TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 162, December 1971

HOMOLOGY IN VARIETIES OF GROUPS. III

BY
C. R. LEEDHAM-GREEN

Abstract. A spectral sequence is used to calculate approximately the homology groups $\mathfrak{B}_2(\Pi, Z)$ as defined in the first paper in this series, for Π a finitely generated abelian group and $\mathfrak{B}$ the variety of all nilpotent groups of class at most c.

Introduction. In previous papers [21] and [25], henceforth referred to as [HI] and [HII] respectively, (co-) homology groups $\mathfrak{B}_n(\Pi, A)$, $\mathfrak{B}^n(\Pi, A)$ were discussed, where $\mathfrak{B}$ is a variety containing Π, and A is a suitable Π-module. If $\mathfrak{W}$ is a variety containing $\mathfrak{B}$, there are homomorphisms

$$\phi_n\colon \mathfrak{W}_n(\Pi, A) \to \mathfrak{B}_n(\Pi, A) \quad \text{and} \quad \phi^n\colon \mathfrak{B}^n(\Pi, A) \to \mathfrak{W}^n(\Pi, A).$$

Their basic properties are discussed in §1, and a spectral sequence with ϕ_n as an edge homomorphism is constructed in §2. Similar spectral sequences have been constructed by various authors; the point of this treatment is to calculate the edge homomorphisms. Using the exact sequence of terms of low degree, the wild behaviour of $\mathfrak{B}_2(\Pi, Z)$ is demonstrated. In so far as one's intuition is based on the homology of groups, this comes near to being a universal counterexample.

The conventions and definitions used in [HI] and [HII] will remain in force. In particular the reader is referred to [HI, §1] for the definition of the (co-) homology groups $\mathfrak{B}_n(\Pi, A)$ and $\mathfrak{B}^n(\Pi, A)$.

1. **Change of variety morphisms.** If $P_*^{\mathfrak{B}} \to \Pi$ and $P_*^{\mathfrak{W}} \to \Pi$ are simplicial resolutions of Π by $\mathfrak{B}$-splitting groups and $\mathfrak{W}$-splitting groups respectively, then since $\mathfrak{W}$ contains $\mathfrak{B}$ there is a simplicial map of $P_*^{\mathfrak{W}}$ into $P_*^{\mathfrak{B}}$ over 1_Π which is unique up to homotopy, (cf. Tierney and Vogel [18]). For example if $\Pi B_*^{\mathfrak{B}} \to \Pi$ and $\Pi B_*^{\mathfrak{W}} \to \Pi$ are the Barr-Beck resolutions of Π in $\mathfrak{B}$ and $\mathfrak{W}$ respectively (see [HI, §2]), so that $\Pi B_n^{\mathfrak{B}}$ is $\mathfrak{B}$-free on $\Pi B_{n-1}^{\mathfrak{B}}$, $n \geqq 0$, $\Pi B_{-1}^{\mathfrak{B}} = \Pi$, and $\Pi B_n^{\mathfrak{W}}$ is similarly defined, then a simplicial map $\eta_*\colon \Pi B_*^{\mathfrak{W}} \to \Pi B_*^{\mathfrak{B}}$ may be defined inductively by $[w]\eta_n = w\eta_{n-1}$, where $w \in \Pi B_{n-1}^{\mathfrak{W}}$, $[w]$ is the corresponding $\mathfrak{W}$-free generator of $\Pi B_n^{\mathfrak{W}}$, and $[w\eta_{n-1}]$ is the $\mathfrak{B}$-free generator of $\Pi B_n^{\mathfrak{B}}$ corresponding to $w\eta_{n-1}$; $\eta_{-1} = 1_\Pi$. If A is a left $\mathfrak{B}\Pi$-module, using these simplicial resolutions to calculate $\mathfrak{B}_*(\Pi, A)$ and $\mathfrak{W}_*(\Pi, A)$, one obtains well-defined homomorphisms

$$\phi_n(\mathfrak{W}, \mathfrak{B}, \Pi, A)\colon \mathfrak{W}_n(\Pi, A) \to \mathfrak{B}_n(\Pi, A), \qquad n \geqq 0.$$

Received by the editors September 28, 1970.

