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BOUNDS ON THE RATIO $n(r, a)/S(r)$ FOR MEROMORPHIC FUNCTIONS(1)

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Abstract. Let f be a meromorphic function in the plane. We prove the existence of an absolute constant K such that if $a_1, a_2, \ldots, a_q$ are distinct elements of the Riemann sphere then $\liminf_{r\to\infty} (\sum_{j=1}^{q} |n(r, a_j)/S(r)-1|) < K$. We show by example that in general no such bound exists for the corresponding upper limit. These results involving the unintegrated functionals of Nevanlinna theory are related to previous work of Ahlfors, Hayman and Stewart, and the author.

Introduction. If f is a nonconstant meromorphic function in $|z|<\infty$, we let $n(r, a)=n(r, a, f)$ denote the number of roots counting multiplicity of the equation $f(z)=a$ in $|z|\leqq r$ and let $S(r)=S(r, f)$ denote the mean covering number of the map f of $|z|\leqq r$ into the Riemann sphere. Our principal result is that there exists an absolute constant K such that if $a_1, a_2, \ldots, a_q$ are distinct elements of the Riemann sphere then

$$\liminf_{r\to\infty}\left(\sum_{j=1}^{q}\left|\frac{n(r, a_j)}{S(r)}-1\right|\right) < K.$$

The lower bound for $\sum_{j=1}^{q} (n(r, a_j)/S(r)-1)$ contained in this result is not new; it is weaker than the bound given in Ahlfors' "unintegrated" second fundamental theorem [1] for meromorphic functions. However, Ahlfors' methods give no upper bound for $\sum_{j=1}^{q} (n(r, a_j)/S(r)-1)$. We give an example of a meromorphic function for which $\limsup_{r\to\infty} n(r, 0)/S(r)=\infty$; thus our result cannot be improved to yield $\limsup_{r\to\infty} (\sum_{j=1}^{q} |n(r, a_j)/S(r)-1|)<K$. Finally we use our main result to show for any nonconstant meromorphic function and any Hausdorff measure Λ that there exists $r_n \to \infty$ such that $n(r_n, a)/S(r_n) \to 1$ for all $a \notin A$ where $\Lambda(A)=0$. This is a strengthening of Theorem 2 in [6].

1. Terminology and notation.

We assume familiarity with the basic notation and definitions of Nevanlinna theory. If Σ denotes the Riemann sphere and m

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denotes normalized area measure on Σ, then the spherical characteristic $T(r) = T(r,f)$ of a function f meromorphic in $|z|<\infty$ is defined by

$$T(r) = \int_0^r \frac{S(t)}{t}\,dt \tag{1.1}$$

where

$$S(r) = \int_\Sigma n(r,a)\,dm(a). \tag{1.2}$$

The difference between the spherical and Nevanlinna characteristics of f is bounded as $r\to\infty$. $S(r)$ is called the mean covering number of the map $f\colon |z|\leqq r\to\Sigma$. If γ is an analytic arc on Σ, s is arc length measure on γ, and $L(\gamma)$ is the length of γ, then the mean covering number of $f\colon |z|\leqq r\to\Sigma$ with respect to γ is

$$S(r,\gamma) = \frac{1}{L(\gamma)}\int_\gamma n(r,a)\,ds(a). \tag{1.3}$$

$L(r)$ denotes the length on Σ of the curve $f(re^{i\theta})$, $-\pi\leqq\theta\leqq\pi$. If we must specify the function under consideration, we use the notation $S(r,\gamma,f)$ and $L(r,f)$.

If $a\in\Sigma$ and $b\in\Sigma$, the distance between a and b is defined to be the length of the shorter great circle arc on Σ joining a and b. This distance is denoted by $\delta(a,b)$. Where there is no danger of confusion we do not distinguish between a complex number and its stereographic projection on Σ.

If $E\subset[1,\infty)$, we denote the logarithmic measure of E by $m_l(E)=\int_E dt/t$. If $E_r = E\cap[1,r]$, by the upper (lower) logarithmic density of E we mean

$$\limsup_{t\to\infty}\,(\inf)\; m_l(E_r)/\log r.$$

Suppose $h(r)$ is a continuous, strictly increasing function such that $h(0)=0$. The Hausdorff measure Λ associated with h is defined as follows. For any set A in the plane and any $\rho>0$, we consider all coverings of A by countable collections of disks D_i with radius $r_i<\rho$ and let $H(\rho)$ be the infimum over all such coverings of $\sum_i h(r_i)$. Evidently $H(\rho)$ increases as ρ decreases. We define $\Lambda(A)$ to be $\lim_{\rho\to 0} H(\rho)$.

