spectrum Γ of A there is an a in A with Gelfand representative $\hat{a}$ supported by V, $\hat{a}(\gamma) = 1$ and $\|a\| \leqq c$, a fixed constant, one has:

(4.1) *If* Γ *is connected* TA *is closed iff* $T = 0$ *or* T *is invertible in* M.

(4.2) *If* T *is multiplication by an element of* A, TA *is closed iff* T *is the product of an idempotent and an invertible in* M.

(4.3) T^2A *is closed iff* T *is a product as in* (4.2).

As is well known, $T \in M$ corresponds to a continuous function φ on Γ satisfying $(Ta)^\wedge = \varphi \hat{a}$, and the arguments of §2, in particular of Lemma 2.1, apply with φ in place of $\hat{\mu}$, as the reader can easily verify.

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University of Washington,
Seattle, Washington 98105