AMS 1970 *subject classifications.* Primary 18H40, 20E10; Secondary 18C15, 18G10, 18H10, 20J05.

Key words and phrases. Homology in varieties of groups.

$\phi_n(\mathfrak{W}, \mathfrak{B}, \Pi, A)$ is the "change of variety morphism"; some or all of $\mathfrak{W}$, $\mathfrak{B}$, Π and A will generally be omitted from the notation.

The following results are routine; we omit the proofs.

LEMMA 1.1. *ϕ_0: $\mathrm{Diff}(\Pi, A) \to \mathrm{Diff}(\Pi, A)$ is the identity map.*

We shall see later that ϕ_1 is a surjection.

LEMMA 1.2. *If $\mathfrak{X}$ is a variety containing $\mathfrak{W}$,*

$$\phi_*(\mathfrak{X}, \mathfrak{B}) = \phi_*(\mathfrak{X}, \mathfrak{W})\phi_*(\mathfrak{W}, \mathfrak{B}).$$

LEMMA 1.3. *$\phi_*(\Gamma, A)$ is natural in $\Gamma \to \Pi \in |(\mathfrak{B}, \Pi)|$ and in left $\mathfrak{B}\Pi$-modules A.*

LEMMA 1.4. *Given a short exact sequence of $\mathfrak{B}\Pi$-modules, ϕ_* commutes with the appropriate connecting homomorphisms.*

LEMMA 1.5. *If $\alpha\colon \Gamma_0 \to \Gamma_1$ is a surjection in $(\mathfrak{B}, \Pi)$, there is a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathfrak{W}_n(\Gamma_1, A) & \longrightarrow & M^{\mathfrak{W}}_{n-1}(\alpha, A) & \longrightarrow & \mathfrak{W}_{n-1}(\Gamma_0, A) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \phi_n} & & \downarrow & & \downarrow{\scriptstyle \phi_{n-1}} & \\
\cdots \longrightarrow & \mathfrak{B}_n(\Gamma_1, A) & \longrightarrow & M^{\mathfrak{B}}_{n-1}(\alpha, A) & \longrightarrow & \mathfrak{B}_{n-1}(\Gamma_0, A) & \longrightarrow \cdots
\end{array}$$

whose rows are Rinehart's exact sequence as in [HI, (2.1)], *which was equated with the Barr-Beck exact sequence in* [HII, §1].

LEMMA 1.6. *There is a commutative diagram*

$$\begin{array}{ccc}
\mathfrak{W}_n(\Pi, A) & \xrightarrow{\theta^{\mathfrak{W}}} & \mathrm{Tor}_n^{\mathfrak{W}\Pi}(D_{\mathfrak{W}}\Pi, A) \\
\downarrow{\scriptstyle \phi_n} & & \downarrow \\
\mathfrak{B}_n(\Pi, A) & \xrightarrow{\theta^{\mathfrak{B}}} & \mathrm{Tor}_n^{\mathfrak{B}\Pi}(D_{\mathfrak{B}}\Pi, A).
\end{array}$$

Here $\theta^{\mathfrak{B}} = \theta^{\mathfrak{B}}_n(\Pi, A)$ and $\theta^{\mathfrak{W}} = \theta^{\mathfrak{W}}_n(\Pi, A)$ as in [HII, §1], *and $\mathrm{Tor}_n^{\mathfrak{W}\Pi}(D_{\mathfrak{W}}\Pi, A) \to \mathrm{Tor}_n^{\mathfrak{B}\Pi}(D_{\mathfrak{B}}\Pi, A)$ is the "change of rings" homomorphism given by the unique δ-morphism of $\mathrm{Tor}_n^{\mathfrak{W}\Pi}(D_{\mathfrak{W}}\Pi, -)$ to $\mathrm{Tor}_n^{\mathfrak{B}\Pi}(D_{\mathfrak{B}}\Pi, -)$, regarded as δ-functors from the category of left $\mathfrak{B}\Pi$-modules to **Ab**, which is the identity in dimension* 0.

Dually there are homomorphisms $\phi^n(\mathfrak{B}, \mathfrak{W}, \Pi, A)\colon \mathfrak{B}^n(\Pi, A) \to \mathfrak{W}^n(\Pi, A)$ (note the change of direction), and Lemmas 1.1 to 1.6 all dualize.