Many of our inequalities are valid only for sufficiently large values of the variable, denoted by $r>r_0$ or $t>t_0$. It is not intended that r_0 and t_0 have the same value each time they occur.

Our proofs rely heavily on the geometry of the curve $f(re^{i\theta})$. We recall in particular that the derivative with respect to θ of $\arg f(re^{i\theta})$ is $\operatorname{Re}\,(re^{i\theta}f'(re^{i\theta})/f(re^{i\theta}))$ if $f(re^{i\theta})\neq 0,\infty$ and that the derivative of the argument of the vector tangent to the curve $f(re^{i\theta})$ is $\operatorname{Re}\,(1+(re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta})))$ if $f'(re^{i\theta})\neq 0,\infty$.

2. **Statement and discussion of results.**

We state our principal result.

THEOREM 1. *There exist absolute constants $K<\infty$ and $C\in(0,1)$ such that if f is any nonconstant meromorphic function in $|z|<\infty$ then there exists $E\subset[1,\infty)$ having* *lower logarithmic density at least C with the property that, if* $\varepsilon>0$, *there exists* $r_0=r_0(\varepsilon)$ *such that if* $a_1, a_2, \ldots, a_q$ *are elements of* Σ *with* $\delta(a_i, a_j) \geqq \varepsilon$ *for* $i \neq j$, *then*

$$\sum_{j=1}^{q} \left| \frac{n(r, a_j)}{S(r)} - 1 \right| < K \tag{2.1}$$

for all $r \in E$, $r > r_0(\varepsilon)$.

Our proof will in fact show that K may be chosen to be less than 800. A careful examination of the reasoning in §4 shows this estimate for K could be considerably reduced.

Theorem 1 is related to the following result of Ahlfors [1, p. 189], which may be regarded as an unintegrated second fundamental theorem.

THEOREM (AHLFORS). *If* f *is a nonconstant meromorphic function in* $|z| < \infty$ *and* $a_1, a_2, \ldots, a_q$ *are distinct elements of* Σ, *then there exists* $h > 0$ *depending on* $a_1, a_2, \ldots, a_q$ *such that*

$$\sum_{j=1}^{q} \left(\frac{n(r, a_j)}{S(r)} - 1 \right) > -2 - h \frac{L(r)}{S(r)} \tag{2.2}$$

for all $r > 0$.

Since, for $\eta > 0$, there exists $E' \subset [1, \infty)$ having finite logarithmic measure such that $L(r) < S(r)^{1/2+\eta}$ for all $r \in [1, \infty) - E'$, we see that the lower bound for $\sum_{j=1}^{q} (n(r, a_j)/S(r) - 1)$ contained in Theorem 1 is weaker than the corresponding bound in Ahlfors' theorem. However, Theorem 1 also provides an upper bound for, $\sum_{j=1}^{q} (n(r, a_j)/S(r) - 1)$.

The principal result concerning upper bounds for $n(r, a)/S(r)$ is due to Hayman and Stewart [5]. It states that if $\alpha > e$, then there exists $E_\alpha \subset [1, \infty)$ having positive lower logarithmic density such that $\sup_{a \in \Sigma} n(r, a) < \alpha S(r)$ for all $r \in E_\alpha$. The Hayman-Stewart result and the upper bound for $\sum_{j=1}^{q} (n(r, a_j)/S(r) - 1)$ contained in Theorem 1 are independent; this is a consequence of the fact that K does not depend on q in Theorem 1. Thus the two results complement one another.

In §5 we give an example of a meromorphic function with order zero for which $\limsup_{r \to \infty} n(r, 0)/S(r) = \infty$. This example shows that the exceptional set of r-values is in general unbounded both in Theorem 1 and in the Hayman-Stewart result. (A second example showing that the exceptional set of Theorem 1 is unbounded is Example 2 of [6].)

We obtain Theorem 2 as a corollary of Theorem 1.

THEOREM 2. *If* f *is a nonconstant meromorphic function in* $|z| < \infty$ *and* Λ *is a Hausdorff measure, then there exists* $r_p \to \infty$ *and there exists a set* A *of complex numbers with* $\Lambda(A) = 0$ *such that* $n(r_p, a)/S(r_p) \to 1$ *for all* $a \notin A$.

The conclusion of Theorem 2 cannot in general hold as $r \to \infty$ through all values. This is a consequence of Example 2 in [6].