2. **A spectral sequence.** The spectral sequence which appears below has the same E^2 terms and limit as can be obtained as a special case of spectral sequences

due to André [1], Bachmann [2], Rinehart [13], and Ulmer [20]. These are obtained by varying the first or nonabelian category (here the variety). The spectral sequence obtained by varying the second or abelian category was discussed in [HII, §3]. The point of our treatment (which will generalize) is to calculate the edge effects. The object is to connect the homology in $\mathfrak{W}$ with the homology in $\mathfrak{V}$; one edge homomorphism will be the "change of variety" morphism of §1; we now introduce the other. Recall that if $T\colon (\mathfrak{V}, \Pi) \to \boldsymbol{Ab}$ is any functor, the derived functors $\mathfrak{V}_n(\Pi, T)$ have been defined as in [HI, §2], $\mathfrak{V}_n(\Pi, A)$ being an abbreviation for $\mathfrak{V}_n(\Pi, \mathrm{Diff}(-, A))$. T need only be defined on the full subcategory of $\mathfrak{V}$-free groups (over Π), in which case $\mathfrak{V}_0(\Pi, T)$ is the Kan extension of T evaluated at Π; however we shall assume that T is defined on $(\mathfrak{V}, \Pi)$. In this case there is a homomorphism $\lambda\colon \mathfrak{V}_0(\Pi, T) \to \Pi T$ defined by various authors. For example the right exact functors from $(\mathfrak{V}, \Pi)$ to $\boldsymbol{Ab}$ in the sense of Rinehart [13] form a reflective subcategory of $\boldsymbol{Ab}^{(\mathfrak{V}, \Pi)}$ (pace set theorists) in the sense of Mitchell [26], and λ is the reflection (evaluated at T and Π), cf. [13, p. 299]. Alternatively, if $P_* \xrightarrow{\varepsilon} \Pi$ is a simplicial resolution of Π by $\mathfrak{V}$-splitting groups, then λ is the unique homomorphism to make

$$\begin{tikzcd}
P_1T \arrow[r, "d_1"] \arrow[d, "="] & P_0T \arrow[r] \arrow[d, "="] & \mathfrak{V}_0(\Pi, T) \arrow[r] \arrow[d, "\lambda"] & 0 \\
P_1T \arrow[r, "d_1"] & P_0T \arrow[r, "\varepsilon T"] & \Pi T &
\end{tikzcd}$$

commute. Here $d_1 = \delta_1^0 T - \delta_1^1 T$ as in [HI, §2]; the top row is exact by definition.

THEOREM 2.1. *There is a spectral sequence*

$$\mathfrak{V}_p(\Pi, \Lambda \cdot \mathfrak{W}_q(-, A)) \underset{p}{\Rightarrow} \mathfrak{W}_n(\Pi, A)$$

where $\Lambda\colon (\mathfrak{V}, \Pi) \to (\mathfrak{W}, \Pi)$ *is the inclusion functor, whose edge effects are given by*:

$$\mathfrak{W}_n(\Pi, A) \to E^2_{n0} = \mathfrak{V}_n(\Pi, \Lambda \cdot \mathfrak{W}_0(-, A)) = \mathfrak{V}_n(\Pi, A)$$

is the "change of variety" morphism ϕ_n, *and*

$$E^2_{0n} = \mathfrak{V}_0(\Pi, \Lambda \cdot \mathfrak{W}_n(-, A)) \to \mathfrak{W}_n(\Pi, A)$$

is the homomorphism λ *above applied to the functor* $\Lambda \cdot \mathfrak{W}_n(-, A)$.

Proof. We first adjust the notation. If $T\colon (\mathfrak{V}, \Pi) \to \boldsymbol{Ab}$ is a functor, the complex of abelian groups from which the derived functors $\mathfrak{V}_n(\Pi, T)$ are calculated from the Barr-Beck resolution will be written as $B^{\mathfrak{V}}_*(\Pi, T)$, unless $T = \mathrm{Diff}(-, A)$ which will be abbreviated to A. The augmentation $B^{\mathfrak{V}}_0\Pi \to \Pi$ will be $\varepsilon^{\mathfrak{V}}$.

Form the first quadrant double complex $T_{pq} = B^{\mathfrak{V}}_p(\Pi, \Lambda \cdot B^{\mathfrak{W}}_q(-, A))$, where $\Lambda\colon (\mathfrak{V}, \Pi) \to (\mathfrak{W}, \Pi)$ is the inclusion functor. Fixing p and taking homology gives $B^{\mathfrak{V}}_p(\Pi, \Lambda \cdot \mathfrak{W}_q(-, A))$, and taking homology again gives $\mathfrak{V}_p(\Pi, \Lambda \cdot \mathfrak{W}_q(-, A))$. Fixing q and taking homology in $\boldsymbol{T}$ gives $\mathfrak{V}_p(\Pi, \Lambda \cdot B^{\mathfrak{W}}_q(-, A))$. Now $\Lambda \cdot B^{\mathfrak{W}}_q(-, A)$

factors through the comma category of sets over the underlying set of Π and hence is flask in the sense of Rinehart [13] (cf. Rinehart [14] and the proof of [HII, Proposition 1.1]). Thus $\mathfrak{B}_p(\Pi, \Lambda\cdot B_q^{\mathfrak{W}}(-, A))=0$ for $p>0$ and

$$\mathfrak{B}_0(\Pi, \Lambda\cdot B_q^{\mathfrak{W}}(-, A)) = B_q^{\mathfrak{W}}(\Pi, A).$$

Taking homology again gives $\mathfrak{W}_q(\Pi, A)$, and so the first part of the theorem is proved. We calculate the edge effects using the same techniques as in the proof of [HII, Theorem 3.1]. Recall [22, Theorem XI, 4.4] which states that if I_{pq}^r is the first spectral sequence of the first quadrant double complex S_{pq} then the edge effects are given by $I_{0n}^1=H_nS_{0*}\to H_n\boldsymbol{S}$ induced by the inclusion of S_{0*} in A, and $H_n\boldsymbol{S}\to H_n(\boldsymbol{S}/\boldsymbol{M})=I_{n0}^2$ induced by the projection of $\boldsymbol{S}$ on $\boldsymbol{S}/\boldsymbol{M}$, where $\boldsymbol{M}$ is the subcomplex of $\boldsymbol{S}$ given by

$$M_n = \sum_{p+q=n;q>0} S_{pq} \cup \partial \sum_{p+q=n+1;q>0} S_{pq}.$$

We apply this first to the second spectral sequence of $\boldsymbol{T}$ (which collapses), that is with $\boldsymbol{S}=\boldsymbol{T}$ transposed. In this case $\boldsymbol{S}/\boldsymbol{M}$ is chain isomorphic to $B_*^{\mathfrak{W}}(\Pi, A)$ via $B_*^{\mathfrak{W}}(\varepsilon^{\mathfrak{V}}, A)$. Hence we have an isomorphism ω of $H_*\boldsymbol{T}$ onto $\mathfrak{W}_*(\Pi, A)$. Now looking at the first spectral sequence and applying the first part of the above theorem gives us the homomorphism ζ in the diagram

$$\begin{tikzcd}
B_0^{\mathfrak{V}}(\Pi, \Lambda\cdot\mathfrak{W}_n(-, A)) \arrow[rr, "\zeta"] \arrow[dr, "\mathfrak{W}_n(\varepsilon^{\mathfrak{V}}, A)"'] & & H_n(\boldsymbol{T}) \arrow[dl, "\omega"] \\
& \mathfrak{W}_n(\Pi, A) &
\end{tikzcd}$$