In [6] it is shown for any nonconstant meromorphic function f that there exists $E\subset[1,\infty)$ having logarithmic density zero and there exists a set A' in the plane having inner logarithmic capacity zero such that $\lim_{r\to\infty; r\notin E} n(r,a)/S(r)=1$ for all $a\notin A'$. We note that Theorem 2 has a smaller exceptional set of a-values than does this result, but has a larger exceptional set of r-values.

3. **A lemma.** The following lemma is used in the proof of Theorem 1. It is similar to a result of Fuchs [2]. Our proof is considerably less involved than is the proof of Lemma 1 in [2] because we are not concerned with

$$\operatorname{Im}(re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta})).$$

LEMMA. *There exist absolute constants $K_1<\infty$ and $C\in(0,1)$ such that if f is a nonconstant meromorphic function in $|z|<\infty$ then there exists $E_1\subset[1,\infty)$ having lower logarithmic density at least C such that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\operatorname{Re}\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right|d\theta < K_1S(r,f) \tag{3.1}$$

for all sufficiently large $r\in E_1$.

Proof. It follows from elementary considerations that if f is rational, then (3.1) holds for all sufficiently large values of r with $K_1=3$. Thus we restrict our attention to transcendental functions.

We consider any $r>0$ and associate with r the positive quantity $h=h_r=T(r,f)/3S(r,f)$. We let $\rho_1=re^h$, $\rho_2=re^{2h}$, and $\rho_3=re^{3h}$. By the differentiated Poisson-Jensen formula [4, p. 22] applied to $f'(z)$ we have that if $|z|=r$ and $f'(z)\neq 0,\infty$, then

$$\begin{aligned}\frac{zf''(z)}{f'(z)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f'(\rho_1e^{i\varphi})|\frac{2z\rho_1e^{i\varphi}}{(\rho_1e^{i\varphi}-z)^2}\,d\varphi\\ &\quad+\sum_{|a_n|<\rho_1}\left(\frac{z}{z-a_n}+\frac{\bar a_nz}{\rho_1^2-\bar a_nz}\right)-\sum_{|b_n|<\rho_1}\left(\frac{z}{z-b_n}+\frac{\bar b_nz}{\rho_1^2-\bar b_nz}\right)\end{aligned} \tag{3.2}$$

where $\{a_n\}$ are the zeros of $f'(z)$ and $\{b_n\}$ are the poles of $f'(z)$.

The function $w=z/(z-a)$ maps $|z|=r>|a|$ onto a circle in $\operatorname{Re} w>\frac{1}{2}$; it maps $|z|=r<|a|$ onto a circle in $\operatorname{Re} w<\frac{1}{2}$. Furthermore

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\operatorname{Re}\frac{re^{i\theta}}{re^{i\theta}-a}\,d\theta &= 1 \quad \text{if } r>|a|,\\ &= 0 \quad \text{if } r<|a|.\end{aligned} \tag{3.3}$$

Combining (3.3) with the above observations about the function $z/(z-a)$ we conclude for all complex a that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\operatorname{Re}\frac{re^{i\theta}}{re^{i\theta}-a}\right|d\theta \leqq 1. \tag{3.4}$$

From (3.2) and (3.4) we deduce

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\operatorname{Re}\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right| d\theta \\
(3.5)\qquad &\leqq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\log|f'(\rho_1 e^{i\varphi})||\,d\varphi\right)\left(\frac{1}{2\pi}\frac{2r\rho_1}{(\rho_1^2-r^2)}\int_{-\pi}^{\pi}\frac{\rho_1^2-r^2}{|\rho_1 e^{i\varphi}-re^{i\theta}|^2}d\theta\right) \\
&\quad +2n(\rho_1, 0, f')+2n(\rho_1, \infty, f').
\end{aligned}$$

Standard estimates yield that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\operatorname{Re}\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right| d\theta \\
(3.6)\qquad &\leqq \frac{2r\rho_1}{\rho_1^2-r^2}\{m(\rho_1, f')+m(\rho_1, 1/f')\}+\frac{2}{h}\{N(\rho_2, 0, f')+N(\rho_2, \infty, f')\}.
\end{aligned}$$

Certainly

$$(3.7)\qquad 2r\rho_1/(\rho_1^2-r^2) = 2e^h/(e^{2h}-1) < 1/h.$$

Thus

$$(3.8)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\operatorname{Re}\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right| d\theta \leqq \frac{2}{h}\{3T(\rho_2, f')+O(1)\}.$$

We have

$$(3.9)\qquad T(\rho_2, f') \leqq 2T(\rho_2, f)+m(\rho_2, f'/f).$$

Suppose $\varepsilon>0$. The lemma on the logarithmic derivative [4, p. 36] in combination with (3.8) and (3.9) implies that for $r>r_0(\varepsilon)$

$$(3.10)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\operatorname{Re}\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right| d\theta \leqq \frac{12+\varepsilon}{h}\left(T(\rho_3, f)+3\log^+\frac{1}{h}\right).$$

We apply the following growth lemma, due to Hayman [3, Lemma 1 and Lemma 6].

LEMMA (HAYMAN). *Suppose that $\varphi(x)$ and $\varphi'(x)$ are positive and nondecreasing for $x \geqq x_0$ and that $\varepsilon>0$. For $x>x_0$ let $h'=h'_x=\varphi(x)/\varphi'(x)$. Then $\varphi(x+h') < h'(e+\varepsilon)\varphi'(x)$ holds for all x in a set having lower density at least $C(\varepsilon)$, where $C(\varepsilon)$ is positive and depends only on ε.*

We apply this lemma to $\varphi(x)=T(e^x, f)$ and note that $h'=T(e^x)/S(e^x)$. Thus

$$(3.11)\qquad T(e^{x+h'}) < h'(e+\varepsilon)S(e^x)$$

holds on a set of values of x having lower density at least $C(\varepsilon)$. Introducing the change of variables $r=e^x$, we note $h'=T(r)/S(r)=3h$. Thus

$$(3.12)\qquad T(\rho_3) < 3h(e+\varepsilon)S(r)$$

holds on a set E_2 of r-values having lower logarithmic density at least $C(\varepsilon)$.

Let $E_3=\{r>1 : S(r)>T(r)^2\}$. Then

$$\int_{E_3} \frac{dt}{t} \leqq \int_{E_3} \frac{S(t)}{tT(t)^2}\, dt \leqq \int_1^\infty \frac{S(t)}{tT(t)^2}\, dt = \frac{1}{T(1)} < \infty. \tag{3.13}$$

Finally let $E_1=E_2-E_3$. Then for $r \in E_1$

$$\log^+ \frac{1}{h} = \log^+ \frac{3S(r)}{T(r)} = o(T(r)) \qquad (r \to \infty). \tag{3.14}$$

By (3.13), the lower logarithmic density of E_1 is at least $C(\varepsilon)$. From (3.10), (3.12), and (3.14) we conclude

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \operatorname{Re} \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})} \right| d\theta \leqq 3(12+2\varepsilon)(e+\varepsilon)S(r) \tag{3.15}$$

for $r>r_0(\varepsilon)$, $r \in E_1$. Setting $K_1=3(12+2\varepsilon)(e+\varepsilon)$ and $C=C(\varepsilon)$ we have the desired conclusion. We note that K_1 may thus be chosen to be less than 98.

4. **Theorems 1 and 2.** We now use the lemma to prove Theorem 1. Since the case of f rational clearly causes no difficulty, we assume f is transcendental. Thus $S(r) \to \infty$ as $r \to \infty$. We let K_1 and C be the constants of the lemma and let E_1^* be the set of all $r>1$ for which (3.1) holds and for which $f'(z)\neq 0, \infty$ for any z of modulus r. We let $E'=\{r : L(r)\geqq S(r)^{2/3}\}$ and $E=E_1^*-E'$. Thus E is an open set having lower logarithmic density at least C such that (3.1) holds for each $r \in E$.

We are given $\varepsilon>0$ and elements $a_1, a_2, \ldots, a_q$ of Σ such that $\delta(a_i, a_j)\geqq \varepsilon$ if $i\neq j$. This condition implies the existence of $k(\varepsilon)<\infty$ such that $q<k(\varepsilon)$. Without loss of generality we may assume $a_j\neq\infty$ for $1\leqq j\leqq q$; this follows from the fact [4, p. 13] that transformations of the form $w=(\exp(i\theta_0))(1+\bar{a}z)/(z-a)$ correspond to rotations of Σ and hence preserve distances and areas on Σ.

For any $r \in E$ we partition $\{1, 2, \ldots, q\}$ into sets H_1^r, H_2^r, and H_3^r by the following definitions:

$$\begin{aligned} H_1^r &= \left\{ j : 1 \leqq j \leqq q \text{ and } \left| \frac{n(r, a_j)}{S(r)} - 1 \right| < 1/q \right\}, \\ H_2^r &= \{ j : 1 \leqq j \leqq q \text{ and } n(r, a_j)/S(r) \geqq 1+1/q \}, \quad \text{and} \\ H_3^r &= \{ j : 1 \leqq j \leqq q \text{ and } n(r, a_j)/S(r) \leqq 1-1/q \}. \end{aligned} \tag{4.1}$$

Clearly we have

$$\sum_{j\in H_1^r} \left| \frac{n(r, a_j)}{S(r)} - 1 \right| < 1. \tag{4.2}$$

For $1\leqq j\leqq q$ we let γ_j be a closed line segment in the complex plane which has a_j as one endpoint, which lies on a ray from 0 to ∞, and which has a stereographic projection on Σ of length $\varepsilon/4$. Thus the distance on Σ between γ_i and γ_j is at least $\varepsilon/2$ if $i\neq j$.