which is clearly commutative. This identifies the edge homomorphism $E_{0n}^1\to H_n(\boldsymbol{T})$ as $\mathfrak{W}_n(\varepsilon^{\mathfrak{V}}, A)$ "up to ω". It follows from the definition of λ above that λ is the edge homomorphism $E_{0n}^2\to H_n(\boldsymbol{T})$, again "up to ω". We now turn to the base. Define the first quadrant double complex $\bar{\boldsymbol{T}}$ by $\bar{T}_{pq}=B_p^{\mathfrak{V}}(\Pi, B_q^{\mathfrak{V}}(-, A))$. Dividing by the verbal subgroup defined by $\mathfrak{V}$ induces a functor of $(\mathfrak{W}, \Pi)$ into $(\mathfrak{V}, \Pi)$ and hence a chain map F: $\boldsymbol{T}\to\bar{\boldsymbol{T}}$. Now let $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\bar{\boldsymbol{M}}_1$ and $\bar{\boldsymbol{M}}_2$ correspond to $\boldsymbol{M}$ in the theorem quoted above, where $\boldsymbol{S}$ is taken as $\boldsymbol{T}$, $\boldsymbol{T}$ transposed, $\bar{\boldsymbol{T}}$ and $\bar{\boldsymbol{T}}$ transposed respectively. Then $\boldsymbol{T}/\boldsymbol{M}_1$, $\bar{\boldsymbol{T}}/\bar{\boldsymbol{M}}_1$, and $\bar{\boldsymbol{T}}/\bar{\boldsymbol{M}}_2$ are naturally isomorphic to $B_*^{\mathfrak{V}}(\Pi, A)$, and $\boldsymbol{T}/\boldsymbol{M}_2$ is isomorphic to $B_*^{\mathfrak{W}}(\Pi, A)$. Using these identifications, F induces a commutative diagram

$$\begin{tikzcd}
 & \boldsymbol{T}/\boldsymbol{M}_1 \arrow[r, equal] & \bar{\boldsymbol{T}}/\bar{\boldsymbol{M}}_1 \arrow[r, equal] & B_*^{\mathfrak{V}}(\Pi, A) \\
 & \boldsymbol{T} \arrow[u] \arrow[r, "F"] \arrow[d] & \bar{\boldsymbol{T}} \arrow[u] \arrow[d] & \\
B_*^{\mathfrak{W}}(\Pi, A) \arrow[r, equal] & \boldsymbol{T}/\boldsymbol{M}_2 \arrow[r] & \bar{\boldsymbol{T}}/\bar{\boldsymbol{M}}_2 \arrow[r, equal] & B_*^{\mathfrak{V}}(\Pi, A).
\end{tikzcd}$$

Taking homology now gives

$$\begin{tikzcd}
\mathfrak{V}_*(\Pi, A) \arrow[r, phantom, "="] & \mathfrak{V}_*(\Pi, A) \\
H_*T \arrow[u, "\zeta"] \arrow[r, "H_*F"] \arrow[d, "\omega", "\cong"'] & H_*\bar{T} \arrow[u, "\cong"'] \arrow[d, "\cong"'] \\
\mathfrak{W}_*(\Pi, A) \arrow[r, "\phi_*"] & \mathfrak{V}_*(\Pi, A)
\end{tikzcd}$$

where ω is the isomorphism of the first part and ζ is the base homomorphism. It only remains to prove that the composite of either homomorphism on the East side with the inverse of the other is the identity. Now $H_*\bar{T}$ and $\mathfrak{V}_*(\Pi, -)$ may clearly be regarded as δ-functors from the category of functors from $\mathfrak{V}$-free groups over Π to $\boldsymbol{Ab}$, and as such they are effaced on the left by the projective functors. Moreover, since we are dealing with δ-functors, it is enough to look at dimension zero. But in this case the result may be read off at once by looking at $\bar{T}$ with its canonical augmentation.

COROLLARY 2.2. *There is an exact sequence*

$$(2.1)\qquad \mathfrak{W}_2(\Pi, A) \xrightarrow{\phi_2} \mathfrak{V}_2(\Pi, A) \longrightarrow \mathfrak{V}_0(\Pi, \Lambda\cdot\mathfrak{W}_1(-, A)) \xrightarrow{\lambda} \mathfrak{W}_1(\Pi, A) \xrightarrow{\phi_1} \mathfrak{V}_1(\Pi, A) \longrightarrow 0.$$

In particular, ϕ_1 is a surjection. Of course the results of this paragraph all dualize. By a well-known folk theorem (but see Beck [5]), $\mathfrak{V}^1(\Pi, A)$ classifies the extensions of A by Π that lie in $\mathfrak{V}$, and it can be shown that with this identification the injection ϕ^1: $\mathfrak{V}^1(\Pi, A) \to \mathfrak{W}^1(\Pi, A)$ is the inclusion of the set of extensions of A by Π that lie in $\mathfrak{V}$ in the set of those that lie in $\mathfrak{W}$.