We consider a particular j. It follows from Theorem 2 of [1, p. 165] that corresponding to γ_j there exists a constant $h>0$ independent of f such that, for all $r>0$,

$$|S(r,f)-S(r,\gamma_j,f)| < hL(r,f). \tag{4.3}$$

The constant h in fact depends only on the length $\varepsilon/4$ of γ_j and is independent of the position of γ_j. This can be deduced either from the proof of Theorem 2 in [1] or from the following elementary argument. If γ is an arc of a great circle on Σ and has length $\varepsilon/4$ then there is a rotation of Σ which maps γ onto γ_j. Hence for an appropriate choice of a and θ_0, if we let $F=(\exp(i\theta_0))(1+\bar{a}f)/(f-a)$, we then have $S(r,\gamma,f)=S(r,\gamma_j,F)$, $S(r,f)=S(r,F)$, and $L(r,f)=L(r,F)$. Applying (4.3) to the function F we conclude $|S(r,f)-S(r,\gamma,f)|<hL(r,f)$. Thus $h=h(\varepsilon)>0$ is such that for all $r>0$

$$|S(r)-S(r,\gamma_j)| < hL(r) \tag{4.4}$$

for $1\leqq j\leqq q$.

We now consider any $j\in H_2^r$. Since $L(r)<S(r)^{2/3}$ for $r\in E$, we conclude from (4.4) that there exists $r_0(\varepsilon)>0$ such that $r>r_0$ and $r\in E$ together imply for $j\in H_2^r$

$$S(r,\gamma_j) < S(r)(1+1/2k(\varepsilon)). \tag{4.5}$$

For $r>r_0(\varepsilon)$, $r\in E$, and $j\in H_2^r$, the fact that $q<k(\varepsilon)$ combined with (4.5) implies the existence of a point $z_{j,r}\in\gamma_j$ satisfying

$$n(r, z_{j,r}) < S(r)(1+1/2q). \tag{4.6}$$

Since E is open, for all $r>r_0(\varepsilon)$ there exists $r'>r$ independent of $j\in H_2^r$ such that

$$\begin{aligned} &\text{(i)} \quad r'\in E,\\ &\text{(ii)} \quad n(r',a_j)=n(r,a_j),\\ &\text{(iii)} \quad n(r',z_{j,r})=n(r,z_{j,r}), \quad \text{and}\\ &\text{(iv)} \quad S(r')<2S(r). \end{aligned} \tag{4.7}$$

Thus $|z|=r'$ implies $f(z)\neq a_j$ and $f(z)\neq z_{j,r}$ for $j\in H_2^r$. From the argument principle we have for continuous determinations of the arguments involved and for all $j\in H_2^r$

$$\begin{aligned} n(r,a_j)-n(r,z_{j,r}) &= (1/2\pi)[\arg(f(r'e^{i\pi})-a_j)-\arg(f(r'e^{-i\pi})-a_j)]\\ &\quad -(1/2\pi)[\arg(f(r'e^{i\pi})-z_{j,r})-\arg(f(r'e^{-i\pi})-z_{j,r})]. \end{aligned} \tag{4.8}$$

Let $[a_j, z_{j,r}]$ be the closed line segment from a_j to $z_{j,r}$. The transformation $g(z)=(z-a_j)/(z-z_{j,r})$ maps $[a_j, z_{j,r}]$ onto a ray from 0 to ∞. From (4.8) we have

$$n(r,a_j)-n(r,z_{j,r}) = (1/2\pi)[\arg g(f(r'e^{i\pi}))-\arg g(f(r'e^{-i\pi}))]. \tag{4.9}$$

Elementary properties of the argument, applied to the function $g(f)$, enable us to conclude from (4.9) that there exist $n_j\geqq n(r,a_j)-n(r,z_{j,r})-1$ disjoint open intervals

$I_1^j, I_2^j, \ldots, I_{n_j}^j$ in $(-\pi, \pi)$ with $I_k^j=(s_k^j, t_k^j)$ such that for $1 \leqq k \leqq n_j$

$$\text{(4.10)}\quad \begin{aligned} &\text{(i)}\quad f(r' \exp(is_k^j)) \in [a_j, z_{j,r}] \subset \gamma_j,\\ &\text{(ii)}\quad f(r' \exp(it_k^j)) \in [a_j, z_{j,r}] \subset \gamma_j, \quad \text{and}\\ &\text{(iii)}\quad [\arg(f(r' \exp(it_k^j))-a_j)-\arg(f(r' \exp(is_k^j))-a_j)]\\ &\qquad -[\arg(f(r' \exp(it_k^j))-z_{j,r})-\arg(f(r' \exp(is_k^j))-z_{j,r})] \geqq 2\pi. \end{aligned}$$

For notational convenience we do not indicate the dependence of n_j or the intervals I_k^j on r. From (4.10) we conclude that the curve $f(r'e^{i\theta})$, restricted to $s_k^j \leqq \theta \leqq t_k^j$, has a tangent parallel to γ_j at some point in (s_k^j, t_k^j) and has a tangent perpendicular to γ_j at some point in (s_k^j, t_k^j). Hence for $1 \leqq k \leqq n_j$,

$$\text{(4.11)}\quad \frac{1}{2\pi}\int_{s_k^j}^{t_k^j} \left| \operatorname{Re} \frac{r'e^{i\theta}f''(r'e^{i\theta})}{f'(r'e^{i\theta})}+1 \right| d\theta \geqq \frac{\pi}{2(2\pi)} = \frac{1}{4}.$$

For $r \in E$, $r>r_0(\varepsilon)$, and $j \in H_2^r$, we let

$$\text{(4.12)}\quad \begin{aligned} U_j &= \{I_k^j : 1 \leqq k \leqq n_j\} \quad \text{and}\\ V_j &= \{I_k^j : \text{there exists } \theta \in (s_k^j, t_k^j) \text{ such that } f(r'e^{i\theta}) \in \gamma_i \text{ for some } i \neq j\}. \end{aligned}$$

We do not indicate the dependence of U_j and V_j on r. From the definition of V_j, (4.10)(i), and (4.10)(ii) it follows that $\bigcup_{j\in H_2^r} U_j - \bigcup_{j\in H_2^r} V_j$ is a disjoint collection of intervals. We note that $I_k^j \in V_j$ implies that the length on Σ of the curve $f(r'e^{i\theta})$, $s_k^j \leqq \theta \leqq t_k^j$, is at least ε since the distance between γ_i and γ_j is at least $\varepsilon/2$ if $i \neq j$. From (4.7) it follows that for each $r \in E$ we have $L(r')<S(r')^{2/3}<2S(r)^{2/3}$. The last two observations together imply for each $j \in H_2^r$ that the number of elements of V_j is at most $L(r')/\varepsilon<2S(r)^{2/3}/\varepsilon$. For $r>r_0(\varepsilon)$ and $r \in E$ we thus have that $\bigcup_{j\in H_2^r} U_j - \bigcup_{j\in H_2^r} V_j$ is a disjoint collection of at least

$$\sum_{j\in H_2^r} \{n(r, a_j)-n(r, z_{j,r})-1-2S(r)^{2/3}/\varepsilon\}$$

intervals, for each of which (4.11) holds. Thus

$$\text{(4.13)}\quad \begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \operatorname{Re} \frac{r'e^{i\theta}f''(r'e^{i\theta})}{f'(r'e^{i\theta})}+1 \right| d\theta\\ &\qquad \geqq \frac{1}{4}\left(\sum_{j\in H_2^r} \{n(r, a_j)-n(r, z_{j,r})-1-2S(r)^{2/3}/\varepsilon\}\right). \end{aligned}$$

From (4.6) and (4.13) we deduce for $r>r_0(\varepsilon)$, $r \in E$,

$$\text{(4.14)}\quad \begin{aligned} &1+\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \operatorname{Re} \frac{r'e^{i\theta}f''(r'e^{i\theta})}{f'(r'e^{i\theta})} \right| d\theta\\ &\qquad \geqq \frac{1}{4}\left(-q+\sum_{j\in H_2^r} \{n(r, a_j)-S(r)-S(r)/2q-2S(r)^{2/3}/\varepsilon\}\right)\\ &\qquad \geqq \frac{1}{4}\left(\sum_{j\in H_2^r} \{n(r, a_j)-S(r)\}\right)-q/4-S(r)/8-qS(r)^{2/3}/2\varepsilon. \end{aligned}$$