3. **The second homology groups.** The striking behaviour of $\mathfrak{V}_2(\Pi, A)$ is illustrated, using (2.1), when Π is a finitely generated abelian group and $A=Z$.

The abelian group Π is of type $(s; n_1, \ldots, n_t)$ if the torsion subgroup T of Π is of direct product of cyclic subgroups of order $n_1, \ldots, n_t$; $n_1>1$, $n_i|n_{i+1}$ for $i=1, \ldots, t-1$; and Π/T is of rank s. $s+t$ is the rank of Π. In quoting (2.1) the symbol Λ will be omitted. Define a function γ of two positive integers by

$$(3.1)\qquad \gamma(r, c) = \frac{1}{c+1} \sum_{d|(c+1)} \mu(d) r^{(c+1)/d}.$$

Here μ is the Möbius function; if n is the product of u distinct primes ($u \geqq 0$), then $\mu(n)=(-1)^u$; else $\mu(n)=0$.

LEMMA 3.1. *If* Π *is a finitely generated abelian group of rank* $r>0$ *and type* $(s; n_1, \ldots, n_t)$, *and* $\mathfrak{V}=\mathfrak{N}_c$, *then* $\mathfrak{V}_0(\Pi, H_2(-, Z))$ *is of rank* ρ *and type*

$(\sigma; \nu_1, \ldots, \nu_\tau)$ where $\rho=\gamma(r, c)$, $\sigma=\gamma(s, c)$, and $\nu_\tau=n_t$ if $s>0$, $\nu_\tau=n_{t-1}$ if $s=0$. (In particular, if $r=s$, $\rho=\sigma$ and if $r=1$, $\rho=0$.)

Proof. Let

$$\Pi = C(a_1)\times\cdots\times C(a_s)\times C_{n_1}(a_{s+1})\times\cdots\times C_{n_t}(a_r)$$

with the obvious notation, let F be $\mathfrak{N}_c$-freely generated by $x_1, \ldots, x_r$, and define $f\colon F\to\Pi$ by $x_if=a_i$, $i=1,\ldots,r$. The fibre product $F\times_\Pi F$ is the subgroup of $F\times F$ consisting of elements (p, q) such that $pf=qf$, and $(p, q)\mapsto(pq^{-1}, q)$ is an isomorphism of $F\times_\Pi F$ onto the split extension RF of R by F, where R is the kernel of f. R is generated *qua subgroup* by $x^{n_{i-s}}$, $i=s+1,\ldots,r$, and $w_1,\ldots,w_k$, say, where w_i is a commutator for all i. Defining a homomorphism of a group G into $F\times_\Pi F$ is equivalent to defining a homomorphism (g_1, g_2) of G into $F\times F$ $(g_i\colon G\to F)$ such that $g_1f=g_2f$. Let $\bar{F}$ be $\mathfrak{N}_c$-freely generated by $y_1,\ldots,y_r$, $z_{s+1},\ldots,z_r$, $v_1,\ldots,v_k$, and define $(g_1, g_2)\colon \bar{F}\to F\times_\Pi F$ by $y_ig_1=x_i$, $z_ig_1=x^{n_{i-s}}$, $v_ig_1=w_i$, $y_ig_2=x_i$, $z_ig_2=1$, $v_ig_2=1$. Then (g_1, g_2) is a surjection, and by [13, p. 299], $\mathfrak{B}_0(\Pi, H_2(-, Z))$ is the cokernel of $H_2(g_1, Z)-H_2(g_2, Z)\colon H_2(\bar{F}, Z)\to H_2(F, Z)$. It is easy to see that the Schur multiplier of the $\mathfrak{N}_c$-free group on a set $\mathfrak{x}$ is the $(c+1)$th lower central factor of the absolutely free group on $\mathfrak{x}$; that is, the free abelian group on the basic commutators of weight $c+1$ in $\mathfrak{x}$; these multipliers will be written additively. One sees easily that $H_2(g_1, Z)-H_2(g_2, Z)$ has the following properties:

(i) the image of a basic commutator (of weight $c+1$) in $\bar{F}$ is a multiple of a basic commutator in F or 0;