From (4.7)(iv), the lemma of §3, and the fact that $q<k(\varepsilon)$ we conclude

$$(4.15)\quad \frac{1}{4}\left(\sum_{j\in H_2^r}\{n(r,a_j)-S(r)\}\right) \leqq k(\varepsilon)/4+S(r)/8+k(\varepsilon)S(r)^{2/3}/2\varepsilon+1+2K_1S(r).$$

Thus for $r>r_0(\varepsilon)$ and $r\in E$,

$$(4.16)\quad \sum_{j\in H_2^r}\{n(r,a_j)-S(r)\} \leqq (8K_1+1)S(r).$$

A similar discussion applies to the set H_3^r. The required inequality in this case also follows from Ahlfors' theorem mentioned in §2 if one checks that the constant h of that theorem depends only on $\min_{i\neq j}\delta(a_i, a_j)$. The proof is completed upon combining (4.2), (4.16), and an inequality for H_3^r similar to (4.16).

We now use Theorem 1 to prove Theorem 2. Let h be the strictly increasing function associated with the Hausdorff measure Λ. Let p be a positive integer and $\rho_p>0$ be such that $h(\rho_p)<1/p^3$. There exists $c_p<\infty$ such that $|a-b|/\delta(a,b)<c_p$ if $|a|<p$, $|b|<p$ and $a\neq b$. Applying Theorem 1 with $\varepsilon=\rho_p/c_p$ we conclude there exists $r_p>2^p$ such that $\{|a|<p : |n(r_p, a)/S(r_p)-1|\geqq 1/p\}$ can be covered by a set of $[Kp]$ disks in the plane each of radius ρ_p where [ ] is the greatest integer function. We denote the union of these disks by D_p. Thus $n(r_p, a)/S(r_p)\to 1$ for all $a\notin A$ where $A=\bigcap_{m=1}^{\infty}\bigcup_{p=m}^{\infty} D_p$. From the definition of Hausdorff measure we see that $\Lambda(A)$ does not exceed $\sum_{p=m}^{\infty}(Kp)h(\rho_p)$ for any value of m. Hence $\Lambda(A)=0$.

5. **An example.** We now exhibit a meromorphic function for which

$$\limsup_{r\to\infty} n(r,0)/S(r) = \infty.$$

We define a sequence $N_n=2^{2^{4n}}$ and verify directly that

$$(5.1)\quad 4^{N_n}\sum_{j=n+1}^{\infty} N_j 2^{-N_j} = o(1),$$

$$(5.2)\quad \sum_{j=1}^{n-1} N_j^2 < N_n^{1/2}, \qquad n=2,3,4,\ldots,$$

and

$$(5.3)\quad 2^{N_{n+1}} > 2(4^{N_n}), \qquad n=1,2,3,\ldots.$$

In (5.1) and throughout §5 all quantities indicated to be $o(1)$ are $o(1)$ as $n\to\infty$.

The required function is

$$(5.4)\quad f(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{4^{N_n}}\right)^{N_n^2}\Big/\prod_{n=1}^{\infty}\left(1-\frac{z}{2^{N_n}}\right)^{N_n}.$$

We adopt the notation $f_n(z)=g_n(z)/h_n(z)$ where

$$(5.5)\quad g_n(z) = (1+z/4^{N_n})^{N_n^2} \quad\text{and}\quad h_n(z) = (1-z/2^{N_n})^{N_n}.$$

For $n \geqq 2$ we let $r_n = 4^{N_n}$ and select $\alpha_n \in (0, \frac{1}{2}]$ in a manner to be specified later. Letting $\rho_n = r_n(1+\alpha_n)$ we have if $|z| = \rho_n$ that

$$\log \left| \prod_{j=1}^{n-1} g_j(z) \right| \leqq \log \prod_{j=1}^{n-1} \left(\frac{3(4^{N_n})}{4^{N_j}} \right)^{N_j^2} = \sum_{j=1}^{n-1} N_j^2(\log 3 + N_n \log 4 - N_j \log 4). \tag{5.6}$$

Likewise we find for $|z| = \rho_n$

$$\log \left| \prod_{j=1}^{n-1} h_j(z) \right| \geqq \log \prod_{j=1}^{n-1} \left(\frac{4^{N_n}}{2(2^{N_j})} \right)^{N_j} = \sum_{j=1}^{n-1} N_j(N_n \log 4 - N_j \log 2 - \log 2). \tag{5.7}$$