(ii) if $[x_{i_1},\ldots,x_{i_{c+1}}]$ is a basic commutator in F (not necessarily left normed), and if $\max(i_1,\ldots,i_{c+1})\leqq s$, then no nonzero multiple of $[x_{i_1},\ldots,x_{i_{c+1}}]$ is the image of a basic commutator, whereas

(iii) if $i_\alpha>s$ for some α and i_α is the least such integer, then $n_{i_\alpha-s}[x_{i_1},\ldots,x_{i_{c+1}}]$ is the image of a basic commutator, and if $m[x_{i_1},\ldots,x_{i_{c+1}}]$ is such an image, then $n_{i_\alpha-s}|m$. Note that if $s>0$, $i_\alpha=r$ for any basic commutator involving x_1 and x_r only; whereas if $s=0$, $i_\alpha\leqq r-1$ and $i_\alpha=r-1$ for any basic commutator involving x_{r-1} and x_r only.

Of course neither (ii) nor (iii) occurs if $r=1$ in which case $H_2(F, Z)$ is trivial, and (iii) only occurs if $r>s$, so degenerate cases give no trouble. Finally the number of basic commutators of weight $c+1$ on r letters is $\gamma(r, c)$ (see [9]). Putting all this together gives the lemma.

It is now easy to prove the following

THEOREM 3.2. *If Π is a finitely generated abelian group of rank $r>0$ and type $(s; n_1,\ldots,n_t)$, and $\mathfrak{B}=\mathfrak{N}_c$, then $\mathfrak{B}_2(\Pi, Z)$ is of rank ρ and type $(\sigma; \nu_1,\ldots,\nu_\tau)$ where*

$0\leqq\rho-\gamma(r, c)\leqq t+r(r-1)(r-2)/6$ (cf. (3.1)),

$0\leqq\sigma-\gamma(s, c)\leqq s(s-1)(s-2)/6$,

ν_τ divides n_t^2, and if $s=0$, ν_τ divides $n_{t-1}n_t$.

In particular, as $c\to\infty$, $\rho\sim r^{c+1}/(c+1)$ and $\sigma\sim s^{c+1}/(c+1)$.

Proof. Apply Corollary 2.2 with $\mathfrak{W}$ the universal variety and $A=Z$. By [HII, Proposition 2.3], $\mathfrak{W}_1(\Pi, Z)$ and $\mathfrak{B}_1(\Pi, Z)$ are isomorphic; and being Hopf groups, ϕ_1 is an isomorphism. (In fact ϕ_1 and $\theta_1^{\mathfrak{B}}(\Pi, Z)$ are inverses, see Lemma 1.6.) Since $\mathfrak{W}_2(\Pi, Z)=H_3(\Pi, Z)$ a routine calculation gives

$$\mathfrak{W}_2(\Pi, Z) = Z^{s(s-1)(s-2)/6} \oplus \bigoplus_{i=1}^{t} Z_{n_i}^{1+(r-i)(r-i-1)/2}$$

where Z_m denotes Z/mZ, and A^k denotes the direct sum of k copies of A. So $\mathfrak{W}_2(\Pi, Z)$ is of rank $r(r-1)(r-2)/6+t$ and type $(s(s-1)(s-2)/6; n_1, n_1, \ldots, n_t, n_t)$. Now applying Lemma 3.1 gives the result.

In particular, the homology in dimension 2 of a product is not related in a simple way to the homology of the factors, whereas in dimension 1 there is a Künneth formula of sorts (cf. [HI, Theorem 5.2]).

To obtain results in cohomology, apply universal coefficients [HI, Lemma 4.1]. If Π is finite, $\mathfrak{B}^2(\Pi, Q/Z) \cong \mathfrak{B}_2(\Pi, Z)$. The wild behaviour of this group if $\mathfrak{B}=\mathfrak{N}_c$ can hardly be reflected in an obstruction theory.

Bibliography

25. C. R. Leedham-Green, *Homology in varieties of groups*. II, Trans. Amer. Math. Soc. **162** (1971), 15–25.

26. B. Mitchell, *Theory of categories*, Pure and Appl. Math., vol. 17, Academic Press, New York, 1965. MR **34** #2647.

Queen Mary College, London, England