Combining (5.2), (5.6), and (5.7) we see that on $|z| = \rho_n$

$$\log^+ \left| \prod_{j=1}^{n-1} f_j(z) \right| = o(N_n^2). \tag{5.8}$$

Similarly if $|z| = \rho_n$ then by (5.1)

$$\log \left| \prod_{j=n+1}^{\infty} g_j(z) \right| \leqq \left(\frac{3}{2} \right) \sum_{j=n+1}^{\infty} N_j^2 4^{N_n - N_j} = o(1) \tag{5.9}$$

and by (5.1) and (5.3)

$$\log \left| \prod_{j=n+1}^{\infty} h_j(z) \right| \geqq -3 \sum_{j=n+1}^{\infty} N_j 4^{N_n} 2^{-N_j} = o(1). \tag{5.10}$$

If $|z| = \rho_n$ we have

$$\log |2^{-N_n^2} h_n(z)| = \log |2^{-N_n} - z/4^{N_n}|^{N_n} \geqq N_n \log |1 + \alpha_n - 2^{-N_n}| \geqq -2N_n 2^{-N_n} = o(1). \tag{5.11}$$

Thus on $|z| = \rho_n$

$$\log |h_n(z)| \geqq N_n^2 \log 2 - o(1). \tag{5.12}$$

From (5.8), (5.9), (5.10), and (5.12) we conclude there exists $\varepsilon_1(n) \to 0$ as $n \to \infty$ such that for any choice of $\alpha_n \in (0, \frac{1}{2}]$ we have everywhere on $|z| = \rho_n = r_n(1+\alpha_n)$

$$\log |f(z)| \leqq N_n^2 \{\log |\tfrac{1}{2}(1 + z/r_n)| + \varepsilon_1(n)\}. \tag{5.13}$$

An elementary calculation shows that $\log |\frac{1}{2}(1 + \rho_n e^{i\theta}/r_n)| > 0$ if and only if $\cos \theta > 1 - (4\alpha_n + \alpha_n^2)/(2 + 2\alpha_n)$. For $0 < \alpha_n \leqq \frac{1}{2}$ we have

$$\cos 4\alpha_n^{1/2} \leqq 1 - 4^2\alpha_n/2! + 4^4\alpha_n^2/4! < 1 - 2\alpha_n < 1 - (4\alpha_n + \alpha_n^2)/(2 + 2\alpha_n). \tag{5.14}$$

We deduce that if $\log |\frac{1}{2}(1+\rho_n e^{i\theta}/r_n)| > 0$ and $\theta \in [-\pi, \pi]$ then $|\theta| \leqq 4\alpha_n^{1/2}$. Combining this fact with (5.13) we have

$$m(\rho_n, f) \leqq N_n^2\left\{\varepsilon_1(n) + \frac{8\alpha_n^{1/2}}{2\pi}\log\left(1+\frac{\alpha_n}{2}\right)\right\} \leqq N_n^2\{\varepsilon_1(n)+\alpha_n^{3/2}\}. \tag{5.15}$$

Letting $\varepsilon_2(n) = N_n^{-1}$ we have

$$\begin{aligned} N(\rho_n, f) - N(r_n, f) &= n(r_n, f)\log(1+\alpha_n) \\ &\leqq 2N_n \log(1+\alpha_n) \leqq N_n = \varepsilon_2(n)N_n^2. \end{aligned} \tag{5.16}$$

Because the difference between the spherical characteristic and the Nevanlinna characteristic is bounded, we see from (5.15), (5.16) and the definition of $\varepsilon_2(n)$ that for $n > n_0$

$$\begin{aligned} S(r_n) &< \frac{T(\rho_n, f) - T(r_n, f)}{\log(1+\alpha_n)} \\ &\leqq \frac{m(\rho_n, f) + N(\rho_n, f) - N(r_n, f) + O(1)}{\log(1+\alpha_n)} \\ &\leqq \frac{N_n^2\{\varepsilon_1(n) + \varepsilon_2(n) + \alpha_n^{3/2}\} + O(1)}{\alpha_n/2} \\ &\leqq \frac{2N_n^2\{\varepsilon_1(n) + 2\varepsilon_2(n) + \alpha_n^{3/2}\}}{\alpha_n}. \end{aligned} \tag{5.17}$$

We choose $\alpha_n = \min(\frac{1}{2}, [\varepsilon_1(n) + 2\varepsilon_2(n)]^{2/3})$ and note that $\alpha_n \to 0$ as $n \to \infty$. Thus for $n > n_0$

$$S(r_n) < 4N_n^2 \frac{\varepsilon_1(n) + 2\varepsilon_2(n)}{(\varepsilon_1(n) + 2\varepsilon_2(n))^{2/3}} = o(N_n^2). \tag{5.18}$$

Since f has N_n^2 zeros on $|z| = r_n$, we conclude that $\limsup_{r\to\infty} n(r, 0)/S(r) = \infty$.

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